TU Dresden, Fakultät Informatik
Stephan Mennicke, Markus Krötzsch

Winter Term 2021/2022

Complexity Theory

Exercise 5: Space Complexity

1st December 2021

Exercise 5.1. A *deterministic linear bounded automaton* (LBA) is a tuple

$$\langle Q, \Sigma, \Gamma, \delta, q_0, Q_F, E_L, E_R \rangle$$

where $\langle Q, \Sigma, \Gamma, \delta, q_0, Q_F \rangle$ is a Turing machine and $E_L, E_R \in \Sigma$ are left/right markers, such that for all $q \in Q$, $\delta(q, E_L) = \langle q, E_L, R \rangle$ and $\delta(q, E_R) = \langle q, E_R, L \rangle$. An LBA receives as input $E_L w_1 w_2 \dots w_n E_R$.

Let $\mathbf{A}_{\text{LBA}}$ be the word problem of deterministic linear bounded automata.

$$\mathbf{A}_{\text{LBA}} = \{\langle \mathcal{M}, w \rangle \mid \mathcal{M} \text{ is a (deterministic) LBA and } w \in \mathbf{L}(\mathcal{M})\}$$

Show that $\mathbf{A}_{\text{LBA}}$ is PSPACE-complete.

Exercise 5.2. Consider the Japanese game *go-moku* that is played by two players X and O on a 19x19 board. Players alternately place markers on the board, and the first one to have five of its markers in a row, column, or diagonal wins.

Consider the generalised version of go-moku on an $n \times n$ board. Say that a *position* of go-moku is a placement of markers on such a board as it could occur during the game, together with a marker which player moves next. We define

$$\mathbf{GM} = \{\langle B \rangle \mid B \text{ is a position of go-moku where X has a winning strategy}\}.$$

Show that **GM** is in PSPACE.

Exercise 5.3. Show that the universality problem of nondeterministic finite automata

$$\mathbf{ALL}_{\text{NFA}} = \{\langle \mathcal{A} \rangle \mid \mathcal{A} \text{ an NFA accepting every valid input}\}$$

is in PSPACE.

Hint:
Prove that, if $\mathbf{L}(\mathcal{A}) \neq \Sigma^*$ and $\mathcal{A}$ has n states, then there exists a word $w \in \Sigma^*$ of length at most 2^n such that $w \notin \mathbf{L}(\mathcal{A})$. Then, use this fact to give a non-deterministic algorithm whose space consumption is polynomially bounded. Finally, apply Savitch's Theorem.

Exercise 5.4. Show that the composition of logspace reductions again yields a logspace reduction.

Exercise 5.5. Show that the word problem $\mathsf{A}_{\mathsf{NFA}}$ of non-deterministic finite automata is NL-complete.

Exercise 5.6. Show that

$$\mathsf{BIPARTITE} = \{ \langle G \rangle \mid G \text{ a finite bipartite graph} \}$$

is in NL. For this show that $\overline{\mathsf{BIPARTITE}} \in \mathsf{NL}$ and use $\mathrm{NL} = \mathrm{coNL}$. **Hint:**
Show that a graph G is bipartite if and only if it does not contain a cycle of odd length.