
In the following exercises we will try to show that the satisfiability problem for FO^2 is NExpTime-hard. To do it, we will provide a reduction from the following tiling problem: given (T, H, V, I, n) with T being a set of tiles, $H \subset T \times T$ and $V \subset T \times T$ being, respectively, sets of horizontal and vertical tiling constants (i.e. specifying that a given can be placed above/to the right of another tile), n being a number encoded in unary and $I \in T$ being an initial time, we ask: is there a correctly tiled grid of size $2^n \times 2^n$ with the position $(0, 0)$ labelled with I?

Exercise 1

Use fresh unary predicates $H_1, H_2, \ldots, H_n$ and $V_1, V_2, \ldots, V_n$ and treat them as bits of some number encoded in binary, e.g. for $n = 3$ a domain element satisfies $\neg V_3 \wedge V_2 \wedge V_1$ if the predicates V encode the number 3 . Write a formula, of size polynomial in n and with two free variables x and y, stating that $y = x + 1$, i.e. the value encoded by y is equal to the value encoded on x plus one. How it helps you to solve the problem?

Exercise 2

Do the routine part of the encoding. Can you prove hardness without use of equality?

Exercise 3

We know that constant-free FO^2 has FMP and is decidable in NExpTime. Show that the same result holds for FO^2 with constants [EASY!].

Exercise 4

During the lecture, in the proof of FMP for FO^2, we presented a construction that creates three sets: C, D and E. Why just two sets are not enough?

Exercise 5

During the lecture we employed the Scott-like normal form for FO^2, namely: $\forall x \forall y\ \varphi \wedge \bigwedge_i \forall x \exists y \varphi_i(x, y)$, where φ, φ_i are quantifier-free. Show that one can do an extra step to ensure that your formula looks as follows:

$$\forall x \forall y\ \varphi \wedge \bigwedge_i \forall x \exists y x \neq y \wedge \varphi_i(x, y)$$

Exercise 6

[Hard, you must solve exercise 1 first.] We now know that if FO^2 formula has a model then it has a model of exponential size. Show that it is not true for FO^2 extended by $\exists^{=1}$ counting quantifiers. More precisely, show that there is a formula of size $O(n)$ whose models are of size at least $O(2^{2^n})$. Hint: How many leaves a binary tree of height n has?