TU Dresden, Fakultät Informatik
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Complexity Theory

Exercise 1: Mathematical Foundations, Decidability, and Recognisability

During the exercise session on November 3, 2020 we discussed additional tasks complementing the original exercise sheet. To provide a complete and commented list of tasks, the following block contains all of them.

Kahoot 1

1. *Show that if* $M \neq \emptyset$ *and* $f : M \to 2^M$*, then* f *is not surjective.*
2. *Show that* $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.
3. *Show that* $|\mathbb{N}| = |\mathbb{Q}|$.
4. *Argue that the set* $A = \{s\}$*, where*

$$s = \begin{cases} 42 & \text{if the velocity of light is independent of direction} \\ 0 & \text{otherwise,} \end{cases}$$

   is decidable. (For the purpose of this problem, assume that the question whether the velocity of light is independent of direction has an unambiguous "yes" or "no" answer.)

5. *Show that Turing-decidable languages are closed under (1) union, (2) concatenation, (3) intersection, (4) star, but not under (5) homomorphisms.*
6. *Show that* $\mathsf{ALL}_{\mathsf{DFA}} = \{ \langle A \rangle \mid A \text{ is a DFA that accepts every word} \}$ *is decidable.*
7. *Construct a Turing machine that shifts a word* w *on tape by one position to the right in at most* $|w| + 1$ *steps (given that the head is at the first position of* w*).*

Exercise 1.1. Show the following claims.

1. $|\mathbb{N}| = |\mathbb{Z}|$.
2. $|\mathbb{N}| \neq |\mathbb{R}|$.

Exercise 1.2. Show the following claims.

1. There exist non-regular languages.
2. There exist undecidable languages.
3. There exist non-Turing-recognizable languages.

Exercise 1.3. Let $G = \{V, E\}$ be a simple undirected graph such that $|V| \geq 2$ (i. e., no self-loops). Show that G contains two or more nodes that have equal degree. That is, show that that there is a pair of nodes that occur in the same number of edges.

$*$ **Exercise 1.4.** Show that the class of Turing-recognizable languages is closed under homomorphisms.

Exercise 1.5. A *Turing machine with two-sided unbounded tape* is a single-tape Turing machine where the tape is unbounded on both sides. Argue that such machines can be simulated by ordinary Turing machines.

Kahoot 2 *CHANGE TO TWO-DIMENSIONAL TURING MACHINE.*

Exercise 1.6. Let $\mathsf{E_{TM}} = \{\, \langle M \rangle \mid M$ is a TM such that $\mathcal{L}(M) = \emptyset \,\}$. Show that $\overline{\mathsf{E_{TM}}}$ is Turing-recognizable.

Exercise 1.7. Let C be a language. Prove that C is Turing-recognizable if and only if a decidable language D exists such that $C = \{\, x \mid \exists y. \langle x, y \rangle \in D \,\}$.

Kahoot 3 *Show that the class of decidable languages is not closed under homomorphisms.*

Hint:
Consider Exercise 1.7. Is D a homomorphic image of C?