Two Restrictions on Contraction

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Abstract. I show two limitations of multiplicative (or context-independent) sequent calculi: contraction can neither be restricted to atoms nor to the bottom of a proof tree.

There is a a set of rules for classical propositional logic, system SKS [1], with two interesting properties:

1. Contraction can be restricted to atoms. As a consequence, no rule in system SKS requires the duplication of formulae of unbounded size.
2. Contraction can be restricted to the bottom of a proof. As a consequence, a proof in SKS can be separated into two phases (seen bottom-up): In the first phase atoms are duplicated and in the second phase all other rules are applied and the size of the frontier of the proof (in terms of number of atoms) does not increase.

System SKS is not a sequent calculus but is presented in a more general formalism, the *calculus of structures* [2]. While the sequent calculus restricts the application of rules to the main connective of a formula, the calculus of structures is more expressive by admitting rules that can be applied anywhere *deep* inside formulae.

This leads to the question whether the extra expressive power of the calculus of structures is needed to observe these properties, or whether they can be observed in sequent systems as well. This question seems quickly answered by sequent system G3cp [3] in which contraction is admissible and can thus trivially be restricted to atoms or to the bottom of a proof. However, the R$\wedge$-rule in G3cp is additive (or context-sharing), so none of the above mentioned interesting consequences for system SKS follow for G3cp. Even though contraction is admissible, formulae of unbounded size still have to be duplicated in the context of an additive R$\wedge$. The separation of a proof into two phases as in system SKS also is not observable in G3cp, simply because there is no rule that just duplicates formulae.

To answer the question whether similar properties to those of system SKS can be achieved in sequent systems, I thus consider systems with a multiplicative R$\wedge$-rule. For the specific case of system GS1p [3], a Gentzen-Schütte formulation of classical logic shown in Fig. 1, I now prove by counterexample that it does not admit the properties of SKS.

Atoms can be either positive or negative and are denoted by a or b. The negation of an atom a is denoted by $\bar{a}$ and is again an atom. In a proof, contraction is considered to be at the bottom if there is no application of a rule different from contraction such that one of its premises is the conclusion of a contraction. An application of the contraction rule is considered atomic if its principal formula is an atom.

Theorem 1. There is a sequent that is valid but for which there is no cut-free proof in GS1p with multiplicative context treatment in which all contractions are at the bottom.

$$\mathsf{Ax}\,\frac{}{\vdash A, \bar{A}} \qquad \mathsf{Cut}\,\frac{\vdash \Gamma, A \quad \vdash \Delta, \bar{A}}{\vdash \Gamma, \Delta}$$

$$\mathsf{R}\vee\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \qquad \mathsf{R}\wedge\,\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B}$$

$$\mathsf{RC}\,\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \qquad \mathsf{RW}\,\frac{\vdash \Gamma}{\vdash \Gamma, A}$$

Fig. 1. GS1p with multiplicative context treatment

Proof. Consider the following sequent:

$$\vdash a \wedge a, \bar{a} \wedge \bar{a} \quad .$$

It suffices to show that

$$\vdash a \wedge a, \ldots, a \wedge a, \bar{a} \wedge \bar{a}, \ldots, \bar{a} \wedge \bar{a}$$

(for any number of occurrences of the formulae $a \wedge a$ and $\bar{a} \wedge \bar{a}$) is not provable in GS1p without contraction and cut. The connective $\vee$ does not occur in the conclusion. Thus, the only rules that can appear in the proof are RW, Ax and R$\wedge$.

A proof has to have all branches closed with axioms $\vdash A, \bar{A}$. Since no rule that may occur introduces new formulae, the only formula that can take the place of A in an axiom is the atom a.

There can be no such proof since there is always one branch in the derivation tree in which there is at most one single atom, as is shown by induction on the derivation tree:

Base Case
There is only one branch in

$$\vdash a \wedge a, \ldots, a \wedge a, \bar{a} \wedge \bar{a}, \ldots, \bar{a} \wedge \bar{a}$$

and it contains at most one (that is, no) single atom.

Inductive Case
Weakening does not increase the number of single atoms. When a R$\wedge$ rule is applied, the context is split, so the only single atom that may occur in the conclusion goes to one branch. Choose the other branch, which has at most one (that is, exactly one) atom. □

Theorem 2. There is a sequent that is valid but for which there is no cut-free proof in GS1p with multiplicative context treatment in which all contractions are atomic.

Proof. Consider the following sequent:

$$\vdash a \wedge b, (\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b}) \quad .$$

There are no single atoms, so contraction cannot be applied. Each applicable rule leads to a premise that is not valid. □

Theorems 1 and 2 rely on the multiplicative treatment of context in the $\mathsf{R}\wedge$-rule, but hold regardless of whether the system in question is propositional or predicative, whether it is two- or one-sided, whether or not rules for implication are in the system, whether it is related to $\mathsf{G1}$ (explicit weakening) or $\mathsf{G2}$ (weakening built into the identity axiom) and whether a multiplicative or additive version of the $\mathsf{R}\vee$-rule is used.

The properties of system SKS, that is, the possibility of restricting contraction to atoms as well as to the bottom of a proof, thus cannot be observed in those sequent systems.

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