Stochastic Local Search

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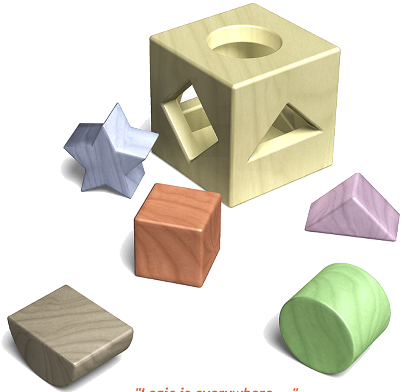

"Logic is everywhere …"

- Stochastic Local Search Algorithms
- Uninformed Random Walk
- Iterative Improvement
- Local Minima and Escape Strategies
- Randomized Iterative Improvement
- Tabu Search
- The GSAT Architecture
- The Walksat Architecture

Stochastic Local Search Algorithms

- A probability distribution for a finite set S is a function $D : S \mapsto [0, 1]$ with
$$\sum_{s \in S} D(s) = 1$$
- Let $\mathcal{D}(S)$ denotes the set of probability distributions over a given set S
- Given a (combinatorial) problem Π, a stochastic local search algorithm for solving an arbitrary instance $\pi \in \Pi$ is defined by the following components:
  - the search space $S(\pi)$,
which is a finite set of candidate solutions $s \in S(\pi)$
  - a set of solutions $S'(\pi) \subseteq S(\pi)$
  - a neighbourhood relation on $S(\pi)$: $N(\pi) \subseteq S(\pi) \times S(\pi)$
  - a finite set of memory states $M(\pi)$
  - an initialization function $\text{init}(\pi) : \to \mathcal{D}(S(\pi) \times M(\pi))$
  - a step function $\text{step}(\pi) : S(\pi) \times M(\pi) \to \mathcal{D}(S(\pi) \times M(\pi))$
  - a termination predicate $\text{terminate}(\pi) : S(\pi) \times M(\pi) \to \{\perp, \top\}$

Some Notation

- We often write step(π, s, m) instead of step(π)(s, m) and, likewise, for terminate and other functions
- We omit $M(\pi)$ and the parameter m if no memory is used

General Outline of a Stochastic Local Search Algorithm

- procedure **SLSDecision(π)**
  input $\pi \in \mathbf{\Pi}$
  output $\boldsymbol{s} \in \boldsymbol{S'}(\pi)$ or "no solution found"
  $(\boldsymbol{s}, \boldsymbol{m}) = \mathbf{selectRandomly}(\boldsymbol{S}(\pi) \times \boldsymbol{M}(\pi), \mathbf{init}(\pi));$
  while not $\mathbf{terminate}(\pi, \boldsymbol{s}, \boldsymbol{m})$ do
  $\quad(\boldsymbol{s}, \boldsymbol{m}) = \mathbf{selectRandomly}(\boldsymbol{S}(\pi) \times \boldsymbol{M}(\pi), \mathbf{step}(\pi, \boldsymbol{s}, \boldsymbol{m}));$
  end
  if $\boldsymbol{s} \in \boldsymbol{S'}(\pi)$ then
  $\quad$return $\boldsymbol{s}$
  else
  $\quad$return "no solution found"
  end
  end

  - **where selectRandomly gets a pair $(\boldsymbol{S}(\pi) \times \boldsymbol{M}(\pi), \boldsymbol{D})$ as input and yields the result of a random experiment selecting an element of $\boldsymbol{S}(\pi) \times \boldsymbol{M}(\pi)$ wrt the probability distribution $\boldsymbol{D} \in \mathcal{D}(\boldsymbol{S}(\pi) \times \boldsymbol{M}(\pi))$**

A Simple SLS Algorithm for SAT: Uninformed Random Walk (URW)

- Let F be a CNF-formula with variables $1, \ldots, n$
- The search space $S(F)$ is the set of all interpretations for F
- The set of solutions $S'(F)$ is the set of models for F
- The neighbourhood relation on $S(F)$ is the one-flip neighbourhood
  $N(F, I, I')$ iff there exists $A \in \{1, \ldots, n\}$ such that $A^I \neq A^{I'}$
  and for all $A' \in \{1, \ldots, n\} \setminus \{A\}$ we find $A'^I = A'^{I'}$
- We will not use memory
- The initialization function yields the uninformed random distribution

$$\text{init}(F, I) = \frac{1}{|S(F)|} = \frac{1}{2^n} \text{ for all } I \in S(F)$$

- The step function maps any I to the uniform distribution over all its neighbours

$$\text{step}(F, I, I') = \frac{1}{|\{I' \mid N(F, I, I')\}|} = \frac{1}{n} \text{ for all } I' \text{ with } N(F, I, I')$$

- terminate(F, I) holds iff $I \models F$

Evaluation Functions

- Given a (combinatorial) problem Π and let $\pi \in \Pi$;
an evaluation function $g(\pi) : S(\pi) \mapsto \mathbb{R}$ is a function which maps each candidate solution to a real number such that the global optima of $g(\pi)$ correspond to the solutions of π
- Optima are usually minima or maxima
- $g(\pi)$ is used to rank candidate solutions
- Concerning SAT: Let F be a CNF-formula and I an interpretation
  - Often, $g(F)(I) = g(F, I)$ is the number of clauses of F not satisfied by I, i.e.,
$$g(F, I) = |\{C \in F \mid I \not\models C\}|$$
  - Consequently, $g(F, I) = 0$ iff $I \models F$

Iterative Improvement

- Given Π, $\pi \in \Pi$, $S(\pi)$, $N(\pi)$ and $g(\pi)$
- We assume that the solutions of π correspond to global minima of $\mathrm{g}(\pi)$
- Iterative improvement (II) starts from a randomly selected point in the search space and tries to improve the current candidate solution wrt $\mathrm{g}(\pi)$
  - Initialization function

$$\mathrm{init}(\pi, s) = \frac{1}{|S(\pi)|} \quad \text{for all } s \in S(\pi)$$

  - Neighbouring candidate solutions

$$N'(s) = \{s' \mid (s, s') \in N(\pi) \text{ and } \mathrm{g}(\pi, s') < \mathrm{g}(\pi, s)\} \quad \text{for all } s \in S(\pi)$$

  - Step function

$$\mathrm{step}(\pi, s, s') = \begin{cases} \frac{1}{|N'(s)|} & \text{if } s' \in N'(s) \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } s,\ s' \in S(\pi)$$

Local Minima and Escape Strategies

- The step function in the definition of iterative improvement is ill-defined!
- Given Π, $\pi \in \Pi$, $S(\pi)$, $N(\pi)$ and $g(\pi)$
- A local minimum is a candidate solution $s \in S(\pi)$ such that for all $(s, s') \in N(\pi)$ we find $g(\pi, s) \leq g(\pi, s')$
- A local minimum $s \in S(\pi)$ is strict if for all $(s, s') \in N(\pi)$ we find $g(\pi, s) < g(\pi, s')$
  - If II encounters a local minimum which does not correspond to a solution, then it "gets stuck"; step(π, s) is not a probability distribution!
- Escape Strategies
  - Restart re-initialize the search whenever a local minimum is encountered
  - Random Walk perform a randomly chosen non-improving step
  - Tabu List forbid steps to recently visited candidate solutions
- Even with these escape strategies there is no guarantee that an SLS-algorithm does eventually find a solution

Randomized Iterative Improvement – Preliminaries

- We want to escape local minima by selecting non-improving steps
- **Walk Probability** wp $\in [0, 1]$
- **stepURW** the step function of uninformed random walk
- **stepII**
  a variant of the step function used in the iterative improvement algorithm, which differs only in that a minimally worsening neighbour is selected if $N'(s) = \emptyset$

The Step Function of Randomized Iterative Improvement

procedure **stepRII**(π, $\boldsymbol{s}$, **wp**)
  input $\pi \in \boldsymbol{\Pi}$, $\boldsymbol{s} \in \boldsymbol{S}(\pi)$, **wp** $\in$ **[0, 1]**
  output $\boldsymbol{s'} \in \boldsymbol{S}(\pi)$
  $\boldsymbol{u}$ = **random([0, 1])**;
  if $\boldsymbol{u} \leq$ **wp** then
    $\boldsymbol{s'}$ = **stepURW**(π, $\boldsymbol{s}$);
  else
    $\boldsymbol{s'}$ = **stepII**(π, $\boldsymbol{s}$);
  end
  return $\boldsymbol{s'}$
end

The Randomized Iterative Improvement Algorithm

- **Termination**
  - after limit on the CPU time
  - after limit on the number of search steps,
    i.e., iterations of the while loop or
  - after a number of search steps have been performed without improvement
- **Properties**
  - Arbitrarily long sequences of random walk steps may occur
  - The algorihm can escape from any local minimum
  - Solutions can be (provably) found with arbitrarily high probability

GUWSAT

- Randomized iterative improvement algorithm for SAT, but
  - instead of stepII
  - a best improvement local search algorithm is applied, i.e.,
    - in each step a variable is flipped that leads to a maximal increase in the evaluation function
- The algorithm does not terminate in a local minima
  - The maximally improving variable flip is a least worsening step in this case
- The search in stepURW is still uninformed

Tabu Search

- Iterative improvement algorithm using a form of short-term memory
- It uses a best improvement strategy
- Forbids steps to recently visited candidate solutions
  - by memorizing recently visited solutions explicitly or
  - by using a parameter tt called tabu tenure

The Step Function of Tabu Search

procedure **stepTS(π, *s*, tt)**
  input $\pi \in \mathbf{\Pi}$, $\boldsymbol{s} \in \boldsymbol{S}(\pi)$, **tt**
  output $\boldsymbol{s}' \in \boldsymbol{S}(\pi)$
  $\boldsymbol{N}' =$ **admissableNeighbours(π, *s*, tt)**;
  $\boldsymbol{s}' =$ **selectBest($\boldsymbol{N}'$)**;
  return $\boldsymbol{s}'$
end

The GSAT Architecture

- **GSAT was one of the first SLS algorithms for SAT**
  Selman, Levesque, Mitchell:
  A New Method for Solving Hard Satisfiability Problems
  In: Proc. AAAI National Conference on Artificial Intelligence, 440-446: 1992
- **Given CNF-formula F and interpretation I, GSAT uses**
  - **the one-flip neighbourhood relation**
  - **the evaluation function**

$$\mathrm{g}(F, I) = |\{C \in F \mid I \not\models C\}|$$

  - **the score $\mathrm{g}(F, I) - \mathrm{g}(F, I')$ of a variable A under I where I' is obtained from I by flipping A**
- **At the time of its introduction GSAT outperformed the best systematic search algorithms for SAT**

The Basic GSAT Algorithm

```
procedure gsat(F, maxtries, maxsteps)
    input F ∈ 𝓛(𝓡), maxtries, maxsteps ∈ ℕ⁺
    output model of F or ”no solution found”
    for try = 1 to maxtries do
        I = randomly chosen interpretation of F;
        for step = 1 to maxsteps do
            if I ⊨ F then
                return I
            end
            A = randomly selected variable with maximal score;
            I = I with A flipped;
        end
    end
    return “no solution found”
end
```

GSAT with Random Walk (GWSAT)

- Consider GSAT, but use a randomised best-improvement search method
- Conflict-directed random walk steps In a random walk step do
  - randomly select a currently unsatisfied clause C
  - randomly select a variable A occurring in C
  - Flip A
- GWSAT
  - Use the basic GSAT algorithm
  - At each local step decide with fixed walk probability wp whether to do
    - a standard GSAT step or
    - a conflict-directed random walk step
- In contrast to GUWSAT, GWSAT performs informed random walk steps
- GWSAT achieves substantially better performance than basic GSAT

GSAT with Tabu Search (GSAT/Tabu)

- Consider GSAT, but after A has been flipped, it cannot be flipped back within the next tt steps
- With each variable A a tabu status is associated as follows
  - Let t be the current search step number
  - Let tt $\in \mathbb{N}$
  - Let t_A be the search step number, when A was flipped for the last time
  - Initialize $t_A = -\mathrm{tt}$
  - Every time variable A is flipped set $t_A = t$
  - Variable A is tabu iff $t - t_A \leq \mathrm{tt}$

The WalkSAT Architecture

procedure **WalkSAT**($\boldsymbol{F}$, **maxtries**, **maxsteps**, **select**)
    input $\boldsymbol{F} \in \mathcal{L}(\mathcal{R})$, **maxtries**, **maxsteps** $\in \mathbb{N}^+$
        heuristic function **select**
    output *model of F* or "no solution found"
    for **try** = **1** to **maxtries** do
        $\boldsymbol{I}$ = randomly choosen interpretation of $\boldsymbol{F}$;
        for **step** = **1** to **maxsteps** do
            if $\boldsymbol{I} \models \boldsymbol{F}$ then
                return $\boldsymbol{I}$
            end
            $\boldsymbol{C}$ = randomly selected clause unsatisfied under $\boldsymbol{I}$;
            $\boldsymbol{A}$ = variable selected from $\boldsymbol{C}$ according to **select**;
            $\boldsymbol{I}$ = $\boldsymbol{I}$ with $\boldsymbol{A}$ flipped;
        end
    end
    return "no solution found"
end

Application of a Solver

- Consider walksat
  - Check out the internet for walksat
  - Walksat accepts .cnf-files and attempts to find a model
  - E.g., `walksat -sol < axioms.cnf`
- WalkSAT as well as GSAT and GWSAT are sound but incomplete

Novelty

- **Considers variables in the selected clauses sorted according to their score**
- **If the best variable is not the most recently flipped one, it is flipped, otherwise, it is flipped with a probability $1 - p$, while in the remaining cases, the second-best variable is flipped,**
  - **where $p \in [0, 1]$ is a parameter called noise setting**
- **Is in many cases substantially better than WalkSAT**
- **It suffers from *essential incompleteness***
- McAllester, Selman, Kautz: Evidence for Invariants in Local Search
  Proc. 14th National Conference on Artificial Intelligence (AAAI), 321-326:1997

Novelty$^+$

- **In each search step, with a user-specified probability wp,**
  the variable to be flipped is randomly selected from the selected clause,
  otherwise, the variable is selected according to the heuristics from Novelty
- **Is probabilistically approximately complete**
- **In practice, wp $= 0.01$ is sufficient**
- Hoos: On the run-time behavior of stochastic local search algorithms for SAT
  Proc. 16th National Conference on Artificial Intelligence (AAAI), 661-666: 1999

Adaptive Novelty$^+$

- **Optimal value for noise p varies significantly between problem instances**
- Idea **Adapt p**
  - **Initially $p = 0$**
  - **Rapid improvement typically leading to stagnation**
  - **Increase the value of p until escape from stagnation**
  - **Gradually decrease the value of p**
  - **Repeat this process until solution is found**
- Hoos: An Adaptive Noise Mechanism for WalkSAT
  Proc. 18th National Conference on Artificial Intelligence (AAAI), 655-660: 2004
- **Implemented in the UBCSAT framework**

Final Remarks

- **This section is based on**
  Hoos, Stützle: Stochastic Local Search
  Morgan Kaufmann/Elsevier, San Francisco: 1998
- **So far: stochastic local search**
  - **Sound but usually incomplete**
  - **Often quite fast**
- **Alternative: systematic search**
  - **Decides SAT problems**
  - **Sound and complete**
  - **May be too slow**
  - **In real applications it is often known that the problem is solvable**