Unifying Sequent Systems for Gödel-Löb Provability Logic via Syntactic Transformations

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Abstract

We demonstrate the inter-translatability of proofs between the most prominent sequent-based formalisms for Gödel-Löb provability logic. In particular, we consider Sambin and Valentini's sequent system $\mathsf{GL}_{\mathsf{seq}}$, Shamkanov's non-wellfounded and cyclic sequent systems GL_{∞} and $\mathsf{GL}_{\mathsf{circ}}$, Poggiolesi's tree-hypersequent system CSGL, and Negri's labeled sequent system G3GL. Shamkanov provided proof-theoretic correspondences between $\mathsf{GL}_{\mathsf{seq}}$, GL_{∞}, and $\mathsf{GL}_{\mathsf{circ}}$, and Goré and Ramanayake showed how to transform proofs between CSGL and G3GL, however, the exact nature of proof transformations between the former three systems and the latter two systems has remained an open problem. We solve this open problem by showing how to restructure tree-hypersequent proofs into an end-active form and introduce a novel *linearization technique* that transforms such proofs into linear nested sequent proofs. As a result, we obtain a new proof-theoretic tool for extracting linear nested sequent systems from tree-hypersequent systems, which yields the first cut-free linear nested sequent calculus LNGL for Gödel-Löb provability logic. We show how to transform proofs in LNGL into a certain normal form, where proofs repeat in stages of modal and local rule applications, and which are translatable into $\mathsf{GL}_{\mathsf{seq}}$ and G3GL proofs. These new syntactic transformations, together with those mentioned above, establish full proof-theoretic correspondences between $\mathsf{GL}_{\mathsf{seq}}$, GL_{∞}, $\mathsf{GL}_{\mathsf{circ}}$, CSGL, G3GL, and LNGL while also giving (to the best of the author's knowledge) the first constructive proof mappings between structural (viz. labeled, tree-hypersequent, and linear nested sequent) systems and a cyclic sequent system.

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1 Introduction

Provability logics are a class of modal logics where the $\Box$ operator is read as 'it is provable that' in some arithmetical theory. One of the most prominent provability logics is Gödel-Löb logic (GL), which arose out of the work of Löb, who formulated a set of conditions on the provability predicate of Peano Arithmetic (PA). The logic GL can be axiomatized as an extension of the basic modal logic K by the single axiom $\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$, called *Löb's axiom.* It is well-known that the axioms of GL are sound and complete relative to transitive and conversely-wellfounded relational models [38]. In a landmark result, Solovay [41] remarkably showed that GL is complete for PA's provability logic, i.e., GL proves everything that PA can prove about its own provability predicate.

The logic GL enjoys a rich structural proof theory, possessing a number of cut-free sequent-style systems. Sequent systems in the style of Gentzen were originally provided by Sambin and Valentini in the early 1980s [36, 37]; see also Avron [2]. (NB. In this work, we take a *Gentzen system* to be a proof system whose rules operate over *Gentzen sequents*, i.e., expressions of the form $\varphi_1, \ldots, \varphi_n \vdash \psi_1, \ldots, \psi_k$ such that φ_i and ψ_j are logical formulae.) Since then, a handful of alternative systems have been introduced, each of which either

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generalizes the structure of sequents or generalizes the notion of proof. The labeled sequent system G3GL was provided by Negri [31] and uses *labeled sequents* in proofs, which are binary graphs whose nodes are Gentzen sequents. In a similar vein, the tree-hypersequent system CSGL was provided by Poggiolesi [35] and uses *tree-hypersequents* in proofs, which are trees whose nodes are Gentzen sequents. The use of *(types of) graphs* of Gentzen sequents in proofs (as in G3GL and CSGL) allows for the systems to possess properties beyond those of the original Gentzen systems [36, 37]. For example, both G3GL and CSGL enjoy invertibility of all rules, rules are symmetric (i.e., for each logical connective, there is at least one rule that introduces it in the antecedent and at least one rule that introduces it in the consequent of the rule's conclusion), and the close connection between the syntax of such sequents and GL's relational semantics makes such systems suitable for counter-model extraction.

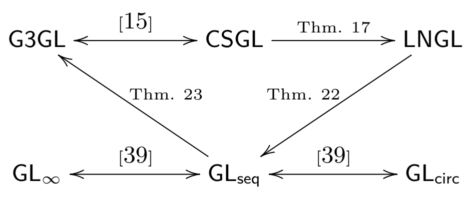
Rather than generalizing the structure of sequents, Shamkanov [39] showed that one could obtain alternative cut-free sequent systems for GL by generalizing the structure of proofs. In particular, by taking the sequent calculus for the modal logic K4 and allowing for *non-wellfounded proofs*, one obtains a non-wellfounded sequent system GL_{∞} for GL. Non-wellfounded proofs were introduced to capture (co)inductive reasoning, and are potentially infinite trees of sequents such that (1) every parent node is the conclusion of a rule with its children the corresponding premises and (2) infinite branches satisfy a certain *progress condition*, which ensures soundness (cf. [5, 11, 32, 39]). For GL, non-wellfounded proofs correspond to regular trees (i.e., only contain finitely many distinct sub-trees), which means such proofs can be 'folded' into finite trees of sequents such that leaves are 'linked' to internal nodes of the tree, giving rise to *cyclic proofs* (cf. [1, 3, 4, 10]). Shamkanov [39] additionally showed that one could obtain a cut-free cyclic sequent system $\mathsf{GL}_{\mathsf{circ}}$ for GL by allowing cyclic proofs in K4's sequent calculus, which was then used to provide the first syntactic proof of the Lyndon interpolation property for GL.

Due to the diversity in GL's proof theory, it is natural to wonder about the relationships between the various systems that have been introduced. Typically, proof systems are related by means of *proof transformations*, which are functions that map proofs from one calculus into another, are sensitive to the structure of the input proof, and operate syntactically by permuting rules, replacing rules, or adding/deleting sequent structure in the input proof to yield the output proof. Studying proof transformations between sequent systems is a beneficial enterprise as it lets one transfer results from one system to another, thus alleviating the need of independent proofs in each system (e.g., [9, 15]). Moreover, one can measure the relative sizes or certain characteristics of proofs, giving insight into which systems are better suited for specific (automated) reasoning tasks, and letting one 'toogle' between differing formalisms when one is better suited for a task than another (e.g., [24, 23]).

Indeed, the question of the relationship between G3GL and CSGL was asked by Poggiolesi [35] and answered in full by Goré and Ramanayake [15], who provided constructive mappings of proofs between the two systems. Similarly, Shamkanov [39] provided syntactic mappings of proofs between the systems GL_{∞} and $\mathsf{GL}_{\mathsf{circ}}$, and the Gentzen system $\mathsf{GL}_{\mathsf{seq}}$ (an equivalent reformulation of Sambin and Valentini's systems [36, 37]). Nevertheless, the inter-translatability of proofs between the former two structural sequent systems (G3GL and CSGL) and the latter three sequent systems ($\mathsf{GL}_{\mathsf{seq}}$, GL_{∞}, and $\mathsf{GL}_{\mathsf{circ}}$) has yet to be identified, and presents a non-trivial open problem that we solve in this paper. Thus, our first contribution in this paper is to 'complete the picture' and establish complete correspondences between the above five mentioned sequent systems by means of syntactic proof transformations.

There is an inherent difficulty in transforming proofs that use structural sequents (e.g., labeled sequents or tree-hypersequents) into proofs that use Gentzen sequents. This is due to the fact that structural sequents are (types of) graphs of Gentzen sequents, and thus, possess a more complicated structure that must be properly ‘shed’ during proof transformations [23, 26]. To overcome this difficulty and define proof transformations from G3GL and CSGL to $\mathsf{GL}_{\mathsf{seq}}$, we rely on three techniques: first, we show how to restructure proofs in CSGL so that they are *end-active* (cf. [21]), meaning rules only affect data at leaves or parents of leaves in tree-hypersequents. Second, we introduce a novel *linearization technique*, whereby we show how to shed the tree structure of tree-hypersequents in end-active proofs, yielding a proof consisting solely of *linear nested sequents* [21], i.e., lines whose nodes are Gentzen sequents. Linear nested sequents were introduced as an alternative (albeit equivalent) formalism to 2-sequents [29, 30] that allows for sequent systems with complexity-optimal proof-search that also retain fundamental admissibility and invertibility properties [21]. The presented linearization technique is new and shows how to extract linear nested sequent systems from tree-hypersequent systems, serving as the second contribution of this paper. We conjecture that this method can be generalized and applied in other settings to provide new linear nested sequent systems for modal and related logics. The technique also yields the first (cut-free) linear nested sequent calculus for GL, which we dub LNGL, and which is the third contribution of this paper. Last, we show that proofs in LNGL can be put into a specific normal form that repeats in stages of modal and local rules. Such proofs are translatable into $\mathsf{GL}_{\mathsf{seq}}$ proofs, which are translatable into G3GL proofs, thus establishing purely syntactic proof transformations between the most prominent sequent systems for GL. These proof transformations and systemic correspondences are summarized in Figure 1 below.

Outline of Paper. In Section 2, we recall the language, semantics, and axioms of Gödel-Löb logic. In Section 3, we discuss Negri’s labeled system G3GL [31], Poggiolesi’s tree-hypersequent system CSGL [35], and Goré and Ramanayake correspondence result for the two systems [15]. In Section 4, we show how to put proofs in CSGL into an end-active form (Theorem 16) and specify our novel linearization method (Theorem 17), which yields the new linear nested sequent system LNGL for GL. In Section 5, we recall the sequent calculus $\mathsf{GL}_{\mathsf{seq}}$ due to Sambin and Valentini [36, 37], Shamkanov’s non-wellfounded system GL_{∞} and cyclic system $\mathsf{GL}_{\mathsf{circ}}$, as well as Shamkanov’s correspondence result for the aforementioned three systems [39]. We then show how to transform proofs in LNGL into proofs in $\mathsf{GL}_{\mathsf{seq}}$ (Theorem 22) and how to transform proofs in $\mathsf{GL}_{\mathsf{seq}}$ into proofs in G3GL (Theorem 23). This establishes a six-way correspondence between the systems G3GL, CSGL, $\mathsf{GL}_{\mathsf{seq}}$, GL_{∞}, $\mathsf{GL}_{\mathsf{circ}}$, and LNGL. Last, in Section 6, we conclude and discuss future work.



Name	Type of System
G3GL	Labeled Sequent System
CSGL	Tree-Hypersequent System
LNGL	Linear Nested Sequent System
$\mathsf{GL}_{\mathsf{seq}}$	Gentzen System
GL_{∞}	Non-Wellfounded Sequent System
$\mathsf{GL}_{\mathsf{circ}}$	Cyclic Sequent System

Figure 1 Proof transformations and correspondences between sequent systems for GL.

2 Gödel-Löb Provability Logic

We let $\mathsf{Prop} := \{p, q, r, \ldots\}$ be a countable set of *propositional atoms* and define the language $\mathcal{L}$ to be the set of all formulae generated via the following grammar in BNF:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box\varphi$$

where p ranges over Prop. We use φ, ψ, χ, ... to denote formulae in $\mathcal{L}$ and define $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$ and $\varphi \to \psi := \neg\varphi \vee \psi$ as usual.

▶ **Definition 1** (Model). *We define a* model *to be a tuple $M = (W, R, V)$ such that*
- *W is a non-empty set of worlds w, u, v, ... (occasionally annotated);*
- *$R \subseteq W \times W$ is transitive and conversely-wellfounded;*[1]
- *$V : \mathsf{Prop} \mapsto 2^W$ is a* valuation function.

▶ **Definition 2** (Semantic Clauses). *We define the* satisfaction *of a formula φ in a model M at world w, written $M, w \models \varphi$, recursively as follows:*
- *$M, w \models p$ iff $w \in V(p)$;*
- *$M, w \models \neg\varphi$ iff $M, w \not\models \varphi$;*
- *$M, w \models \varphi \vee \psi$ iff $M, w \models \varphi$ or $M, w \models \psi$;*
- *$M, w \models \Box\varphi$ iff $\forall u \in W$, if $(w, u) \in R$, then $M, u \models \varphi$;*
- *$M \models \varphi$ iff $\forall w \in W$, $M, w \models \varphi$.*

We write $\models \varphi$ and say that φ is valid *iff for all models M, $M \models \varphi$.* Gödel-Löb logic (GL) *is defined to be the set $\mathsf{GL} \subset \mathcal{L}$ of all valid formulae.*

As shown by Segerberg [38], the logic GL can be axiomatized by extending the axioms of the modal logic K with Löb's axiom $\Box(\Box\varphi \to \varphi) \to \Box\varphi$.

3 Labeled and Tree Sequent Systems

In this section, we review the labeled sequent calculus G3GL by Negri [31] and its correspondence (proven by Goré and Ramanayake [15]) with a notational variant of Poggiolesi's tree-hypersequent system for GL [35]. *Labeled sequents* are binary graphs of traditional Gentzen sequents, which encode the relational semantics of a logic directly in the syntax of sequents. The formalism of labeled sequents has been extensively studied with the inception of the formalism dating back to the work of Kanger [18] and achieving its modern form in the work of Simpson [40]. It has been shown that labeled sequent systems can capture sizable and diverse classes of logics in a cut-free manner while exhibiting fundamental properties such as the admissibility of various structural rules and the invertibility of rules [40, 43, 31].

By contrast, *tree-hypersequents*, which are more traditionally known as *nested sequents*, are trees of Gentzen sequents. The formalism was introduced independently by Kashima [19] and Bull [7] with further influential works provided by Brünnler [6] and Poggiolesi [34, 35]. Such systems arose out of a call for cut-free sequent-style systems for logics not known to possess a cut-free Gentzen system, such as the tense logic $\mathsf{K_t}$ and the modal logic S5. Like labeled sequents, tree-hypersequent systems exhibit fundamental admissibility and invertibility properties, having been defined for large classes of various logics such as tense logics [14], intuitionistic modal logics [42, 25], and first-order non-classical logics [13, 27].

As observed by Goré and Ramanayake [15], restricting labeled sequents to be trees, rather than more general, binary graphs (which may be disconnected or include cycles), yields labeled tree sequents (cf. [17]), which are a notational variant of tree-hypersequents/nested sequents. Via this observation, the authors established bi-directional proof transformations between Negri's labeled sequent calculus and Poggiolesi's tree-hypersequent calculus for GL. Proof theoretic correspondences between labeled and nested systems for various other logics have been established in recent years as well; e.g., for tense logics [9], first-order intuitionistic

[1] We note that R is conversely-wellfounded *iff* it does not contain any infinite ascending R-chains.

$$\frac{}{\mathcal{R}, \Gamma, x : p \vdash x : p, \Delta}\ \mathsf{id}_1 \qquad \frac{}{\mathcal{R}, \Gamma, x : \Box\varphi \vdash x : \Box\varphi, \Delta}\ \mathsf{id}_2 \qquad \frac{}{\mathcal{R}, xRx, \Gamma \vdash \Delta}\ \mathsf{ir}$$

$$\frac{\mathcal{R}, xRy, yRz, xRz, \Gamma \vdash \Delta}{\mathcal{R}, xRy, yRz, \Gamma \vdash \Delta}\ \mathsf{tr} \qquad \frac{\mathcal{R}, \Gamma, x : \varphi \vdash \Delta \qquad \mathcal{R}, \Gamma, x : \psi \vdash \Delta}{\mathcal{R}, \Gamma, x : \varphi \vee \psi \vdash \Delta}\ \vee\mathsf{L}$$

$$\frac{\mathcal{R}, \Gamma \vdash x : \varphi, \Delta}{\mathcal{R}, \Gamma, x : \neg\varphi \vdash \Delta}\ \neg\mathsf{L} \qquad \frac{\mathcal{R}, \Gamma, x : \varphi \vdash \Delta}{\mathcal{R}, \Gamma \vdash x : \neg\varphi, \Delta}\ \neg\mathsf{R} \qquad \frac{\mathcal{R}, \Gamma \vdash x : \varphi, x : \psi, \Delta}{\mathcal{R}, \Gamma \vdash x : \varphi \vee \psi, \Delta}\ \vee\mathsf{R}$$

$$\frac{\mathcal{R}, xRy, \Gamma, x : \Box\varphi, y : \varphi \vdash \Delta}{\mathcal{R}, xRy, \Gamma, x : \Box\varphi \vdash \Delta}\ \Box\mathsf{L} \qquad \frac{\mathcal{R}, xRy, \Gamma, y : \Box\varphi \vdash y : \varphi, \Delta}{\mathcal{R}, \Gamma \vdash x : \Box\varphi, \Delta}\ \Box\mathsf{R}^{\dagger}$$

Figure 2 Labeled Sequent Calculus G3GL for GL. The □R rule is subject to a side condition †, namely, the rule is applicable only if the label y is fresh.

logics [23, 22], and intuitionistic modal logics [25]. Recently, it was proven in a general setting that correspondences between labeled and nested systems are a product of two underlying proof transformation techniques, structural rule elimination and introduction, and that (Horn) labeled and nested systems tend to come in pairs, being dual to one another [28].

Reducing Negri's labeled sequent system to one that uses trees, as opposed to binary graphs, is the first step in establishing syntactic correspondences between the various sequent systems for GL. As shown in the sequel, we will systematically reduce the structure of sequents in proofs: first, going from binary graphs of Gentzen sequents to trees of Gentzen sequents (this section), then from trees of Gentzen sequents to lines of Gentzen sequents (Section 4), and last from lines of Gentzen sequents to Gentzen sequents themselves, which are easily embedded in labeled sequent proofs, completing the circuit of correspondences (Section 5). This yields correspondences between the most widely regarded sequent systems for GL, as depicted in Figure 1.

3.1 Labeled Sequents

We let Lab $= \{x, y, z, \ldots\}$ be a countably infinite set of *labels*, define a *relational atom* to be an expression of the form xRy with $x, y \in$ Lab, and define a *labeled formula* to be an expression of the form $x : \varphi$ such that $x \in$ Lab and $\varphi \in \mathcal{L}$. We use upper-case Greek letters $\Gamma, \Delta, \Sigma, \ldots$ to denote finite multisets of labeled formulae. For a set $\mathcal{R}$ of relational atoms and multiset Γ of labeled formulae, we let $\mathrm{Lab}(\mathcal{R})$, $\mathrm{Lab}(\Gamma)$, and $\mathrm{Lab}(\mathcal{R}, \Gamma)$ be the sets of all labels occurring therein. For a multiset Γ of labeled formulae, we define the multiset $\Gamma(x) := \{\varphi \mid x : \varphi \in \Gamma\}$, for a multiset of formulae $\Gamma := \varphi_1, \ldots, \varphi_n$, we define $x : \Gamma := x : \varphi_1, \ldots, x : \varphi_n$, and for multisets Γ and Δ of labeled formulae, we let Γ, Δ denote the multiset union of the two. We define a *labeled sequent* to be an expression of the form $\mathcal{R}, \Gamma \vdash \Delta$ with $\mathcal{R}$ a set of relational atoms and Γ, Δ a multiset of labeled formulae. Given a labeled sequent $\mathcal{R}, \Gamma \vdash \Delta$, we refer to $\mathcal{R}, \Gamma$ as the *antecedent* and Δ as the *consequent*. Below, we clarify the interpretation of labeled sequents by explaining their evaluation over models.

▶ **Definition 3** (Labeled Sequent Semantics). *Let $M = (W, R, V)$ be a model. We define an M-*assignment *to be a function μ: Lab $\to W$. A labeled sequent $\mathcal{R}, \Gamma \vdash \Delta$ is* satisfied *on M with M-assignment μ iff if for all $xRy \in \mathcal{R}$ and $x : \varphi \in \Gamma$, $(\mu(x), \mu(y)) \in R$ and $M, \mu(x) \models \varphi$, then there exists a $y : \psi \in \Delta$ such that $M, \mu(y) \models \psi$. A labeled sequent is defined to be* valid *iff it is satisfied on all models M with all M-assignments; a labeled sequent is defined to be* invalid *otherwise.*

$$\frac{\mathcal{R}, \Gamma \vdash \Delta}{\mathcal{R}(x/y), \Gamma(x/y) \vdash \Delta(x/y)} (x/y) \qquad \frac{\mathcal{R}, \Gamma \vdash \Delta}{\mathcal{R}, \mathcal{R}', \Gamma, \Gamma' \vdash \Delta, \Delta'} \mathsf{w} \qquad \frac{\mathcal{R}, \Gamma, x : \varphi, x : \varphi \vdash \Delta}{\mathcal{R}, \Gamma, x : \varphi \vdash \Delta} \mathsf{cL}$$

$$\frac{\mathcal{R}, \Gamma \vdash x : \varphi, x : \varphi, \Delta}{\mathcal{R}, \Gamma \vdash x : \varphi, \Delta} \mathsf{cR} \qquad \frac{\mathcal{R}, \Gamma \vdash x : \varphi, \Delta \qquad \mathcal{R}, \Gamma, x : \varphi \vdash \Delta}{\mathcal{R}, \Gamma \vdash \Delta} \mathsf{cut}$$

Figure 3 Admissible rules.

Negri's labeled sequent calculus G3GL (adapted to our signature) is shown in Figure 2. The labeled calculus consists of three initial rules id_1, id_2, and ir. We refer to the conclusion of an initial rule as a *initial sequent*. The tr rule is a structural rule that bottom-up adds transitive edges to labeled sequents, and the remaining rules form pairs of left and right logical rules, introducing complex logical formulae into either the antecedent or consequent of the rule's conclusion. We note that the $\Box$R rule is subject to a side condition, namely, the label y must be *fresh* in any application of the rule, i.e., the label y is forbidden to occur in the conclusion. We refer to the distinguished formulae in the conclusion (premises) of a rule as the *principal formulae* (*auxiliary formulae*, respectively). For example, $x : \Box\varphi$ is principal in $\Box$R and $xRy, x : \Box\varphi, y : \varphi$ are auxiliary.

▶ Remark 4. Negri's original labeled system G3GL includes the following $\Box$L$'$ rule rather than the $\Box$L rule. However, the left premise of the $\Box$L$'$ rule is provable in G3GL using $\Box$L, $\Box$R, and tr [31]. We therefore opt to use the simpler $\Box$L rule in G3GL rather than the $\Box$L$'$ rule to simplify our work.

$$\frac{\mathcal{R}, xRy, \Gamma, x : \Box\varphi \vdash y : \Box\varphi, \Delta \qquad \mathcal{R}, xRy, \Gamma, x : \Box\varphi, y : \varphi \vdash \Delta}{\mathcal{R}, xRy, \Gamma, x : \Box\varphi \vdash \Delta} \Box\mathsf{L}'$$

A *derivation* of a labeled sequent $\mathcal{R}, \Gamma \vdash \Delta$ is defined to be a (potentially infinite) tree whose nodes are labeled with labeled sequents such that (1) $\mathcal{R}, \Gamma \vdash \Delta$ is the root of the tree and (2) each parent node is the conclusion of a rule with its children the corresponding premises. A *proof* is a finite derivation such that every leaf is an instance of an initial sequent. We use π (potentially annotated) to denote derivations and proofs throughout the remainder of the paper, and use this notation to denote derivations and proofs in other systems as well with the context determining the usage. The *height* of a proof is defined as usual to be equal to the length of a maximal path from the root of the proof to an initial sequent.

▶ Remark 5. We assume w.l.o.g. that every fresh variable used in a proof is *globally fresh*, meaning there is a one-to-one correspondence between $\Box$R applications and their fresh variables. This assumption is helpful, yet benign (cf. [31]).

As shown by Negri [31], the various rules displayed in Figure 3 are *admissible* in G3GL. Note that the (x/y) rule applies a *label substitution* to the premise which replaces every occurrence of the label y in a relational atom or labeled formula by x. We define a rule to be *admissible* (*height-preserving admissible*) *iff* if the premises of the rule have proofs (of height $h_1, \ldots, h_n$), then the conclusion of the rule has a proof (of height $h \leq \max\{h_1, \ldots, h_n\}$). We refer to a height-preserving admissible rule as *hp-admissible*. Moreover, the non-initial rules of G3GL are *height-preserving invertible*. If we let r_i^{-1} be the i-inverse of the rule r whose conclusion is the i^{th} premise of the n-ary rule r and premise is the conclusion of r, then we say that r is *(height-preserving) invertible iff* r_i^{-1} is (height-preserving) admissible for each $1 \leq i \leq n$. We refer to height-preserving invertible rules as *hp-invertible*. The following theorem is due to Negri [31].

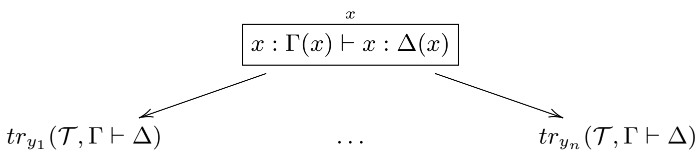
▶ **Theorem 6** (G3GL Properties [31]). *The labeled sequent calculus* G3GL *satisfies the following:*

(1) *Each labeled sequent of the form* $\mathcal{R}, \Gamma, x : \varphi \vdash x : \varphi, \Delta$ *is provable in* G3GL*;*
(2) *All non-initial rules are hp-invertible in* G3GL*;*
(3) *The* (x/y), w*,* cL*, and* cR *rules are hp-admissible in* G3GL*;*
(4) *The* cut *rule is admissible in* G3GL*;*
(5) φ *is valid iff* $\vdash x : \varphi$ *is provable in* G3GL*.*

3.2 Labeled Tree Sequents

A set $\mathcal{T}$ of relational atoms is a *tree iff* the graph $G(\mathcal{T}) = (V, E)$ forms a tree, where $V = \{x \mid x \in \mathrm{Lab}(\mathcal{T})\}$ and $E = \{(x, y) \mid xRy \in \mathcal{T}\}$. A *tree sequent* is defined to be an expression of the form $\mathcal{T}, \Gamma \vdash \Delta$ such that (1) $\mathcal{T}$ is a tree, (2) if $\mathcal{T} \neq \emptyset$, then $\mathrm{Lab}(\Gamma, \Delta) \subseteq \mathrm{Lab}(\mathcal{T})$, and (3) if $\mathcal{T} = \emptyset$, then $|\mathrm{Lab}(\Gamma, \Delta)| = 1$, i.e. all labeled formulae in Γ, Δ share the same label. We note that conditions (1)–(3) ensure that each tree sequent forms a connected graph that is indeed of a tree shape. We use T and annotated versions thereof to denote tree sequents. We define a *flat sequent* to be a tree sequent of the form $\Gamma \vdash \Delta$, that is, a flat sequent is a sequent $\Gamma \vdash \Delta$ without relational atoms and where every labeled formula in Γ, Δ shares the same label. The *root* of a tree sequent $\mathcal{T}, \Gamma \vdash \Delta$ is the label x such that there exists a unique directed path of relational atoms in $\mathcal{T}$ from x to every other label $y \in \mathrm{Lab}(\mathcal{T}, \Gamma, \Delta)$; if $\mathcal{T} = \emptyset$, then the root is the single label x shared by all formulae in Γ, Δ.

Every tree sequent encodes a tree whose vertices are flat sequents. In other words, each tree sequent $T = \mathcal{T}, xRy_1, \ldots, xRy_n, \Gamma \vdash \Delta$ such that x is the root and $y_1, \ldots, y_n$ are all children of x can be graphically depicted as a tree $tr_x(T)$ of the form shown below:



▶ **Definition 7** (Tree Sequent Calculus CSGL)**.** *We define* $\mathsf{CSGL} := (\mathsf{G3GL} \setminus \{\mathsf{ir}, \mathsf{tr}\}) \cup \{\mathsf{4L}\}$*, where the* 4L *is shown below and the rules of the calculus only operate over tree sequents.*

$$\frac{\mathcal{T}, xRy, \Gamma, x : \Box\varphi, y : \Box\varphi \vdash \Delta}{\mathcal{T}, xRy, \Gamma, x : \Box\varphi \vdash \Delta}\ \mathsf{4L}$$

The system CSGL is a notational variant of Poggiolesi's eponymous tree-hypersequent system [35], and thus, we identify the two systems with one another. The main difference between the two systems is notational: the system defined above uses tree sequents, which are tree-hypersequents 'dressed' as labeled sequents [15]. We also note that Poggiolesi's original tree-hypersequent system CSGL uses a notational variant of the binary rule $\Box\mathsf{L}'$ discussed in Remark 4. However, as with the labeled system G3GL, the left premise of this rule is provable in CSGL. We have therefore opted to use the unary $\Box$L rule in CSGL to simplify our work and note that this change is benign.

▶ Remark 8. Derivations, proofs, the height of a proof, (hp-)admissibility, the *i*-inverse of a rule, and (hp-)invertibility are defined for CSGL analogous to how such notions are defined for G3GL. We will apply these terms and concepts in the expected way to other sequent systems as well to avoid repeating similar definitions.

The system CSGL differs from G3GL in that CSGL only allows for tree sequents in proofs, lacks the structural rules ir and tr, and includes the 4L rule. We refer to 4L and $\Box$L as *propagation rules* (cf. [12, 8]) since the rules bottom-up propagate data forward along relational atoms, and we refer to ¬L, ¬R, ∨L, and ∨R as *local rules* since they only affect formulae locally at a single label. For any rule, we call the label x labeling the principal formula in the conclusion the *principal label*, for the 4L, □L, and □R rules, we refer to the label y labeling the auxiliary formula(e) in the premise(s) as the *auxiliary label*, and for local rules, the *auxiliary label* is taken to be the same as the principal label since they are identical. Note that we define a label x in a tree sequent $\mathcal{T}, \Gamma \vdash \Delta$ to be a *leaf iff* x is a leaf in $\mathcal{T}$, and we define a label x to be a *pre-leaf iff* for all $y \in \text{Lab}(\mathcal{T})$, if $xRy \in \mathcal{T}$, then y is a leaf.

Since the tree sequent calculus CSGL is isomorphic to Poggiolesi's tree-hypersequent system, the two systems share the same properties. We note that in the setting of tree sequents the (x/y) and w rules are less general than for labeled sequents. In particular, such rules are assumed to preserve the 'tree shape' of tree sequents when applied. Nevertheless, the restricted forms of these rules are still hp-admissible in CSGL.

▶ **Theorem 9** (CSGL Properties [35]). *The tree sequent calculus* CSGL *satisfies the following:*
(1) *Each tree sequent of the form* $\mathcal{T}, \Gamma, x : \varphi \vdash x : \varphi, \Delta$ *is provable in* CSGL*;*
(2) *All non-initial rules are hp-invertible in* CSGL*;*
(3) *The* (x/y)*,* w*,* cL*, and* cR *rules are hp-admissible in* CSGL*;*
(4) *The* cut *rule is admissible in* CSGL*;*
(5) φ *is valid iff* $\vdash x : \varphi$ *is provable in* CSGL.

Proofs in G3GL and CSGL are inter-translatable with one another. This correspondence was established by Goré and Ramanayake [15] and is based on a couple observations. First, the ir rule does not occur in G3GL proofs where the end sequent is a tree sequent. It is not difficult to see why this is the case: if one takes a proof of a tree sequent, then bottom-up applications of rules from G3GL will not allow for directed cycles to enter a sequent in a proof. This follows from the fact that the conclusion of the proof is a tree sequent, which is free of directed cycles, and each rule of G3GL either preserves relational atoms bottom-up, adds a single relational atom from a label x to a fresh label y in the case of the □R rule, or adds an undirected cycle in the case of tr. Since the conclusion of ir contains a directed cycle xRx, such a sequent will never occur in such a proof.

▶ **Observation 1** ([15]). *The* ir *rule does not occur in any* G3GL *proof of a tree sequent.*

Second, Goré and Ramanayake [15] show that instances of tr can be eliminated from G3GL proofs not containing ir and replaced by instances of 4L. This elimination procedure can be used to map G3GL proofs such that the end sequent is a tree sequent to CSGL proofs. Conversely, if we let arbitrary labeled sequent appear in CSGL proofs, it can be shown that 4L can be eliminated from CSGL proofs and replaced by instances of tr. These elimination results are proven syntactically, showing that proof transformations exist between CSGL and G3GL for proofs of tree sequents as summarized in the theorem below. We refer to the reader to [15] for the details.

▶ **Theorem 10** ([15]). *A tree sequent* T *is provable in* G3GL *iff it is provable in* CSGL.

4 Linearizing Tree Sequents in Proofs

We now show how to extract a *linear nested sequent calculus* from CSGL, dubbed LNGL (see Figure 4). To the best of the author's knowledge, this is the first linear nested sequent calculus for Gödel-Löb provability logic. Linear nested sequents were introduced by Lellmann [21] and are a finite representation of Masini's 2-sequents [29]. Such systems operate over *lines* of Gentzen sequents and have been used to provide cut-free systems for intermediate and modal logics [20, 21, 29, 30].

$$\frac{}{\mathcal{G} \mathbin{/\!/} \Gamma, p \vdash p, \Delta}\ \mathsf{id}_1 \qquad \frac{}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Box\varphi, \Delta}\ \mathsf{id}_2 \qquad \frac{\mathcal{G} \mathbin{/\!/} \Gamma, \varphi \vdash \Delta \qquad \mathcal{G} \mathbin{/\!/} \Gamma, \psi \vdash \Delta}{\mathcal{G} \mathbin{/\!/} \Gamma, \varphi \vee \psi \vdash \Delta}\ \vee\mathsf{L}$$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma \vdash \varphi, \psi, \Delta}{\mathcal{G} \mathbin{/\!/} \Gamma \vdash \varphi \vee \psi, \Delta}\ \vee\mathsf{R} \qquad \frac{\mathcal{G} \mathbin{/\!/} \Gamma \vdash \varphi, \Delta}{\mathcal{G} \mathbin{/\!/} \Gamma, \neg\varphi \vdash \Delta}\ \neg\mathsf{L} \qquad \frac{\mathcal{G} \mathbin{/\!/} \Gamma, \varphi \vdash \Delta}{\mathcal{G} \mathbin{/\!/} \Gamma \vdash \neg\varphi, \Delta}\ \neg\mathsf{R}$$

$$\frac{\mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Delta \mathbin{/\!/} \Sigma, \Box\varphi \vdash \Pi}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Delta \mathbin{/\!/} \Sigma \vdash \Pi}\ 4\mathsf{L} \qquad \frac{\mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Delta \mathbin{/\!/} \Sigma, \varphi \vdash \Pi}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Delta \mathbin{/\!/} \Sigma \vdash \Pi}\ \Box\mathsf{L} \qquad \frac{\mathcal{G} \mathbin{/\!/} \Gamma \vdash \Delta \mathbin{/\!/} \Box\varphi \vdash \varphi}{\mathcal{G} \mathbin{/\!/} \Gamma \vdash \Box\varphi, \Delta}\ \Box\mathsf{R}$$

Figure 4 Linear Nested Sequent Calculus LNGL for GL.

The extraction of linear nested sequent proofs from tree sequent proofs takes place in three phases. In the first phase, we show how to transform any CSGL proof into an *end-active* proof, i.e., a proof such that principal and auxiliary formulae only occur at (pre-)leaves in tree sequents (cf. [21]). In the second phase, we define our novel linearization technique, where we identify specific paths in tree sequents and 'prune' sub-trees, yielding a linear nested sequent proof as the result. This technique is an additional contribution of this paper, and we conjecture that this technique can be used in other settings to extract linear nested sequent systems from tree sequent/nested sequent systems. In the third phase, we show how to 'reshuffle' a linear nested sequent proof so that the proof proceeds in repetitive stages of local rules, propagation rules, and □R rules, which we refer to as a proof in *normal form*. This transformation is motivated by one provided in [33] for so-called *basic nested systems*, which transforms proofs in a similar manner to extract Gentzen sequent proofs. Our transformation is distinct however as it works within the context of linear nested sequents. The simpler data structure used in linear nested sequents and the 'end-active' shape of the rules in LNGL simplifies the process of reshuffling proofs into normal form.

A *linear nested sequent* is an expression of the form $\mathcal{G} := \Gamma_1 \vdash \Delta_1 \mathbin{/\!/} \cdots \mathbin{/\!/} \Gamma_n \vdash \Delta_n$ such that Γ_i and Δ_i are multisets of formulae from $\mathcal{L}$ for $1 \leq i \leq n$. We use $\mathcal{G}$, $\mathcal{H}$, ... to denote linear nested sequents and note that such sequents admit a formula interpretation:

$$f(\Gamma \vdash \Delta) := \bigwedge \Gamma \rightarrow \bigvee \Delta \qquad f(\Gamma \vdash \Delta \mathbin{/\!/} \mathcal{G}) := \bigwedge \Gamma \rightarrow (\bigvee \Delta \vee \Box f(\mathcal{G}))$$

We define a linear nested sequent $\mathcal{G}$ to be (in)valid *iff* $f(\mathcal{G})$ is (in)valid. The linear nested sequent calculus LNGL consists of the rules shown in Figure 4. We take the ¬L, ¬R, ∨L, and ∨R rules to be *local rules* and the 4L and □L rules to be *propagation rules* in LNGL. For $1 \leq i \leq n$, we refer to $\Gamma_i \vdash \Delta_i$ as the i-component (or, as a *component* more generally) of the linear nested sequent $\mathcal{G} = \Gamma_1 \vdash \Delta_1 \mathbin{/\!/} \cdots \mathbin{/\!/} \Gamma_n \vdash \Delta_n$, we refer to $\Gamma_n \vdash \Delta_n$ as the *end component*, and we define the *length* of $\mathcal{G}$ to be $||\mathcal{G}|| := n$, i.e., the length of a linear nested sequent is equal to the number of its components. Comparing LNGL to CSGL, one can see that LNGL is the calculus CSGL restricted to lines of Gentzen sequents and where rules only operate in the last two components. Making use of the formula translation, it is a basic exercise to show that if the conclusion of any rule is invalid, then at least one premise is invalid, i.e., LNGL is sound.

▶ **Theorem 11.** *If $\mathcal{G}$ is provable in* LNGL*, then $\mathcal{G}$ is valid.*

When transforming CSGL proofs into LNGL proofs later on, it will be helpful to use the weakening rule w shown in the lemma below. Observe that any application of w to id_1 or id_2 yields an initial sequent, and w permutes above every other rule of LNGL. As an immediate consequence, we have that w is hp-admissible in LNGL.

▶ **Lemma 12.** *The following weakening rule* w *is hp-admissible in* LNGL*.*

$$\dfrac{\mathcal{G} \sslash \Gamma \vdash \Delta \sslash \mathcal{H}}{\mathcal{G} \sslash \Gamma, \Sigma \vdash \Pi, \Delta \sslash \mathcal{H}}\,\mathsf{w}$$

Furthermore, we have that the 4L and □L rules are hp-invertible in LNGL since the premises of each rule may be obtained from the conclusion by w.

▶ **Lemma 13.** *The* 4L *and* □L *rules are hp-invertble in* LNGL*.*

From Tree Sequents to Linear Nested Sequents

To extract LNGL from CSGL, we first establish a set of rule permutation results, i.e., we show that rules of a certain form in CSGL can always be permuted below other rules of a specific form. We note that a rule r *permutes below* a rule r′ whenever an application of r followed by r′ in a proof can be replaced by an application of r′ (potentially preceded by an application of an i-inverse of r) followed by an application of r to derive the same conclusion. To make this definition more concrete, we show (1) the permutation of a unary rule r below a binary rule r′ below top-left, (2) the permutation of a binary rule r below a unary rule r′ below top-right, (3) the permutation of a unary rule r below a unary rule r′ below bottom-left, and (4) the permutation of a binary rule r below a binary rule r′ below bottom-right. In (1) and (4), we note that the case where r is applied to the right premise of r′ is symmetric. Furthermore, recall that r_1^{-1} and r_2^{-1} are the 1- and 2-inverses of r, respectively. (NB. For the definition of the *i-inverse* of a rule, see Section 3.1.) We use (annotated versions of) the symbol S below to indicate not only tree sequents, but linear nested sequents since we consider permutations of rules in LNGL later on as well.

$$\dfrac{\dfrac{S_0}{S_2}\,\mathsf{r} \qquad S_1}{S_3}\,\mathsf{r}' \quad\rightsquigarrow\quad \dfrac{\dfrac{S_0 \qquad \dfrac{S_1}{S'}\,\mathsf{r}^{-1}}{S}\,\mathsf{r}'}{S_3}\,\mathsf{r} \qquad\qquad \dfrac{\dfrac{S_0 \qquad S_1}{S_2}\,\mathsf{r}}{S_3}\,\mathsf{r}' \quad\rightsquigarrow\quad \dfrac{\dfrac{S_0}{S}\,\mathsf{r}' \qquad \dfrac{S_1}{S'}\,\mathsf{r}'}{S_3}\,\mathsf{r}$$

$$\dfrac{\dfrac{S_0}{S_1}\,\mathsf{r}}{S_2}\,\mathsf{r}' \quad\rightsquigarrow\quad \dfrac{\dfrac{S_0}{S}\,\mathsf{r}'}{S_2}\,\mathsf{r} \qquad \dfrac{\dfrac{S_0 \qquad S_1}{S_2}\,\mathsf{r} \qquad S_3}{S_4}\,\mathsf{r}' \quad\rightsquigarrow\quad \dfrac{\dfrac{S_0 \qquad \dfrac{S_3}{S}\,\mathsf{r}_1^{-1}}{S'}\,\mathsf{r}' \qquad \dfrac{S_1 \qquad \dfrac{S_3}{S''}\,\mathsf{r}_2^{-1}}{S'''}\,\mathsf{r}'}{S_4}\,\mathsf{r}$$

The various admissible rule permutations we describe are based on the notion of *end-activity*, which is a property of rule applications where principal and auxiliary formulae only occur at (pre-)leaves in sequents. End-activity was first discussed by Lellmann [21] in the context of mapping Gentzen sequent proofs into linear nested sequent proofs for non-classical logics.

▶ **Definition 14** (End Active)**.** *A* CSGL *proof is* end-active *iff the following hold:*
(1) *The principal label in every instance of* id_1 *and* id_2 *is a leaf;*
(2) *The principal label of each local rule is a leaf;*
(3) *The principal and auxiliary label of a propagation rule is a pre-leaf and leaf, respectively.*
A rule r *is* end-active *iff it satisfies its respective condition above; otherwise, the rule is* non-end-active*. We note that we always take the* □R *rule to be end-active.*

As stated in the following lemma, the end-activity of sequential rule applications determines a set of permutation relationships between the rules of CSGL. The lemma is straightforward to prove, though tedious due to the number of cases; its proof can be found in the appendix.

▶ **Lemma 15.** *The following permutations hold in* CSGL*:*

(1) *If* r *is a non-end-active local rule and* r′ *is non-initial and end-active, then* r *permutes below* r′ *and* r′ *remains end-active after this permutation;*
(2) *if* r *is a non-end-active propagation rule and* r′ *is non-initial and end-active, then* r *permutes below* r′ *and* r′ *remains end-active after this permutation.*

Using the above lemma, every proof π in CSGL of a sequent of the form $\vdash x : \varphi$ can be transformed into an end-active proof as follows: first, observe that the last inference in π must be end-active since the proof ends with $\vdash x : \varphi$. By successively considering bottom-most instances of non-end-active local and propagation rules r in π, we can repeatedly apply Lemma 15 to permute r lower in the proof because all rules below r are guaranteed to be end-active. By inspecting the rules of CSGL, we know that the trees within the tree sequents in π will never 'grow' but will 'shrink' as they get closer to the conclusion of the proof, meaning, each non-end-active rule r will eventually become end-active through successive downward permutations.[2] This process will eventually terminate and yield a proof where all non-initial rules are end-active for the following two reasons: (1) As stated in the lemma above, permuting a non-end-active local or propagation rule r′ below an end-active rule r preserves the end-activity of the rule r. (2) Although downward permutations may require the i-inverse of a rule to be applied above the permuted inferences, the hp-invertibility of all non-initial rules in CSGL (see Theorem 9) ensures that the height of the proof does not grow after a downward rule permutation.

After all such downward permutations have been performed, the resulting proof is *almost* end-active with the exception that initial rules may not be end-active. For example, as shown below left, it may be the case that id_1 is non-end-active and followed by a rule r. Since r is guaranteed to be end-active at this stage, we know that the auxiliary label of r is distinct from y, meaning, the conclusion will be an instance of id_1 as shown below right.

$$\dfrac{\dfrac{}{\mathcal{T}, \Gamma, y : p \vdash y : p, \Delta}\,\mathsf{id}_1 \quad (\mathcal{T}', \Gamma', y : p \vdash y : p, \Delta')}{\mathcal{T}'', \Gamma'', y : p \vdash y : p, \Delta''}\,\mathsf{r} \qquad \dfrac{}{\mathcal{T}'', \Gamma'', y : p \vdash y : p, \Delta''}\,\mathsf{id}_1$$

By replacing such rule applications r by id_1 (or id_2) instances, effectively 'pushing' initial rules down in the proof, we will eventually obtain initial rules such that the label y is auxiliary in the subsequent rule application, which will then be a leaf since all non-initial rules of the proof are end-active. Thus, every proof in CSGL of a sequent of the form $\vdash x : \varphi$ can be transformed into an end-active proof.

▶ **Theorem 16.** *Each proof in* CSGL *of a sequent of the form* $\vdash x : \varphi$ *can be transformed into an end-active proof.*

▶ **Theorem 17.** *Each end-active proof in* CSGL *can be transformed into a proof in* LNGL*.*

Proof. We prove that if there exists an end-active proof in CSGL of a tree sequent $\mathcal{T}, \Gamma \vdash \Delta$, then there exists a path $x_1, \ldots, x_n$ of labels from the root x_1 to a leaf x_n in $\mathcal{T}, \Gamma \vdash \Delta$ such that $\Gamma(x_1) \vdash \Delta(x_1) \,/\!\!/\, \cdots \,/\!\!/\, \Gamma(x_n) \vdash \Delta(x_n)$ is provable in LNGL. We argue this by induction on the height of the end-active proof π in CSGL.

Base case. Suppose π consists of a single application of id_1 or id_2, as shown below:

$$\dfrac{}{\mathcal{T}, \Gamma, x : p \vdash x : p, \Delta}\,\mathsf{id}_1 \qquad \dfrac{}{\mathcal{T}, \Gamma, x : \Box\varphi \vdash x : \Box\varphi, \Delta}\,\mathsf{id}_2$$

[2] Observe that ¬L, ¬R, ∨L, ∨R, □L, and 4L only affect the formulae associated with the label of a tree sequent, whereas □R top-down removes a relational atom from a tree sequent. Therefore, the number of relational atoms in tree sequents will decrease as we move down paths in CSGL proofs from initial sequents to the conclusion.

We know that each initial sequent is end-active, i.e., the label x is a leaf in both tree sequents. Therefore, since each sequent is a tree sequent, there exists a path $y_1, \ldots, y_n = x$ of labels from the root y_1 to the leaf x. By using this path, we obtain respective instances of id_1 and id_2 as shown below:

$$\dfrac{}{\Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!\!/} \cdots \mathbin{/\!\!/} \Gamma(x), p \vdash p, \Delta(x)}\ \mathsf{id}_1 \qquad \dfrac{}{\Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!\!/} \cdots \mathbin{/\!\!/} \Gamma(x), \Box\varphi \vdash \Box\varphi, \Delta(x)}\ \mathsf{id}_2$$

Inductive step. For the inductive hypothesis (IH), we assume that the claim holds for every end-active proof in CSGL of height $h' \leq h$, and aim to show that the claim holds for proofs of height $h + 1$. We let π be of height $h + 1$ and argue the cases where π ends with $\Box\mathsf{L}$ or $\Box\mathsf{R}$ as the remaining cases are shown similarly. Some additional cases are given in the appendix.

$\Box\mathsf{L}$. Let us suppose that π ends with an instance of $\Box\mathsf{L}$ as shown below.

$$\dfrac{\mathcal{T}, xRy, \Gamma, x : \Box\varphi, y : \varphi \vdash \Delta}{\mathcal{T}, xRy, \Gamma, x : \Box\varphi \vdash \Delta}\ \Box\mathsf{L}$$

By IH, we know there exists a path $y_1, \ldots, y_n$ of labels from the root y_1 to the leaf y_n in the premise of $\Box\mathsf{L}$ such that $\mathcal{G} = \Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!\!/} \cdots \mathbin{/\!\!/} \Gamma(y_n) \vdash \Delta(y_n)$ is provable in LNGL. Since $\Box\mathsf{L}$ is end-active, we have three cases to consider: (1) neither x nor y occur along the path in the premise, (2) only x occurs along the path in the premise, or (3) both x and y occur along the path in the premise. In cases (1) and (2), we translate the entire $\Box\mathsf{L}$ inference as the linear nested sequent $\mathcal{G}$. In case (3), $\mathcal{G}$ has the form of the premise shown below with $\Gamma(y_{n-1}) = \Sigma_1, \Box\varphi$ and $\Gamma(y_n) = \Sigma_2, \varphi$. A single application of $\Box\mathsf{L}$ gives the desired result.

$$\dfrac{\Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!\!/} \cdots \mathbin{/\!\!/} \Sigma_1, \Box\varphi \vdash \Delta(y_{n-1}) \mathbin{/\!\!/} \Sigma_2, \varphi \vdash \Delta(y_n)}{\Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!\!/} \cdots \mathbin{/\!\!/} \Sigma_1, \Box\varphi \vdash \Delta(y_{n-1}) \mathbin{/\!\!/} \Sigma_2 \vdash \Delta(y_n)}\ \Box\mathsf{L}$$

$\Box\mathsf{R}$. Let us suppose that π ends with an instance of $\Box\mathsf{R}$ as shown below.

$$\dfrac{\mathcal{T}, xRy, \Gamma, y : \Box\varphi \vdash y : \varphi, \Delta}{\mathcal{T}, \Gamma \vdash x : \Box\varphi, \Delta}\ \Box\mathsf{R}$$

By IH, we know there exists a path $y_1, \ldots, y_n$ of labels from the root y_1 to the leaf y_n in the premise of $\Box\mathsf{R}$ such that $\Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!\!/} \cdots \mathbin{/\!\!/} \Gamma(y_n) \vdash \Delta(y_n)$ is provable in LNGL. We have three cases to consider: either (1) neither x nor y occur along the path, (2) only x occurs along the path, or (3) both x and y occur along the path. In case (1), we translate the entire $\Box\mathsf{R}$ instance as the single linear nested sequent $\mathcal{G}$. In case (2), we know that $x = y_i$ for some $1 \leq i \leq n$. To obtain the desired conclusion, we apply the hp-admissible w rule as shown below. Observe that the conclusion of the w application below corresponds to the linear nested sequent obtained from the path $y_1, \ldots, y_n$ in the conclusion of the $\Box\mathsf{R}$ instance above.

$$\dfrac{\Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!\!/} \cdots \mathbin{/\!\!/} \Gamma(y_i) \vdash \Delta(y_i) \mathbin{/\!\!/} \cdots \mathbin{/\!\!/} \Gamma(y_n) \vdash \Delta(y_n)}{\Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!\!/} \cdots \mathbin{/\!\!/} \Gamma(y_i) \vdash \Box\varphi, \Delta(y_i) \mathbin{/\!\!/} \cdots \mathbin{/\!\!/} \Gamma(y_n) \vdash \Delta(y_n)}\ \mathsf{w}$$

Last, in case (3), we know that $x = y_{n-1}$ and $y = y_n$ due to the freshness condition imposed on the $\Box\mathsf{R}$ rule. In this case, $\mathcal{G}$ has the form of the premise shown below, meaning, a single application of the $\Box\mathsf{R}$ rule gives the linear nested sequent corresponding to the path $y_1, \ldots, y_n$ in the conclusion of the $\Box\mathsf{R}$ instance above.

$$\dfrac{\Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!\!/} \cdots \mathbin{/\!\!/} \Gamma(y_{n-1}) \vdash \Delta(y_{n-1}) \mathbin{/\!\!/} \Box\varphi \vdash \varphi}{\Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!\!/} \cdots \mathbin{/\!\!/} \Gamma(y_{n-1}) \vdash \Box\varphi, \Delta(y_{n-1})}\ \Box\mathsf{R}$$

◀

The following is an immediate consequence of Theorems 11, 16, and 17.

▶ **Corollary 18** (LNGL Soundness and Completeness). *φ is valid iff $\vdash \varphi$ is provable in* LNGL*.*

Last, we show that every LNGL proof can be put into a *normal form* (see Definition 19 and Theorem 16 below) such that (reading the proof bottom-up) $\Box$R instances are preceded by 4L instances, which are preceded by $\Box$L instances, which are preceded by local rule instances (or, initial rules). We will utilize this normal form in the next section to show that every LNGL proof can be transformed into a Gentzen sequent proof (Theorem 22). We let B be a set of LNGL rules and define a *block* to be a derivation that only uses rules from B. We use the following notation to denote blocks, showing that the set B of rules derives $\mathcal{G}$ from $\mathcal{G}_1, \ldots, \mathcal{G}_n$, and refer to $\mathcal{G}_1, \ldots, \mathcal{G}_n$ as the *premises* of the block B.

$$\frac{\mathcal{G}_1, \ldots, \mathcal{G}_n}{\mathcal{G}} \, \mathrm{B}$$

▶ **Definition 19** (Normal Form). *A proof in* LNGL *is in* normal form *iff each bottom-up* $\Box$R *application is derived from a block* B *of* 4L *rules, whose premise is derived from a block* B′ *of* $\Box$L *rules, whose premise is derived from a block* B″ *of local rules, as indicated below.*

$$\cfrac{\cfrac{\cfrac{\cfrac{\mathcal{G} \mathbin{/\!\!/} \Gamma \vdash \Delta \mathbin{/\!\!/} \Sigma_1 \vdash \Pi_1 \qquad \ldots \qquad \mathcal{G} \mathbin{/\!\!/} \Gamma \vdash \Delta \mathbin{/\!\!/} \Sigma_n \vdash \Pi_n}{\mathcal{G} \mathbin{/\!\!/} \Gamma \vdash \Delta \mathbin{/\!\!/} \Gamma', \Gamma'', \Box\varphi \vdash \varphi} \, \mathrm{B}''}{\mathcal{G} \mathbin{/\!\!/} \Gamma \vdash \Delta \mathbin{/\!\!/} \Gamma', \Box\varphi \vdash \varphi} \, \mathrm{B}'}{\mathcal{G} \mathbin{/\!\!/} \Gamma \vdash \Delta \mathbin{/\!\!/} \Box\varphi \vdash \varphi} \, \mathrm{B}}{\mathcal{G} \mathbin{/\!\!/} \Gamma \vdash \Box\varphi, \Delta} \, \Box\mathsf{R}$$

We refer to block of rules of the above form as a complete block*, and refer to the portion of a complete block consisting of only* $\Box$R*,* B*, and* B′ *as a* modal block*.*

As proven in the next section (Theorem 22), every normal form proof in LNGL can be transformed into a proof in Sambin and Valentini's Gentzen calculus $\mathsf{GL}_{\mathsf{seq}}$. Therefore, we need to show that every proof in LNGL can be put into normal form. We prove this by making an observation about the structure of proofs in LNGL. Observe that local and propagation rules in LNGL only affect the end component of linear nested sequents and preserve the length of such sequents, whereas the $\Box$R rule increases the length of a linear nested sequent by 1 when applied bottom-up. This implies that any LNGL proof π bottom-up proceeds in repetitive stages, as we now describe. Let π be a proof in LNGL with conclusion $\mathcal{G}$ such that $||\mathcal{G}|| = n$. The conclusion $\mathcal{G}$ is derived with a block B of local and propagation rules that only affect the n-component in inferences with the premises of the block B being initial rules or derived by applications of $\Box$R rules. These applications of $\Box$R rules will have premises of length $n + 1$ and will be preceded by blocks B_i of local and propagation rules that only affect the $(n + 1)$-component in inferences. The premises of the B_i blocks will then either be initial rules or derived by applications of $\Box$R rules that have premises of length $n + 2$, which are preceded by blocks of local and propagation rules that only affect the $(n + 2)$-component in inferences, and so on. Every proof in LNGL will have this repetitive structure.

Let $\Box$R be applied in an LNGL proof π with premise $\mathcal{G} \mathbin{/\!\!/} \Gamma \vdash \Delta \mathbin{/\!\!/} \Box\varphi \vdash \varphi$ of length n. We say that an instance of a local or propagation rule r in π is *length-consistent* with $\Box$R *iff* the length of the conclusion of r is equal to n. Based on the discussion above, we can see that for any $\Box$R application in a proof π, all length-consistent local and propagation rules will occur in a block B above the $\Box$R application with B free of other $\Box$R rules. It is not difficult to show that B can be transformed into a *complete block* by (1) successively permuting 4L rules down into a block above $\Box$R, and (2) successively permuting $\Box$L rules down above the 4L block. After the permutations from (1) and (2) have been carried out, the premise of the

$$\frac{}{\Gamma, \varphi \vdash \varphi, \Delta}\ \mathsf{id} \qquad \frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta}\ \neg\mathsf{L} \qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}\ \neg\mathsf{R} \qquad \frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vee \psi \vdash \Delta}\ \vee\mathsf{L}$$

$$\frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\ \vee\mathsf{R} \qquad \frac{\Box\Gamma, \Gamma, \Box\varphi \vdash \varphi}{\Sigma, \Box\Gamma \vdash \Box\varphi, \Delta}\ \Box_{\mathsf{GL}} \qquad \frac{\Box\Gamma, \Gamma \vdash \varphi}{\Sigma, \Box\Gamma \vdash \Box\varphi, \Delta}\ \Box_{4}$$

Figure 5 Sequent calculus rules.

$\Box$L block will be derived by length-consistent local rule applications, showing that $\Box$R is preceded by a complete block. As these permutations can be performed for every $\Box$R rule in a proof, every proof can be put into normal form.

▶ **Theorem 20.** *Every proof in* LNGL *can be transformed into a proof in normal form.*

5 Sequent Systems and Correspondences

5.1 Gentzen, Cyclic, and Non-Wellfounded Systems

We use Γ, Δ, Σ, ... to denote finite multisets of formulae within the context of sequent systems. For a multiset $\Gamma := \varphi_1, \ldots, \varphi_n$, we define $\Box\Gamma := \Box\varphi_1, \ldots, \Box\varphi_n$. A *sequent* is defined to be an expression of the form $\Gamma \vdash \Delta$. The sequent calculus $\mathsf{GL}_{\mathsf{seq}}$ for GL consists of the rules id, ¬L, ¬R, ∨L, ∨R, and $\Box_{\mathsf{GL}}$ shown in Figure 5 and is an equivalent variant of the Gentzen calculi GLSC and GLS introduced by Sambin and Valentini for GL [36, 37].[3] The system $\mathsf{GL}_{\mathsf{seq}}$ is sound and complete for GL, admits syntactic cut-elimination, and the weakening and contraction rules w, cL, and cR (shown below) are admissible (cf. [16, 39]).

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Sigma \vdash \Pi, \Delta}\ \mathsf{w} \qquad \frac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\ \mathsf{cL} \qquad \frac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma \vdash \varphi, \Delta}\ \mathsf{cR} \qquad \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma, \varphi \vdash \Delta}{\Gamma \vdash \Delta}\ \mathsf{cut}$$

Shamkanov [39] showed that equivalent non-wellfounded and cyclic sequent systems could be obtained for GL by taking the sequent calculus for the modal logic K4 and generalizing the notion of proof. The sequent calculus $\mathsf{K4}_{\mathsf{seq}}$ is obtained by replacing the $\Box_{\mathsf{GL}}$ rule in $\mathsf{GL}_{\mathsf{seq}}$ with the $\Box_4$ rule shown in Figure 5. Let us now recall Shamkanov's non-wellfounded sequent calculus GL_∞ and cyclic sequent calculus $\mathsf{GL}_{\mathsf{circ}}$ for GL. We present Shamkanov's systems in a *two-sided format*, i.e., using two-sided sequents $\Gamma \vdash \Delta$ rather than one-sided sequents of the form Γ. This makes the correspondence between Shamkanov's systems and $\mathsf{GL}_{\mathsf{seq}}$ clearer as well as saves us from having to introduce a new language for GL since one-sided sequents use formulae in negation normal form. Translating proofs with two-sided sequents to proofs with one-sided sequents and vice-versa can be easily obtained by standard techniques, and so, this minor modification causes no problems.

A *derivation* of a sequent $\Gamma \vdash \Delta$ is defined to be a (potentially infinite) tree whose nodes are labeled with sequents such that (1) $\Gamma \vdash \Delta$ is the root of the tree, and (2) each parent node is taken to be the conclusion of a rule in $\mathsf{K4}_{\mathsf{seq}}$ with its children the corresponding premises. A *non-wellfounded proof* is a derivation such that all leaves are initial sequents. GL_∞ is the non-wellfounded sequent system obtained by letting the set of provable sequents be determined by non-wellfounded proofs.

[3] $\mathsf{GL}_{\mathsf{seq}}$ differs from Sambin and Valentini's original systems in that multisets are used instead of sets, rules for superfluous logical connectives (e.g., conjunction ∧ and implication →) have been omitted as these are definable in terms of other rules, and the weakening rules have been absorbed into id and $\Box_{\mathsf{GL}}$.

A *cyclic derivation* is a pair $\pi = (\kappa, c)$ such that κ is a finite derivation in $\mathsf{K4_{seq}}$ and c is a function with the following properties: (1) c is defined on a subset of the leaves of κ, (2) the image $c(x)$ lies on the path from the root of κ to x and does not coincide with x, and (3) both x and $c(x)$ are labeled by the same sequent. If the function c is defined at a leaf x, then we say that a *back-link* exists from x to $c(x)$. A *cyclic proof* is a cyclic derivation $\pi = (\kappa, c)$ such that every leaf x is labeled by an instance of id or there exists a back-link from x to the node $c(x)$. $\mathsf{GL_{circ}}$ is the cyclic sequent system obtained by letting the set of provable sequents be determined by cyclic proofs.

Shamkanov established a three-way correspondence between $\mathsf{GL_{seq}}$, $\mathsf{GL_{\infty}}$, and $\mathsf{GL_{circ}}$, providing syntactic transformations mapping proofs between the three systems.[4]

▶ **Theorem 21** ([39])**.** $\Gamma \vdash \Delta$ *is provable in* $\mathsf{GL_{seq}}$ *iff* $\Gamma \vdash \Delta$ *is provable in* $\mathsf{GL_{\infty}}$ *iff* $\Gamma \vdash \Delta$ *is provable in* $\mathsf{GL_{circ}}$*.*

5.2 Completing the Correspondences

▶ **Theorem 22.** *If* π *is a normal form proof of* $\vdash \varphi$ *in* LNGL*, then* π *can be transformed into a proof of* $\vdash \varphi$ *in* $\mathsf{GL_{seq}}$*.*

Proof. We show how to transform the normal form proof π of $\vdash \varphi$ in LNGL into a proof π' of $\vdash \varphi$ in $\mathsf{GL_{seq}}$ in a bottom-up manner. For the conclusion $\vdash \varphi$ of the proof π, we take $\vdash \varphi$ to be the conclusion of π'. We now make a case distinction on bottom-up applications of rules applied in π. For each rule $\neg\mathsf{L}$, $\neg\mathsf{R}$, $\vee\mathsf{L}$, or $\vee\mathsf{R}$, we translate each premise of the rule as its end component. For example, the $\vee\mathsf{L}$ rule will be translated as shown below.

$$\frac{\mathcal{G} \mathbin{/\!\!/} \Gamma, \varphi \vdash \Delta \qquad \mathcal{G} \mathbin{/\!\!/} \Gamma, \psi \vdash \Delta}{\mathcal{G} \mathbin{/\!\!/} \Gamma, \varphi \vee \psi \vdash \Delta}\ \vee\mathsf{L} \qquad \frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vee \psi \vdash \Delta}\ \vee\mathsf{L}$$

Suppose now that we encounter a $\Box\mathsf{R}$ rule while bottom-up translating the proof π into a proof in $\mathsf{GL_{seq}}$. Since π is in normal form, we know that $\Box\mathsf{R}$ is preceded by a modal block (see Definition 19), that is, $\Box\mathsf{R}$ is (bottom-up) preceded by a block B_{4L} of $\mathsf{4L}$ rules, which is preceded by a block $\mathrm{B}_{\Box\mathsf{L}}$ of $\Box\mathsf{L}$ rules, i.e., the modal block has the shape shown below. We suppose that $\Box\Sigma_1$ are the principal formulae of the $\mathsf{4L}$ applications, $\Box\Sigma_2$ are those formulae principal in both $\mathsf{4L}$ and $\Box\mathsf{L}$ applications, and $\Box\Sigma_3$ are those formulae principal only in $\Box\mathsf{L}$ applications.

$$\dfrac{\dfrac{\dfrac{\mathcal{G} \mathbin{/\!\!/} \Gamma, \Box\Sigma_1, \Box\Sigma_2, \Box\Sigma_3 \vdash \Delta \mathbin{/\!\!/} \Box\Sigma_1, \Box\Sigma_2, \Sigma_2, \Sigma_3, \Box\varphi \vdash \varphi}{\mathcal{G} \mathbin{/\!\!/} \Gamma, \Box\Sigma_1, \Box\Sigma_2, \Box\Sigma_3 \vdash \Delta \mathbin{/\!\!/} \Box\Sigma_1, \Box\Sigma_2, \Box\varphi \vdash \varphi}\ \mathrm{B}_{\Box\mathsf{L}}}{\mathcal{G} \mathbin{/\!\!/} \Gamma, \Box\Sigma_1, \Box\Sigma_2, \Box\Sigma_3 \vdash \Delta \mathbin{/\!\!/} \Box\varphi \vdash \varphi}\ \mathrm{B}_{\mathsf{4L}}}{\mathcal{G} \mathbin{/\!\!/} \Gamma, \Box\Sigma_1, \Box\Sigma_2, \Box\Sigma_3 \vdash \Box\varphi, \Delta}\ \Box\mathsf{R}$$

We bottom-up translate the entire block as shown below, where the conclusion is obtained from the end component of the modal block's conclusion. Note that we may apply the w rule because it is admissible in $\mathsf{GL_{seq}}$.

$$\dfrac{\dfrac{\Box\Sigma_1, \Box\Sigma_2, \Sigma_2, \Sigma_3, \Box\varphi \vdash \varphi}{\Box\Sigma_1, \Box\Sigma_2, \Box\Sigma_3, \Sigma_1, \Sigma_2, \Sigma_3, \Box\varphi \vdash \varphi}\ \mathsf{w}}{\Gamma, \Box\Sigma_1, \Box\Sigma_2, \Box\Sigma_3 \vdash \Box\varphi, \Delta}\ \Box_{\mathsf{GL}}$$

Last, suppose an instance of id_1 or id_2 is reached in the translation as shown below left.

[4] We note that Shamkanov's proof transformation from $\mathsf{GL_{seq}}$ to $\mathsf{GL_{\infty}}$ relies on the admissibility of the cut rule in $\mathsf{GL_{seq}}$. This is not problematic however since $\mathsf{GL_{seq}}$ admits syntactic cut-elimination.

$$\frac{}{\mathcal{G} \,/\!\!/\, \Gamma, p \vdash p, \Delta}\ \mathsf{id}_1 \qquad \frac{}{\mathcal{G} \,/\!\!/\, \Gamma, \Box\varphi \vdash \Box\varphi, \Delta}\ \mathsf{id}_2 \qquad \frac{}{\Gamma, p \vdash p, \Delta}\ \mathsf{id} \qquad \frac{}{\Gamma, \Box\varphi \vdash \Box\varphi, \Delta}\ \mathsf{id}$$

In each case, we translate the linear nested sequent as its end component, yielding the respective Gentzen sequents shown above right, both of which are instances of id. ◀

Last, the following theorem completes the circuit of proof transformations and establishes syntactic correspondences between G3GL, CSGL, LNGL, $\mathsf{GL_{seq}}$, $\mathsf{GL_{\infty}}$ and $\mathsf{GL_{circ}}$.

▶ **Theorem 23.** *If $\Gamma \vdash \Delta$ is provable in $\mathsf{GL_{seq}}$, then $x : \Gamma \vdash x : \Delta$ is provable in* G3GL.

Proof. By induction on the height of the proof π in $\mathsf{GL_{seq}}$. The base case immediately follows from Theorem 6-(1), and the ¬L, ¬R, ∨L, and ∨R cases of the inductive step straightforwardly follow by applying IH and then the corresponding rule in G3GL. Therefore, we need only show the case where π ends with an application of $\Box_{\mathsf{GL}}$, as shown below left.

$$\frac{\Box\Gamma, \Gamma, \Box\varphi \vdash \varphi}{\Sigma, \Box\Gamma \vdash \Box\varphi, \Delta}\ \Box_{\mathsf{GL}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x : \Box\Gamma, x : \Gamma, x : \Box\varphi \vdash x : \varphi}{yRx, y : \Box\Gamma, x : \Box\Gamma, x : \Gamma, x : \Box\varphi \vdash x : \varphi}\ \mathsf{w}}{yRx, y : \Box\Gamma, x : \Box\Gamma, x : \Box\varphi \vdash x : \varphi}\ \Box\mathsf{L}}{yRx, y : \Box\Gamma, x : \Box\varphi \vdash x : \varphi}\ \mathsf{4L}}{y : \Box\Gamma \vdash y : \Box\varphi}\ \Box\mathsf{R}}{x : \Box\Gamma \vdash x : \Box\varphi}\ (x/y)}{x : \Sigma, x : \Box\Gamma \vdash x : \Box\varphi, x : \Delta}\ \mathsf{w}$$

To obtain the desired proof, we first apply the hp-admissible w rule (Theorem 6), followed by applications of the □L rule and admissible 4L rule (cf. [15]). Applying the □R rule, followed by applications of the hp-admissible (x/y) and w rules (Theorem 6), gives the desired conclusion. ◀

6 Concluding Remarks

There are various avenues for future research: first, it would be interesting to look into the properties of the new linear nested sequent calculus LNGL, investigating additional admissible structural rules, how the system can be amended to allow for the hp-invertibility of all rules, and also looking into syntactic cut-elimination. Second, by employing a methodology for extracting nested sequent systems from relational semantics [28], we can integrate this approach with the linearization technique to develop a general method for extracting (cut-free) linear nested systems from the semantics of various modal, intuitionistic, and related logics. Third, it seems worthwhile to see if the proof transformation techniques discussed in this paper can be applied to structural cyclic systems (e.g., cyclic labeled sequent systems for classical and intuitionistic Gödel-Löb logic [11]) to remove extraneous structure and extract simpler (cyclic) Gentzen systems.

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A Proofs for Section 4

▶ **Lemma 15.** *The following permutations hold in* CSGL*:*
(1) *If* r *is a non-end-active local rule and* r′ *is non-initial and end-active, then* r *permutes below* r′ *and* r′ *remains end-active after this permutation;*
(2) *if* r *is a non-end-active propagation rule and* r′ *is non-initial and end-active, then* r *permutes below* r′ *and* r′ *remains end-active after this permutation.*

Proof. Follows from Lemmas 24 and 25 below. ◀

▶ **Lemma 24.** *If* r *is a non-end-active local rule and* r′ *is non-initial and end-active, then* r *permutes below* r′ *and* r′ *remains end-active after this permutation.*

Proof. We let r be an instance of ¬R as the cases where r is either ¬L, ∨L, or ∨R are shown similarly. We show that r can be permuted down r′ and consider a representative number of cases when r′ is either ∨R, 4L, or □R as the remaining cases are similar.

∨R. By our assumption that ¬R is non-end-active and ∨R is end-active, we know that the labels x and y are distinct. Hence, we can permute the ¬R instance below the ∨R instance. Observe that ∨R remains end-active after the permutation.

$$\dfrac{\dfrac{\mathcal{T}, \Gamma, x : \varphi \vdash y : \psi, y : \chi, \Delta}{\mathcal{T}, \Gamma \vdash x : \neg\varphi, y : \psi, y : \chi, \Delta}\,\neg\mathsf{R}}{\mathcal{T}, \Gamma \vdash x : \neg\varphi, y : \psi \lor \chi, \Delta}\,\lor\mathsf{R} \qquad \dfrac{\dfrac{\mathcal{T}, \Gamma, x : \varphi \vdash y : \psi, y : \chi, \Delta}{\mathcal{T}, \Gamma, x : \varphi \vdash y : \psi \lor \chi, \Delta}\,\lor\mathsf{R}}{\mathcal{T}, \Gamma \vdash x : \neg\varphi, y : \psi \lor \chi, \Delta}\,\neg\mathsf{R}$$

4L. By our assumption, we know that z is distinct from y in the inferences shown below left, meaning, we can permute ¬R below 4L as shown below right. Observe that 4L remains end-active after the permutation.

$$\dfrac{\dfrac{\mathcal{T}, xRy, \Gamma, x : \Box\psi, y : \Box\psi, z : \varphi \vdash \Delta}{\mathcal{T}, xRy, \Gamma, x : \Box\psi, y : \Box\psi \vdash z : \neg\varphi, \Delta}\,\neg\mathsf{R}}{\mathcal{T}, xRy, \Gamma, x : \Box\psi \vdash z : \neg\varphi, \Delta}\,\mathsf{4L} \qquad \dfrac{\dfrac{\mathcal{T}, xRy, \Gamma, x : \Box\psi, y : \Box\psi, z : \varphi \vdash \Delta}{\mathcal{T}, xRy, \Gamma, x : \Box\psi, z : \varphi \vdash \Delta}\,\mathsf{4L}}{\mathcal{T}, xRy, \Gamma, x : \Box\psi \vdash z : \neg\varphi, \Delta}\,\neg\mathsf{R}$$

□R. By our assumption, we know that z is distinct from y in the inferences shown below left, meaning, we can permute ¬R below □R as shown below right. Trivially, the □R rule remains end-active after the permutation.

$$\dfrac{\dfrac{\mathcal{T}, xRy, \Gamma, y : \Box\psi, z : \varphi \vdash y : \psi, \Delta}{\mathcal{T}, xRy, \Gamma, y : \Box\psi \vdash y : \psi, z : \neg\varphi, \Delta}\,\neg\mathsf{R}}{\mathcal{T}, \Gamma \vdash x : \Box\psi, z : \neg\varphi, \Delta}\,\Box\mathsf{R} \qquad \dfrac{\dfrac{\mathcal{T}, xRy, \Gamma, y : \Box\psi, z : \varphi \vdash y : \psi, \Delta}{\mathcal{T}, \Gamma, z : \varphi \vdash x : \Box\psi, \Delta}\,\Box\mathsf{R}}{\mathcal{T}, \Gamma \vdash x : \Box\psi, z : \neg\varphi, \Delta}\,\neg\mathsf{R}$$

◀

▶ **Lemma 25.** *If* r *is a non-end-active propagation rule and* r′ *is non-initial and end-active, then* r *permutes below* r′ *and* r′ *remains end-active after this permutation.*

Proof. We consider the case where r is an instance of □L as the 4L case is similar. We show that r can be permuted down r′ and consider a representative number of cases when r′ is either ¬L, 4L, or □R as the remaining cases are similar.

¬L. By our assumption, we know that z is distinct from y in the inferences below left. We can therefore permute □L below ¬L as shown below right and we observe that ¬L remains end-active.

$$\dfrac{\dfrac{\mathcal{T}, xRy, \Gamma, x : \Box\varphi, y : \varphi \vdash z : \psi, \Delta}{\mathcal{T}, xRy, \Gamma, x : \Box\varphi \vdash z : \psi, \Delta}\,\Box\mathsf{L}}{\mathcal{T}, xRy, \Gamma, x : \Box\varphi, z : \neg\psi \vdash \Delta}\,\neg\mathsf{L} \qquad \dfrac{\dfrac{\mathcal{T}, xRy, \Gamma, x : \Box\varphi, y : \varphi \vdash z : \psi, \Delta}{\mathcal{T}, xRy, \Gamma, x : \Box\varphi, y : \varphi, z : \neg\psi \vdash \Delta}\,\neg\mathsf{L}}{\mathcal{T}, xRy, \Gamma, x : \Box\varphi, z : \neg\psi \vdash \Delta}\,\Box\mathsf{L}$$

4L. Let us suppose we have a □L instance followed by a 4L instance. There are two cases to consider: either the principal formula of □L is the same as for 4L, or the principal formulae are distinct. We show the first case as the second case is similar. Then, our inferences are of the form shown below left, where y and z are distinct due to our assumption. We may permute □L below 4L as shown below right and we observe that 4L remains end-active.

$$\dfrac{\dfrac{\mathcal{T}, xRy, xRz, \Gamma, x : \Box\varphi, y : \varphi, z : \Box\varphi \vdash \Delta}{\mathcal{T}, xRy, xRz, \Gamma, x : \Box\varphi, z : \Box\varphi \vdash \Delta}\,\Box\mathsf{L}}{\mathcal{T}, xRy, xRz, \Gamma, x : \Box\varphi \vdash \Delta}\,4\mathsf{L} \qquad \dfrac{\dfrac{\mathcal{T}, xRy, xRz, \Gamma, x : \Box\varphi, y : \varphi, z : \Box\varphi \vdash \Delta}{\mathcal{T}, xRy, xRz, \Gamma, x : \Box\varphi, y : \varphi \vdash \Delta}\,4\mathsf{L}}{\mathcal{T}, xRy, xRz, \Gamma, x : \Box\varphi \vdash \Delta}\,\Box\mathsf{L}$$

□R. Suppose we have an instance of □L followed by an application of □R as shown below.

$$\dfrac{\dfrac{\mathcal{T}, xRy, zRu, \Gamma, x : \Box\varphi, y : \varphi, u : \Box\psi \vdash u : \psi, \Delta}{\mathcal{T}, xRy, zRu, \Gamma, x : \Box\varphi, u : \Box\psi \vdash u : \psi, \Delta}\,\Box\mathsf{L}}{\mathcal{T}, xRy, \Gamma, x : \Box\varphi \vdash z : \Box\psi, \Delta}\,\Box\mathsf{R}$$

By our assumption, the labels y and u are distinct, meaning, we can permute □L below □R as shown below. Trivially, □R remains end-active after the permutation is performed.

$$\dfrac{\dfrac{\mathcal{T}, xRy, zRu, \Gamma, x : \Box\varphi, y : \varphi, u : \Box\psi \vdash u : \psi, \Delta}{\mathcal{T}, xRy, \Gamma, x : \Box\varphi, y : \varphi \vdash z : \Box\psi, \Delta}\,\Box\mathsf{R}}{\mathcal{T}, xRy, \Gamma, x : \Box\varphi \vdash z : \Box\psi, \Delta}\,\Box\mathsf{L}$$

◀

▶ **Theorem 17.** *Each end-active proof in* CSGL *can be transformed into a proof in* LNGL.

Proof. We have included additional cases of the inductive step that are not included in the main text.

4L. Let us suppose that π ends with an application of 4L as shown below.

$$\dfrac{\mathcal{T}, xRy, \Gamma, x : \Box\varphi, y : \Box\varphi \vdash \Delta}{\mathcal{T}, xRy, \Gamma, x : \Box\varphi \vdash \Delta}\,4\mathsf{L}$$

By IH, we know there exists a path $y_1, \ldots, y_n$ of labels from the root y_1 to the leaf y_n in the premise of 4L such that $\mathcal{G} = \Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!\!/} \cdots \mathbin{/\!\!/} \Gamma(y_n) \vdash \Delta(y_n)$ is provable in LNGL. Since 4L is end-active, there are three cases to consider: either (1) neither x nor y occur along the path, (2) only x occurs along the path, or (3) both x and y occur along the path. In each case, the conclusion is obtained by taking the linear nested sequent corresponding to the path $y_1, \ldots, y_n$ in the conclusion of the 4L instance above. In the first and second cases, we translate the entire 4L instance as the single linear nested sequent $\mathcal{G}$. In the third case, we have that $x = y_{n-1}$ and $y = y_n$, meaning, the premise of the 4L instance shown below is provable in LNGL by IH, where $\Gamma(y_{n-1}) = \Sigma_1, \Box\varphi$ and $\Gamma(y_n) = \Sigma_2, \Box\varphi$. As shown below, a single application of 4L yields the desired conclusion.

$$\frac{\Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!/} \cdots \mathbin{/\!/} \Sigma_1, \Box\varphi \vdash \Delta(y_{n-1}) \mathbin{/\!/} \Sigma_2, \Box\varphi \vdash \Delta(y_n)}{\Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!/} \cdots \mathbin{/\!/} \Sigma_1, \Box\varphi \vdash \Delta(y_{n-1}) \mathbin{/\!/} \Sigma_2 \vdash \Delta(y_n)} \mathsf{4L}$$

$\vee$L. Let us suppose that π ends with an instance of $\vee$L as shown below.

$$\frac{\mathcal{T}, \Gamma, x : \varphi \vdash \Delta \qquad \mathcal{T}, \Gamma, x : \psi \vdash \Delta}{\mathcal{T}, \Gamma, x : \varphi \vee \psi \vdash \Delta} \vee\mathsf{L}$$

By IH, we know there exist paths $v = y_1, \ldots, y_n$ and $v = z_1, \ldots, z_k$ of labels from the root v to the leaves y_n and z_k in the premises of $\vee$L such that $\mathcal{G} = \Gamma(v) \vdash \Delta(v) \mathbin{/\!/} \cdots \mathbin{/\!/} \Gamma(y_n) \vdash \Delta(y_n)$ and $\mathcal{H} = \Gamma(v) \vdash \Delta(v) \mathbin{/\!/} \cdots \mathbin{/\!/} \Gamma(z_k) \vdash \Delta(z_k)$ are provable in LNGL. There are two cases to consider: either (1) $x \neq y_n$ or $x \neq z_k$, or (2) $x = y_n = z_k$. In the first case, if $x \neq y_n$, then we translate the entire $\vee$L inference as the single linear nested sequent $\mathcal{G}$, and if $x \neq z_k$, then we translate the entire $\vee$L inference as $\mathcal{H}$. In the second case, we know that the left premise $\mathcal{G}$ and right premise $\mathcal{H}$ of the $\vee$L inference below are provable with $\Gamma(y_n) = \Sigma, \varphi$ and $\Gamma(z_k) = \Sigma, \psi$, and so, a single application of $\vee$L gives the desired result.

$$\frac{\Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!/} \cdots \mathbin{/\!/} \Sigma, \varphi \vdash \Delta(y_n) \qquad \Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!/} \cdots \mathbin{/\!/} \Sigma, \psi \vdash \Delta(y_n)}{\Gamma(y_1) \vdash \Delta(y_1) \mathbin{/\!/} \cdots \mathbin{/\!/} \Sigma, \varphi \vee \psi \vdash \Delta(y_n)} \vee\mathsf{L}$$

◀

▶ **Theorem 20.** *Every proof in* LNGL *can be transformed into a proof in normal form.*

Proof. Let π be a proof in LNGL. We consider an arbitrary instance of a $\Box$R rule in π and first show that every length-consistent 4L rule above $\Box$R can be permuted down into a block B of 4L rules above $\Box$R. Afterward, we will show that every length-consistent $\Box$L rule can be permuted down into a block B$'$ of $\Box$L rules above B. As a result, all length-consistent local rules will occurs in a block B$''$ above the premise of the block B$'$, showing that $\Box$R is preceded be a complete block. As these permutations can be performed for every $\Box$R instance in π, we obtain a normal form proof as the result.

Let us choose an application of $\Box$R in π, as shown below, proceeded by a (potentially empty) block R of 4L rules.

$$\dfrac{\dfrac{\vdots}{\mathcal{G} \mathbin{/\!/} \Gamma \vdash \Delta \mathbin{/\!/} \Box\varphi \vdash \varphi} \mathrm{R}}{\mathcal{G} \mathbin{/\!/} \Gamma \vdash \Box\varphi, \Delta} \Box\mathsf{R}$$

We now select a bottom-most, length-consistent application of a 4L rule above the chosen $\Box$R application that does not occur within the block R of 4L rules. We show that 4L can be permuted below every local rule and $\Box$L rule until it reaches and joins the R block. We show that 4L can be permuted below $\neg$R, $\vee$L, and $\Box$L as the remaining cases are similar. Note that we are guaranteed that no other $\Box$R applications occur below 4L and above R since then 4L would not be length-consistent with the chosen $\Box$R application.

Suppose 4L occurs above a $\neg$R application as shown below left. The rules can be permuted as shown below right.

$$\dfrac{\dfrac{\mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Delta \mathbin{/\!/} \Sigma, \psi, \Box\varphi \vdash \Pi}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Delta \mathbin{/\!/} \Sigma, \psi \vdash \Pi} \mathsf{4L}}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Delta \mathbin{/\!/} \Sigma \vdash \neg\psi, \Pi} \neg\mathsf{R} \qquad \dfrac{\dfrac{\mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Delta \mathbin{/\!/} \Sigma, \psi, \Box\varphi \vdash \Pi}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Delta \mathbin{/\!/} \Sigma, \Box\varphi \vdash \neg\psi, \Pi} \neg\mathsf{R}}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Delta \mathbin{/\!/} \Sigma \vdash \neg\psi, \Pi} \mathsf{4L}$$

Suppose that we have 4L followed by an application of the $\vee$L rule.

$$\dfrac{\dfrac{\mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Delta \mathbin{/\!/} \Sigma, \Box\varphi, \psi \vdash \Pi}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Delta \mathbin{/\!/} \Sigma, \psi \vdash \Pi} \mathsf{4L} \qquad \mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Delta \mathbin{/\!/} \Sigma, \chi \vdash \Pi}{\mathcal{G} \mathbin{/\!/} \Gamma, \Box\varphi \vdash \Delta \mathbin{/\!/} \Sigma, \psi \vee \chi \vdash \Pi} \vee\mathsf{L}$$

Invoking the hp-invertibility of 4L (Lemma 13), we can permute 4L below ∨L as shown below.

$$\dfrac{\dfrac{\mathcal{G} \sslash \Gamma, \Box\varphi \vdash \Delta \sslash \Sigma, \Box\varphi, \psi \vdash \Pi \qquad \dfrac{\mathcal{G} \sslash \Gamma, \Box\varphi \vdash \Delta \sslash \Sigma, \chi \vdash \Pi}{\mathcal{G} \sslash \Gamma, \Box\varphi \vdash \Delta \sslash \Sigma, \Box\varphi, \chi \vdash \Pi}\,\mathsf{4L}^{-1}}{\mathcal{G} \sslash \Gamma, \Box\varphi \vdash \Delta \sslash \Sigma, \Box\varphi, \psi \lor \chi \vdash \Pi}\,\lor\mathsf{L}}{\mathcal{G} \sslash \Gamma, \Box\varphi \vdash \Delta \sslash \Sigma, \psi \lor \chi \vdash \Pi}\,\mathsf{4L}$$

Last, we show (below left) one of the cases where 4L is applied above a □L rule. We can permute the rules as shown below right.

$$\dfrac{\dfrac{\mathcal{G} \sslash \Gamma, \Box\psi, \Box\varphi \vdash \Delta \sslash \Sigma, \Box\psi, \varphi \vdash \Pi}{\mathcal{G} \sslash \Gamma, \Box\psi, \Box\varphi \vdash \Delta \sslash \Sigma, \varphi \vdash \Pi}\,\mathsf{4L}}{\mathcal{G} \sslash \Gamma, \Box\psi, \Box\varphi \vdash \Delta \sslash \Sigma \vdash \Pi}\,\Box\mathsf{L} \qquad \dfrac{\dfrac{\mathcal{G} \sslash \Gamma, \Box\psi, \Box\varphi \vdash \Delta \sslash \Sigma, \Box\psi, \varphi \vdash \Pi}{\mathcal{G} \sslash \Gamma, \Box\psi, \Box\varphi \vdash \Delta \sslash \Sigma, \Box\psi \vdash \Pi}\,\Box\mathsf{L}}{\mathcal{G} \sslash \Gamma, \Box\psi, \Box\varphi \vdash \Delta \sslash \Sigma \vdash \Pi}\,\mathsf{4L}$$

We can repeat the above downward permutations of bottom-most, length-consistent 4L rules, so that all length-consistent 4L rules occur in a block B above □R as shown below, where we let R′ be a (potentially empty) block of □L rules above the B block of 4L rules.

$$\dfrac{\dfrac{\dfrac{\vdots}{\mathcal{H}}\,\mathrm{R}'}{\mathcal{G} \sslash \Gamma \vdash \Delta \sslash \Box\varphi \vdash \varphi}\,\mathrm{B}}{\mathcal{G} \sslash \Gamma \vdash \Box\varphi, \Delta}\,\Box\mathsf{R}$$

Next we show that every length-consistent □L rule occurring above the block R′ can be permuted down to the block R′. Let □L occur above the block R′ be length-consistent with the chosen □R rule. Notice that we need only consider downward permutations of □L rules with local rules as all 4L rules have already been permuted downward and no other □R rule can occur between □L and R′ because then □L would not be length-consistent. We show how to permute the □L rule below a ∨L instance; the remaining cases are simple and similar.

$$\dfrac{\dfrac{\mathcal{G} \sslash \Gamma, \Box\varphi \vdash \Delta \sslash \Sigma, \psi, \varphi \vdash \Pi}{\mathcal{G} \sslash \Gamma, \Box\varphi \vdash \Delta \sslash \Sigma, \psi \vdash \Pi}\,\Box\mathsf{L} \qquad \mathcal{G} \sslash \Gamma, \Box\varphi \vdash \Delta \sslash \Sigma, \chi \vdash \Pi}{\mathcal{G} \sslash \Gamma, \Box\varphi \vdash \Delta \sslash \Sigma, \psi \lor \chi \vdash \Pi}\,\lor\mathsf{L}$$

By using the hp-invertibility of □L (Lemma 13), we can permute □L below the ∨L rule.

$$\dfrac{\dfrac{\mathcal{G} \sslash \Gamma, \Box\varphi \vdash \Delta \sslash \Sigma, \varphi, \psi \vdash \Pi \qquad \dfrac{\mathcal{G} \sslash \Gamma, \Box\varphi \vdash \Delta \sslash \Sigma, \chi \vdash \Pi}{\mathcal{G} \sslash \Gamma, \Box\varphi \vdash \Delta \sslash \Sigma, \varphi, \chi \vdash \Pi}\,\Box\mathsf{L}^{-1}}{\mathcal{G} \sslash \Gamma, \Box\varphi \vdash \Delta \sslash \Sigma, \varphi, \psi \lor \chi \vdash \Pi}\,\lor\mathsf{L}}{\mathcal{G} \sslash \Gamma, \Box\varphi \vdash \Delta \sslash \Sigma, \psi \lor \chi \vdash \Pi}\,\Box\mathsf{L}$$

By successively permuting all □L rules down into a block above B, we have that □R is preceded by a complete block in the proof. As argued above, this implies that every proof in LNGL can be put into normal form. ◀