Concurrency Theory

Lecture 2: Linear Time vs. Branching Time

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Process (Equivalence) Relations

Definition 11 Any binary relation $\mathcal{R} \subseteq \mathsf{Pr} \times \mathsf{Pr}$ is called a *process relation*. $\mathcal{R}$ is a *process equivalence* if it is a process relation and an equivalence.

We have seen now two instances of process equivalences.

Theorem 12 $\leftrightarrow$ and $\equiv_{\mathsf{tr}}$ are process equivalences.

Proof: in a few slides … ∎

Throughout the course, we will explore many more process equivalences, each time with a different set of requirements.

Isomorphic equivalence ($\leftrightarrow$) and trace equivalence ($\equiv_{\mathsf{tr}}$) form meaninful boundaries.

Trivial boundaries: $\mathcal{U} = \mathsf{Pr} \times \mathsf{Pr}$ (the *universal equivalence*) and $\emptyset$ (the *non-equivalence*).

A Proof of Theorem 12

Theorem 12 $\leftrightarrow$ and $\equiv_{\mathsf{tr}}$ are process equivalences.

Proof: For all processes $p, q, r \in \mathsf{Pr}$,

1. $p \leftrightarrow p$ by $\mathrm{id} : \mathsf{Pr} \to \mathsf{Pr}$ ($\mathrm{id}(q) = q$ for all $q \in \mathsf{Pr}$) being an isomorphism.
2. $p \leftrightarrow q$ implies $q \leftrightarrow p$ since the inverse f^{-1} of an isomorphism f is an isomorphism (cf. Lemma 7).
3. $p \leftrightarrow q$ and $q \leftrightarrow r$ implies $p \leftrightarrow r$ since isomorphisms f and g compose to an isomorphism $g \circ f$ (if unclear, let's make it another exercise 😀).

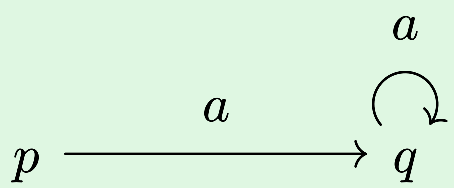
For all processes $p, q, r \in \mathsf{Pr}$,

1. $p \equiv_{\mathsf{tr}} p$ iff $\mathsf{traces}(p) = \mathsf{traces}(p)$ by reflexivity of $=$.
2. $p \equiv_{\mathsf{tr}} q$ iff $\mathsf{traces}(p) = \mathsf{traces}(q)$ iff $\mathsf{traces}(q) = \mathsf{traces}(p)$ iff $q \equiv_{\mathsf{tr}} p$ by symmetry of $=$.
3. $p \equiv_{\mathsf{tr}} q$ and $q \equiv_{\mathsf{tr}} r$ iff □□ iff $p \equiv_{\mathsf{tr}} r$ by transitivity of $=$.

∎

Reminder: $\leftrightarrow$ and $\equiv_{\mathrm{tr}}$

Example. *Reconsider processes p and q and find that $p \equiv_{\mathrm{tr}} q$*



We have $p \not\leftrightarrow q$ but $p \equiv_{\mathrm{tr}} q$.

- this means, $\leftrightarrow \neq \equiv_{\mathrm{tr}}$
- but does $\equiv_{\mathrm{tr}} \subseteq \leftrightarrow$? ✘
- or $\leftrightarrow \subseteq \equiv_{\mathrm{tr}}$? ✓

Process equivalence $\mathcal{E}_1$ .... $\mathcal{E}_2$

- *is finer (than)* if $\mathcal{E}_1 \subseteq \mathcal{E}_2$ *strictly* if if $\mathcal{E}_1 \subsetneq \mathcal{E}_2$
- *is coarser (than)* if $\mathcal{E}_1 \supseteq \mathcal{E}_1$
- *is incomparable with* if **neither** finer **nor** coarser

Towards a Spectrum of Process Equivalences

Theorem 13

$$\emptyset \overset{(1)}{\subsetneq} \leftrightarrow \overset{(2)}{\subsetneq} \equiv_{\mathsf{tr}} \overset{(3)}{\subsetneq} \mathcal{U} = \mathsf{Pr} \times \mathsf{Pr}$$

Towards a Spectrum of Process Equivalences

Theorem 13

$$\emptyset \overset{(1)}{\subsetneq} \leftrightarrow \overset{(2)}{\subsetneq} \equiv_{\mathsf{tr}} \overset{(3)}{\subsetneq} \mathcal{U} = \mathsf{Pr} \times \mathsf{Pr}$$

Proof: Parts (1) and (3) are clear. Proper inclusions stem from the examples we have seen.

Regarding (2), let $p, q \in \mathsf{Pr}$ such that $p \leftrightarrow q$. Then there is an isomorphism f between the graphs $G(p)$ and $G(q)$, meaning

1. $f(p) = q$ (since p and q are the roots of their respective process graphs) and
2. $p_1 \xrightarrow{a} p_2$ ($p_1 \in \mathrm{Reach}(p)$) if and only if $f(p_1) \xrightarrow{a} f(p_2)$ ($f(p_1) \in \mathrm{Reach}(q)$)

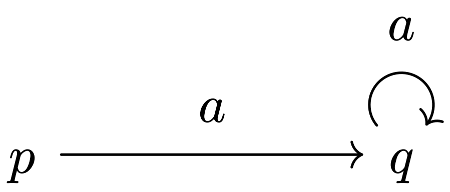
... to be continued ∎

Towards a Spectrum of Process Equivalences

Proof: For every trace $\sigma = a_1 a_2 \ldots a_n \in \mathsf{Act}^\star$,

$$
\begin{aligned}
\sigma \in \mathsf{traces}(p) &\text{ iff } \exists p_1, \ldots, p_n \in \mathsf{Pr}\ .p \xrightarrow{a_1} p_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} p_n && \text{(by definition)}\\
&\text{ iff } \exists p_1, \ldots, p_n \in \mathsf{Pr}\ .f(p) \xrightarrow{a_1} f(p_1) \xrightarrow{a_2} \cdots \xrightarrow{a_n} f(p_n) && \text{(} f \text{ is an isomorphism)}\\
&\text{ iff } \exists q_1, \ldots, q_n \in \mathsf{Pr}\ .q \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} q_n && \text{(take } q_1 = f(p_1) \ldots q_n = f(p_n)\text{)}\\
&\text{ iff } \sigma \in \mathsf{traces}(q) && \text{(by definition)}
\end{aligned}
$$

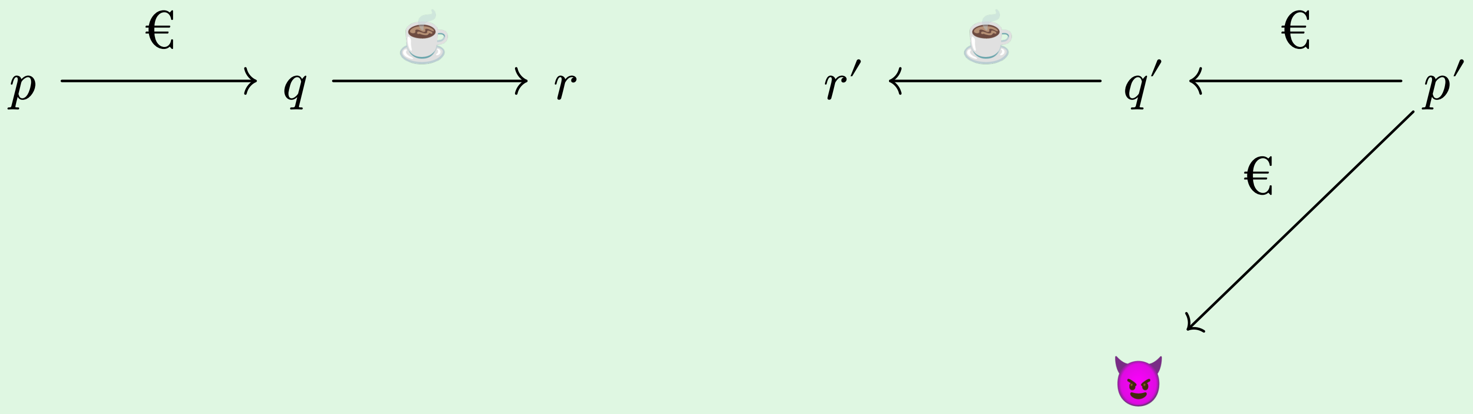
For $\leftrightarrow \neq \equiv_{\mathsf{tr}}$, reconsider p and q below, having $p \equiv_{\mathsf{tr}} q$ but $p \not\leftrightarrow q$.



$\blacksquare$

Trace Equivalence: End of Story?

Example.



$$\mathsf{traces}(p) = \{\varepsilon, \text{€}, \text{€☕}\} = \{\varepsilon, \text{€}, \text{€}, \text{€☕}\} = \mathsf{traces}(p')$$

There is one trace, namely €, that is a completed trace of p' but not of p.

In other words, trace equivalence (i.e., $\equiv_{\mathsf{tr}}$) is **not** sensitive to deadlocks.

The Completed Trace Semantics

Definition 14 A process $p \in \mathsf{Pr}$ is a *deadlock* if $p \overset{a}{\not\longrightarrow}$ for all $a \in \mathsf{Act}$.

The set of *completed traces* of a process $p \in \mathsf{Pr}$, denoted by $\text{traces}_c(p)$ is the set of all traces $\sigma \in \mathsf{ctraces}(p)$ such that $p \overset{\sigma}{\longrightarrow} q$ and q is a deadlock.

Processes $p, q \in \mathsf{Pr}$ are *completed trace equivalent*, denoted by $p \equiv_{\mathsf{ctr}} q$, if $p \equiv_{\mathsf{tr}} q$ and $\mathsf{ctraces}(p) = \mathsf{ctraces}(q)$.

Theorem 15

$$\leftrightarrow \quad \overset{(1)}{\subsetneq} \quad \equiv_{\mathsf{ctr}} \quad \overset{(2)}{\subsetneq} \quad \equiv_{\mathsf{tr}}$$

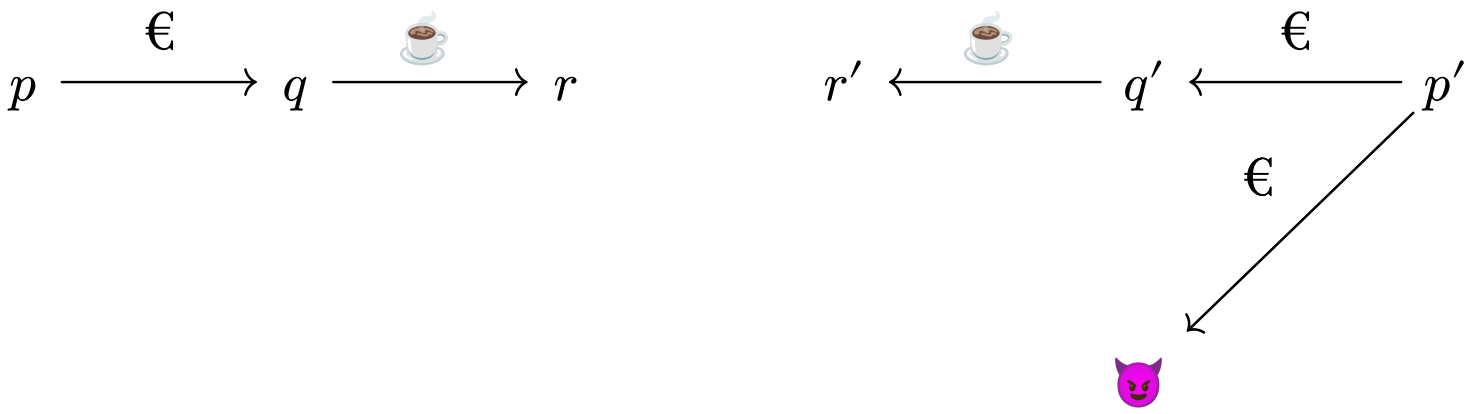
Proof of Theorem 15

Theorem 15

$$\leftrightarrow \quad \overset{(1)}{\subsetneq} \quad \equiv_{\mathsf{ctr}} \quad \overset{(2)}{\subsetneq} \quad \equiv_{\mathsf{tr}}$$

Regarding (2),

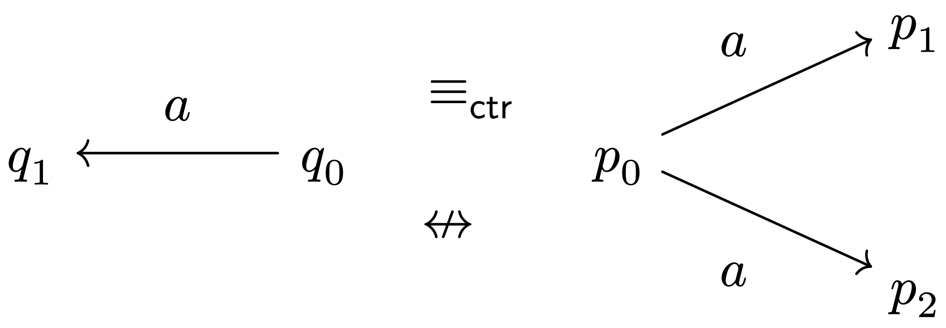
- observe that trace equivalence is part of the definition of $\equiv_{\mathsf{ctr}}$;
- in fact, $\mathsf{ctraces}(p) \subseteq \mathsf{traces}(p)$ for all processes $p \in \mathsf{Pr}$;
- furthermore, 😈 serves as a counterexample, proving $\equiv_{\mathsf{ctr}} \neq \equiv_{\mathsf{tr}}$.



Proof of Theorem 15

Towards (1),

- observe that a deadlock process $p \in \mathsf{Pr}$ can only be isomorphic to other deadlock processes;
- in fact, $p \leftrightarrow q$ for all processes $p, q \in \mathsf{Pr}$ that are deadlocks;
- hence, any completed trace of $p \in \mathsf{Pr}$ must be a a completed trace of $f(p)$ (by the same arguments as in proof of Theorem 13);
- also, $\leftrightarrow \neq \equiv_{\mathsf{ctr}}$ (e.g., p_0 and q_0 below).



∎

Completed Traces: End of Story?

Definition 14 A process $p \in \mathsf{Pr}$ is a *deadlock* if $p \stackrel{a}{\not\longrightarrow}$ for all $a \in \mathsf{Act}$.

The set of *completed traces* of a process $p \in \mathsf{Pr}$, denoted by $\text{traces}_c(p)$ is the set of all traces $\sigma \in \mathsf{ctraces}(p)$ such that $p \xrightarrow{\sigma} q$ and q is a deadlock.

Processes $p, q \in \mathsf{Pr}$ are *completed trace equivalent*, denoted by $p \equiv_{\mathsf{ctr}} q$, if $p \equiv_{\mathsf{tr}} q$ and $\mathsf{ctraces}(p) = \mathsf{ctraces}(q)$.

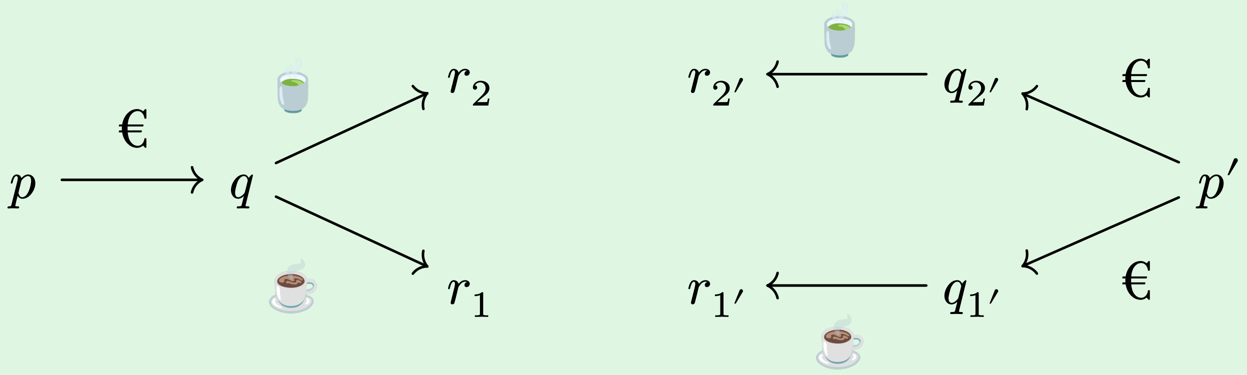
Theorem 15

$$\leftrightarrow \quad \overset{(1)}{\subsetneq} \quad \equiv_{\mathsf{ctr}} \quad \overset{(2)}{\subsetneq} \quad \equiv_{\mathsf{tr}}$$

$\equiv_{\mathsf{ctr}}$ preserves traces (2) and deadlocks (😈)

Completed Traces are Insensitive to Nondeterminism

Example.



What more do we need?

1. We are looking for the intimate connection between nondeterminism and interaction.
2. We are aiming at equivalences going beyond *linear-time* ($\equiv_{\mathsf{tr}}$ and $\equiv_{\mathsf{ctr}}$ are linear-time).

Recall

Definition 11 Any binary relation $\mathcal{R} \subseteq \mathsf{Pr} \times \mathsf{Pr}$ is called a *process relation*. $\mathcal{R}$ is a *process equivalence* if it is a process relation and an equivalence.

Theorem 15

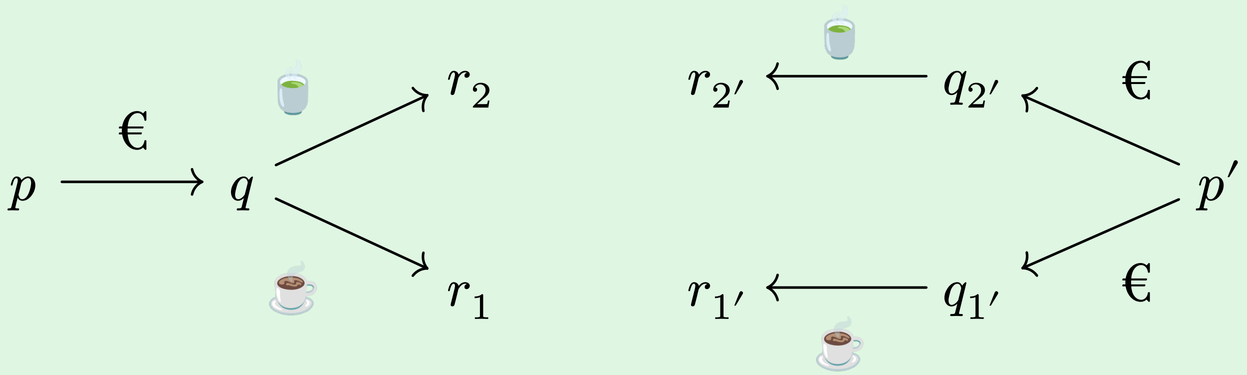
$$\leftrightarrow \overset{(1)}{\subsetneq} \equiv_{\mathsf{ctr}} \overset{(2)}{\subsetneq} \equiv_{\mathsf{tr}}$$

If, between two process equivalences $\mathcal{R}_1$ and $\mathcal{R}_2$, it holds that $\mathcal{R}_1 \subseteq \mathcal{R}_2$, we say that $\mathcal{R}_1$ *is finer than* $\mathcal{R}_2$, and $\mathcal{R}_2$ *is coarser than* $\mathcal{R}_1$.

The coarsest process equivalence of all is $\mathcal{U} \subseteq \mathsf{Pr} \times \mathsf{Pr}$.

Towards More Meaningful Equivalences

Example.



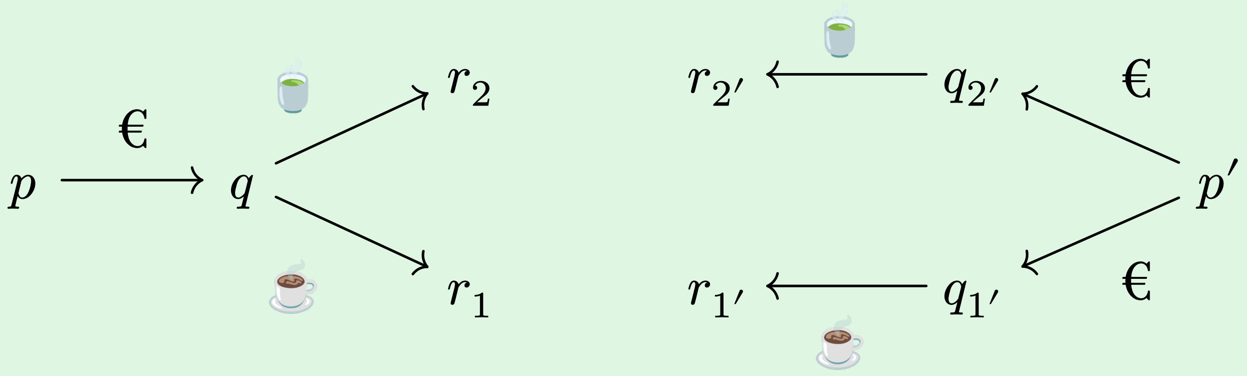
Maybe induction helps?

Suppose, $p \equiv p'$ ($\leftarrow$ **claim**);

1. since $p \xrightarrow{€} q$, p' needs to have a *similar* step
2. $p' \xrightarrow{€} q_{1'}$ and $p' \xrightarrow{€} q_{2'}$
3. thus, the **claim** holds if $q \equiv q_{1'}$ and/or(?) $q \equiv q_{2'}$
4. but as $q \xrightarrow{☕}$ and $q_{2'} \not\xrightarrow{☕}$, $q \not\equiv q_{2'}$; similarly, $q \xrightarrow{🍵}$ but $q_{1'} \not\xrightarrow{🍵}$, $q \not\equiv q_{1'}$

Induction Seems to Work

Example.



$p \not\equiv p'$ because $q \not\equiv q_{1'}$ and $q \not\equiv q_{2'}$.

Cooking up Equivalence $\equiv$

$p \equiv q$ if, for all $a \in \mathsf{Act}$,

1. for all p' with $p \xrightarrow{a} p'$, there is a q' with $q \xrightarrow{a} q'$ and $p' \equiv q'$;
2. for all q' with $q \xrightarrow{a} q'$, there is a p' with $p \xrightarrow{a} p'$ and $p' \equiv q'$.

Induction Seems to Work

$p \equiv q$ if, for all $a \in \mathsf{Act}$,

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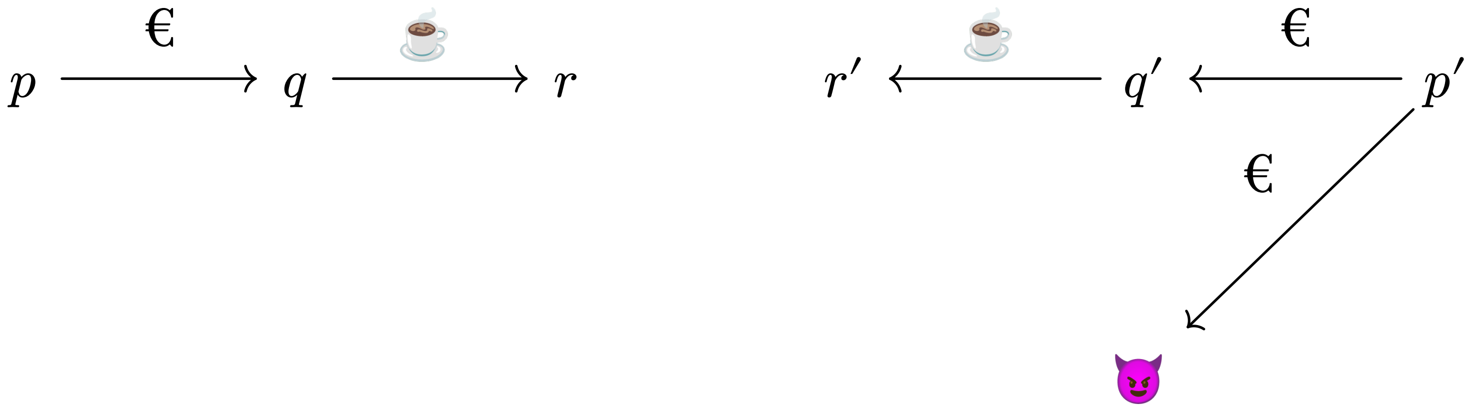


$p \not\equiv p'$ because $q \not\equiv$ 😈

Note, $r \equiv r' \equiv$ 😈

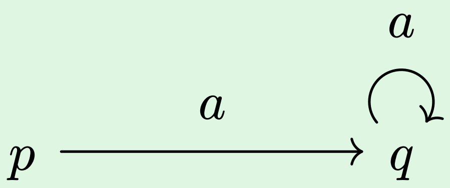
All deadlock processes are equivalent under $\equiv$.

Where Does Induction Fail?

$p \equiv q$ if, for all $a \in \mathsf{Act}$,

1. for all p' with $p \xrightarrow{a} p'$, there is a q' with $q \xrightarrow{a} q'$ and $p' \equiv q'$;
2. for all q' with $q \xrightarrow{a} q'$, there is a p' with $p \xrightarrow{a} p'$ and $p' \equiv q'$.

Example. *Reconsider processes p and q and find that $p \equiv_{\mathsf{tr}} q$*



To prove that $p \equiv q$, we have to show that $q \equiv q$ because

1. *$p \xrightarrow{a} q$ and there is a q' such that $q \xrightarrow{a} q'$, namely $q' = q$, for which $q \equiv q' = q$, and*
2. *$q \xrightarrow{a} q$ and there is a p' such that $p \xrightarrow{a} q'$, namely $q' = q$, … $q \equiv q' = q$.*

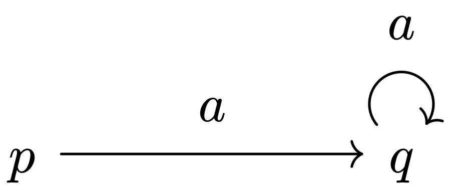
To prove that $q \equiv q$, we have to show that $q \equiv q$ … To prove that $q \equiv q$, we have to show that $q \equiv q$ … To prove that $q \equiv q$, we have to show that $q \equiv q$ … … ∎

Why Does Induction Fail?

$p \equiv q$ if, for all $a \in \mathsf{Act}$,

1. for all p' with $p \xrightarrow{a} p'$, there is a q' with $q \xrightarrow{a} q'$ and $p' \equiv q'$;
2. for all q' with $q \xrightarrow{a} q'$, there is a p' with $p \xrightarrow{a} p'$ and $p' \equiv q'$.

- Induction requires a **base case** start with **nothing**: $\mathcal{R}_0 = \{\}$
- By definition, in order to know that $p \equiv q$, we have to already know that $p' \equiv q'$
- In the example, to know/prove that $p \equiv q$, we have to already know that $q \equiv q$



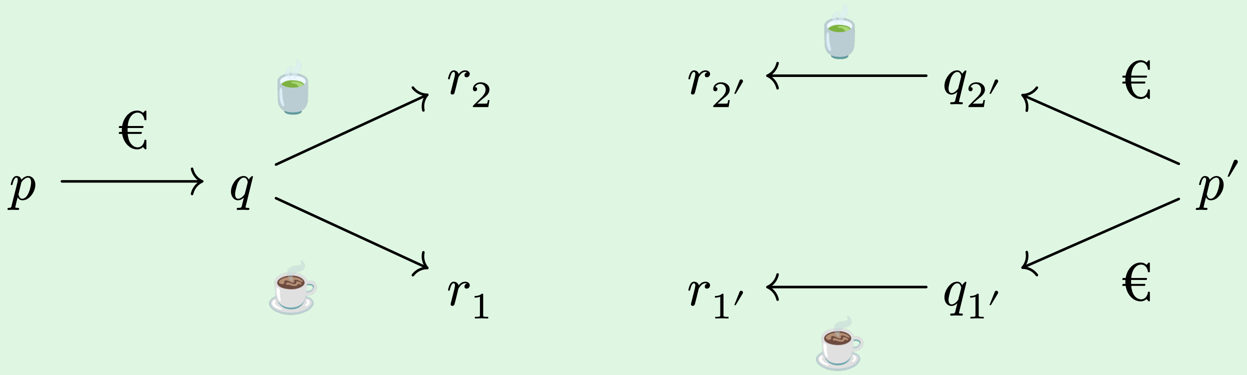
What went wrong?

What went well?

$p \equiv q$ if, for all $a \in \mathsf{Act}$,

1. for all p' with $p \xrightarrow{a} p'$, there is a q' with $q \xrightarrow{a} q'$ and $p' \equiv q'$;
2. for all q' with $q \xrightarrow{a} q'$, there is a p' with $p \xrightarrow{a} p'$ and $p' \equiv q'$.

Example.



An Inductive Approach to Process Equivalence in Reverse

> **Note**
>
> The coarsest process equivalence of all is $\mathcal{U} \subseteq \mathsf{Pr} \times \mathsf{Pr}$.

Compute $\simeq_0, \simeq_1, \ldots$ and define $\simeq_\omega := \bigcap_{i \geq 0} \simeq_i$

1. set $\simeq_0 = \mathcal{U}$
2. $p \simeq_{n+1} q$ for $n \geq 0$ if for all $a \in \mathsf{Act}$:
   a. for all p' with $p \xrightarrow{a} p'$, there is a q' with $q \xrightarrow{a} q'$ and $p' \simeq_n q'$;
   b. for all q' with $q \xrightarrow{a} q'$, there is a p' with $p \xrightarrow{a} p'$ and $p' \simeq_n q'$.

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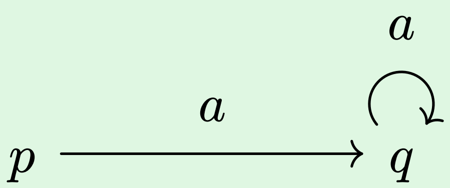
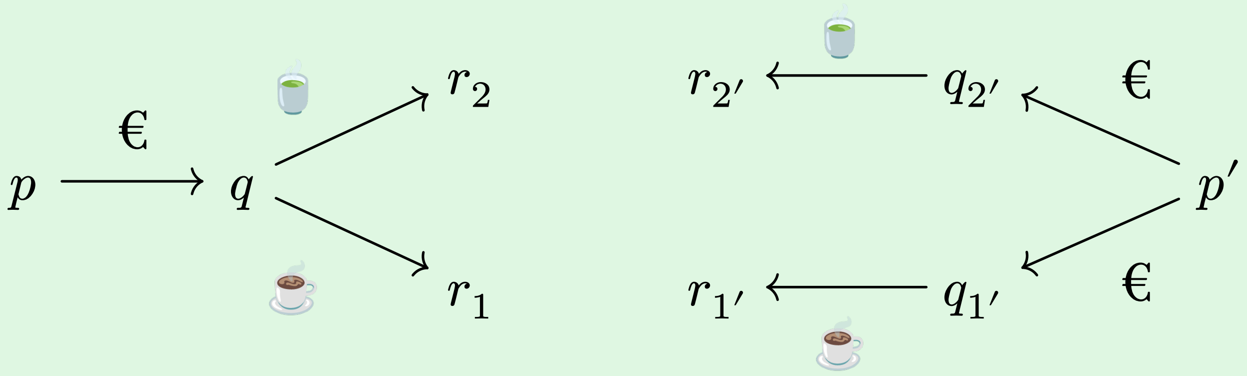
Example.



$$\simeq_0 = \{(p,p),(p,q),(q,p),(q,q)\}$$

$$\simeq_1 = \{(p,p),(p,q),(q,p),(q,q)\} = \simeq_0 = \simeq_\omega$$

Example.



$$\simeq_0 = \{(p,p), \cancel{(p,q)}, \cancel{(p,r_1)}, \cancel{(p,r_2)}, \ldots\}$$

$$\simeq_1 = \{(p,p), (p,p'), \ldots, \cancel{(q,q_{2'})}, \cancel{(q,q_{1'})}, \ldots, (r_1,r_{1'}), (r_1,r_{2'}), \ldots\}$$

$$\simeq_2 = \{(p,p), \cancel{(p,p')}, \cancel{(p',p)}, (p',p'), (q,q), (q_{1'},q_{1'}), (q_{2'},q_{2'}), \ldots\}$$

$$\simeq_3 = \{(p,p), (p',p'), (q,q), (q_{1'},q_{1'}), (q_{2'},q_{2'}), \ldots\} = \simeq_\omega$$

$$p \not\simeq_\omega p'$$

Rebooting Process Equivalence

A process relation $\mathcal{R} \subseteq \mathsf{Pr} \times \mathsf{Pr}$ is called a *(strong) bisimulation* if, for all $p, q \in \mathsf{Pr}$, $p \mathrel{\mathcal{R}} q$ implies

1. for all p' with $p \xrightarrow{a} p'$, there is a q' with $q \xrightarrow{a} q'$ and $p' \mathrel{\mathcal{R}} q'$, and
2. for all q' with $q \xrightarrow{a} q'$, there is a p' with $p \xrightarrow{a} p'$ and $p' \mathrel{\mathcal{R}} q'$

for all $a \in \mathsf{Act}$. We call p and q *bisimilar*, denoted $p \simeq q$, if there is a bisimulation $\mathcal{R}$ such that $p \mathrel{\mathcal{R}} q$.

$\simeq$ is called *the bisimilarity*.

Rebooting Process Equivalence

Definition 16 (Bisimulation, Bisimilarity) A process relation $\mathcal{R} \subseteq \mathsf{Pr} \times \mathsf{Pr}$ is called a *(strong) bisimulation* if, for all $p, q \in \mathsf{Pr}$, $p \mathrel{\mathcal{R}} q$ implies

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Rebooting Process Equivalence

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Consequences

1. bisimilarity $\simeq$ is the union of all bisimulations
2. showing that $p \simeq q$ holds reduces to finding a bisimulation $\mathcal{R}$ such that $p \mathrel{\mathcal{R}} q$
3. conversely, $p \not\simeq q$ can be shown by excluding the existence of any such bisimulation $\mathcal{R}$

Bisimilarity – Two Examples

Definition 16 (Bisimulation, Bisimilarity) A process relation $\mathcal{R} \subseteq \mathsf{Pr} \times \mathsf{Pr}$ is called a *(strong) bisimulation* if, for all $p, q \in \mathsf{Pr}$, $p \mathrel{\mathcal{R}} q$ implies

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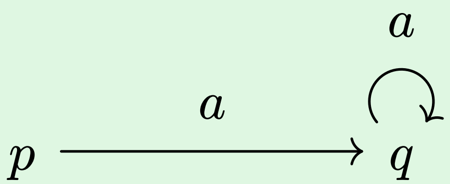
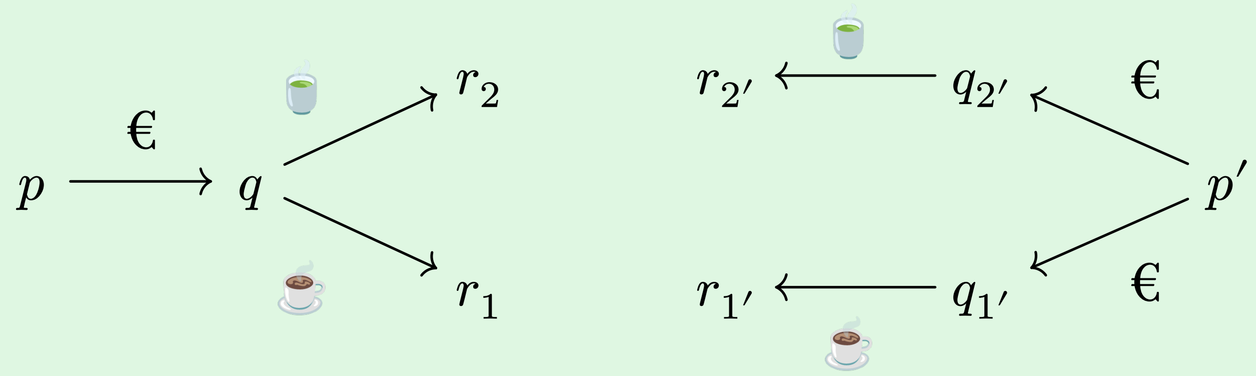
Example.



$p \simeq q$ by $\mathcal{R} = \{(p,q),(q,q)\}$, but $\mathcal{R}' = \{(p,q),(q,p)\}$ is not a bisimulation. ∎

Bisimilarity – Two Examples

Example.



Towards a contradiction, suppose $p \simeq p'$. *Then there is a bisimulation* $\mathcal{R}$ *with* $p \mathrel{\mathcal{R}} p'$. *As* $\mathcal{R}$ *is a bisimulation,* $q \mathrel{\mathcal{R}} q_{1'}$ *since* $p' \xrightarrow{€} q_{1'}$ *and* $p \xrightarrow{€} q$. *But* $q \mathrel{\mathcal{R}} q_{1'}$ *cannot hold since* $q \xrightarrow{🍵} r_2$ *whereas* $q_{1'} \not\xrightarrow{🍵}$. ∎

Disecting Bisimilarity

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Proofs of bisimilarity are

- *local* — checks performed on states separately
- *non-hierarchical* — no fixed temporal order
- require no **base case** — this is **not** *induction*

It is, in fact, an example of **coinduction**

(We had already seen what happens if we read Definition 16 inductively.)

Disecting Bisimilarity

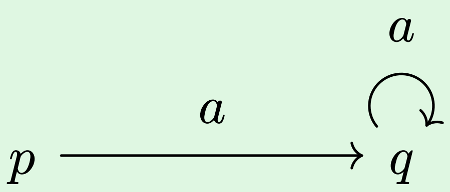
Theorem 17 $\simeq$ is a process equivalence that is itself a bisimulation.

Proof: We have to show that $\simeq$ is (1) an equivalence and (2) a bisimulation.

to be continued... ∎

Not every bisimulation is an equivalence:

Example.



$p \simeq q$ by $\mathcal{R} = \{(p, q), (q, q)\}$ which is ***neither*** *reflexive* ***nor*** *symmetric.*

Disecting Bisimilarity

Theorem 17 $\simeq$ is a process equivalence that is itself a bisimulation.

Proof: We have to show that $\simeq$ is (1) an equivalence and (2) a bisimulation.

Reflexivity id : $\mathsf{Pr} \to \mathsf{Pr}$ is, in fact, a bisimulation. For p id q (i.e., $\mathrm{id}(p) = q$), we get $p \xrightarrow{a} p'$ iff $q = \mathrm{id}(p) = p \xrightarrow{a} p' = \mathrm{id}(p') = q'$. The same holds for steps from $\mathrm{id}(p)$.

Symmetry If $\mathcal{R}$ is a bisimulation, then $\mathcal{R}^{-1} := \{(q, p) \mid p \; \mathcal{R} \; q\}$ is a bisimulation.

Transitivity Let $\mathcal{R}_1, \mathcal{R}_2$ be bisimulations. We subsequently show that $\mathcal{R}_1 \circ \mathcal{R}_2 := \{(x, z) \mid \exists y. x \; \mathcal{R}_1 \; y \wedge y \; \mathcal{R}_2 \; z\}$ is a bisimulation. For $p \; \mathcal{R}_1 \circ \mathcal{R}_2 \; q$ and $p \xrightarrow{a} p'$,

1. there is an r such that $x \; \mathcal{R}_1 \; r$ and $r \; \mathcal{R}_2 \; q$; — by definition of $\mathcal{R}_1 \circ \mathcal{R}_2$
2. there is an r' such that $r \xrightarrow{a} r'$ and $p' \; \mathcal{R}_1 \; r'$ — since $\mathcal{R}_1$ is a bisimulation
3. there is a q' such that $q \xrightarrow{a} q'$ and $r' \; \mathcal{R}_2 \; q'$ — since $\mathcal{R}_2$ is a bisimulation
4. hence, by taking that q', we get $p' \; \mathcal{R}_1 \circ \mathcal{R}_2 \; q'$ — by definition of $\mathcal{R}_1 \circ \mathcal{R}_2$

Since bisimulations are union-closed (by Lemma 18, cf. next slide) and $\simeq$ is the union of all bisimulations, $\simeq$ is itself a bisimulation. ∎

Disecting Bisimilarity

Lemma 18 Bisimulations are closed under set unions: If $\{\mathcal{R}_i\}_i$ is a (at most countable) family of bisimulations, then $\bigcup_i \mathcal{R}_i$ is a bisimulation.

Disecting Bisimilarity

Lemma 18 Bisimulations are closed under set unions: If $\{\mathcal{R}_i\}_i$ is a (at most countable) family of bisimulations, then $\bigcup_i \mathcal{R}_i$ is a bisimulation.

Towards a special case, take two bisimulations $\mathcal{R}_1$ and $\mathcal{R}_2$ and consider $\mathcal{R}_1 \cup \mathcal{R}_2$:

Take $p\ \mathcal{R}_1 \cup \mathcal{R}_2\ q$ and consider $p \xrightarrow{a} p'$.

1. if $p\ \mathcal{R}_1\ q$, then there is a q' such that $q \xrightarrow{a} q'$ and $p'\ \mathcal{R}_1\ q'$ — $\mathcal{R}_1$ is a bisimulation
2. if $p\ \mathcal{R}_2\ q$, then there is a q' such that $q \xrightarrow{a} q'$ and $p'\ \mathcal{R}_2\ q'$ — $\mathcal{R}_2$ is a bisimulation

In both cases, there is a q' such that $q \xrightarrow{a} q'$ and $p\ \mathcal{R}_1 \cup \mathcal{R}_2\ q$. Same for $q \xrightarrow{a} q'$.

Proof: If each $\mathcal{R}_i$ is a bisimulation, then $\mathcal{R} = \bigcup_i \mathcal{R}_i$ is a bisimulation. For each pair $p\ \mathcal{R}\ q$, there is a $\mathcal{R}_i$ such that $p\ \mathcal{R}_i\ q$.

1. if $p \xrightarrow{a} p'$, there is a q' such that $q \xrightarrow{a} q'$ and $p'\ \mathcal{R}_i\ q'$ — $\mathcal{R}_i$ is a bisimulation
2. if $q \xrightarrow{a} q'$, there is a p' such that $p \xrightarrow{a} p'$ and $p'\ \mathcal{R}_i\ q'$ — $\mathcal{R}_i$ is a bisimulation

In each case $p'\ \mathcal{R}_i\ q'$ and, thus, $p'\ \mathcal{R}\ q'$. ∎

Yet Another Characterization of $\simeq$

Theorem 19 $\simeq$ is the largest bisimulation, i.e., the largest process relation $\simeq$ such that $p \simeq q$ implies for all $a \in \mathsf{Act}$:

1. for all p' with $p \xrightarrow{a} p'$, there is a q' with $q \xrightarrow{a} q'$ and $p' \mathcal{R} q'$, and
2. for all q' with $q \xrightarrow{a} q'$, there is a p' with $p \xrightarrow{a} p'$ and $p' \mathcal{R} q'$.

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2. for all q' with $q \xrightarrow{a} q'$, there is a p' with $p \xrightarrow{a} p'$ and $p' \mathcal{R} q'$.

Proof: By Theorem 17, $\simeq$ is a bisimulation. It remains to be shown that it is the largest one.

Consider two largest bisimulations $\underset{1}{\simeq}$ and $\underset{2}{\simeq}$. Since bisimulations are union-closed (by Lemma 18), $\underset{1}{\simeq} \cup \underset{2}{\simeq}$ is a bisimulation as well, implying that $\underset{1}{\simeq} = \underset{1}{\simeq} \cup \underset{2}{\simeq}$ and $\underset{2}{\simeq} = \underset{1}{\simeq} \cup \underset{2}{\simeq}$ to not contradict the assumption that $\underset{1}{\simeq}$ and $\underset{2}{\simeq}$ were chosen to be largest. Thus, $\simeq$ is the *unique* largest bisimulation. ∎

Bisimilarity is an Example for Branching-Time

Theorem 20

$$\leftrightarrow \overset{(1)}{\subsetneq} \simeq \overset{(2)}{\subsetneq} \equiv_{\mathsf{ctr}} \subsetneq \equiv_{\mathsf{tr}}$$

Proof:

(1) Let $f : \mathsf{Pr} \to \mathsf{Pr}$ be an isomorphism. We show, f is a bisimulation.
For $p\ f\ q$ (i.e., $f(p) = q$),

$$p \xrightarrow{a} p' \text{ iff } f(p) \xrightarrow{a} f(p') \quad \text{since } f \text{ is an isomorphism}$$

$$\text{iff } \exists q'.q \xrightarrow{a} q' \quad \text{by } f(p) = q \text{ take } q' = f(p')$$

We have $p'\ f\ q'$ since $f(p') = q'$. The second direction is analogous.

Towards $\leftrightarrow \neq \simeq$, $\simeq$ is insensitive to branch duplicates.

∎

Bisimilarity is an Example for Branching-Time

Theorem 20

$$\leftrightarrow \overset{(1)}{\subsetneq} \simeq \overset{(2)}{\subsetneq} \equiv_{\mathsf{ctr}} \subsetneq \equiv_{\mathsf{tr}}$$

Proof:

(2) Let $p, q \in \mathsf{Pr}$ such that $p \simeq q$. We need to show that $p \equiv_{\mathsf{ctr}} q$, meaning $\mathsf{ctraces}(p) = \mathsf{ctraces}(q)$. It is sufficient to show that $\mathsf{ctraces}(p) \subseteq \mathsf{ctraces}(q)$ since the other direction follows by symmetry (process equivalences are symmetric).

Let $\sigma \in \mathsf{ctraces}(p)$ with $\sigma = a_1 a_2 \ldots a_n$. Then there are states $p_1, p_2, \ldots, p_n$ such that $p \xrightarrow{a_1} p_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} p_n$ and p_n is a deadlock.

Since $p \simeq q$, there are $q_1, q_2, \ldots, q_n$ such that $q \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} q_n$ such that $p_i \simeq q_i$ $(i = 1, \ldots, n)$. In particular, q_n is a deadlock. Thus, $a_1 a_2 \ldots a_n = \sigma \in \mathsf{ctraces}(q)$.

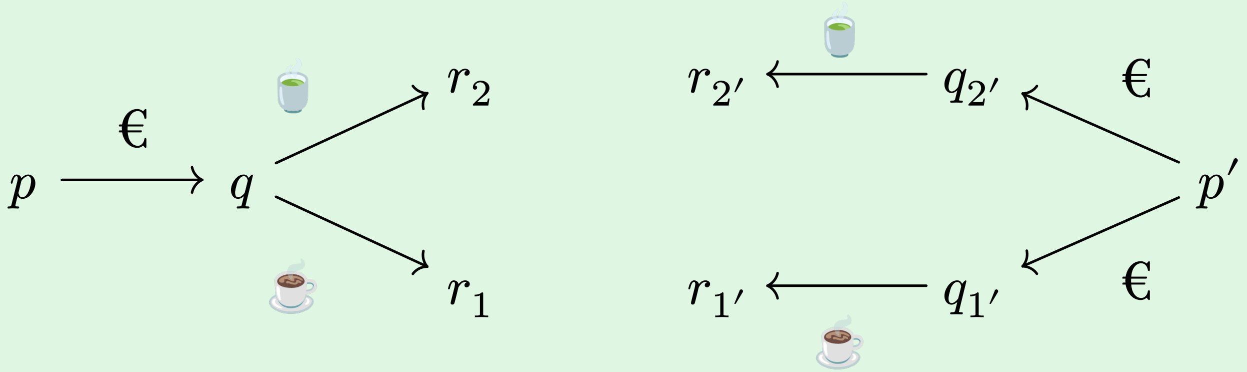
∎

Counterexample for $\simeq = \equiv_{\mathsf{ctr}}$

Theorem 20

$$\leftrightarrow \overset{(1)}{\subsetneq} \simeq \overset{(2)}{\subsetneq} \equiv_{\mathsf{ctr}} \subsetneq \equiv_{\mathsf{tr}}$$

Example.



$$p \not\simeq p' \text{ but } p \equiv_{\mathsf{ctr}} p'$$

What about $\simeq_\omega$?

Do the two views on process equivalence, $\simeq$ and $\simeq_\omega$, coincide?

Definition 16 (Bisimulation, Bisimilarity) A process relation $\mathcal{R} \subseteq \mathsf{Pr} \times \mathsf{Pr}$ is called a *(strong) bisimulation* if, for all $p, q \in \mathsf{Pr}$, $p \,\mathcal{R}\, q$ implies

1. for all p' with $p \xrightarrow{a} p'$, there is a q' with $q \xrightarrow{a} q'$ and $p' \,\mathcal{R}\, q'$, and
2. for all q' with $q \xrightarrow{a} q'$, there is a p' with $p \xrightarrow{a} p'$ and $p' \,\mathcal{R}\, q'$

for all $a \in \mathsf{Act}$. We call p and q *bisimilar*, denoted $p \simeq q$, if there is a bisimulation $\mathcal{R}$ such that $p \,\mathcal{R}\, q$.

$\simeq$ is called *bisimilarity*.

$\simeq_\omega := \bigcap_{i \geq 0} \simeq_i$

1. set $\simeq_0 = \mathcal{U}$
2. $p \simeq_{n+1} q$ for $n \geq 0$ if for all $a \in \mathsf{Act}$:
   a. for all p' with $p \xrightarrow{a} p'$, there is a q' with $q \xrightarrow{a} q'$ and $p' \simeq_n q'$;
   b. for all q' with $q \xrightarrow{a} q'$, there is a p' with $p \xrightarrow{a} p'$ and $p' \simeq_n q'$.

What about $\simeq_\omega$?

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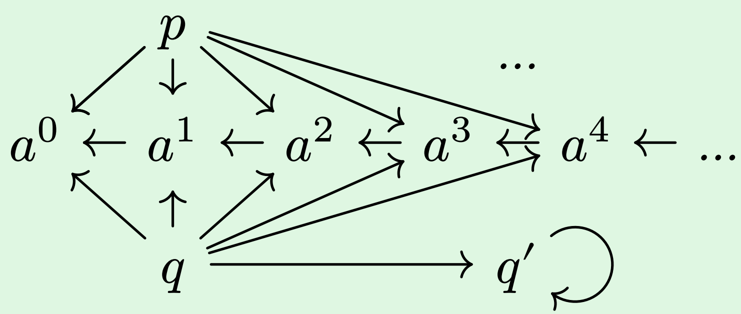
Example.



Claim: *For each $n \in \mathbb{N}$, we get $p \simeq_n q$.*

1. *$n = 0$, $p \simeq_n q$ since $\simeq_0 = \mathsf{Pr} \times \mathsf{Pr}$ is the universal process equivalence.*

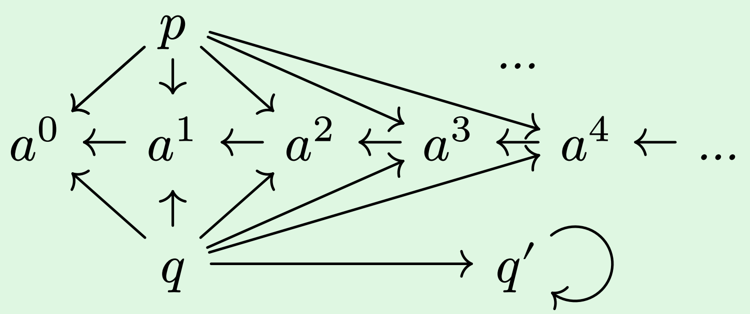
What about $\simeq_\omega$?

2. $n \to n+1$,
   - *if* $q \longrightarrow q'$, p *answers by* $p \longrightarrow a^n$; *for which we get* $a^n \simeq_n q'$ *by another induction on* n.
   - *if* $q \longrightarrow a^k$, *answer by* $p \longrightarrow a^k$, *and vice versa. Exploit reflexivity of* $\simeq_n$.

Claim: For each $n \in \mathbb{N}$, $a^n \simeq_n q'$

1. $n = 0$, ✓
2. $n \to n+1$, a^{n+1} *still has* $n+1$ *steps to go until it deadlocks* a^0.

Another Fact: *For each* $m, n \in \mathbb{N}$, $a^m \simeq_n q'$ *if* $m \geq n$.



What about $\simeq_\omega$?

Definition 16 (Bisimulation, Bisimilarity) A process relation $\mathcal{R} \subseteq \mathsf{Pr} \times \mathsf{Pr}$ is called a *(strong) bisimulation* if, for all $p, q \in \mathsf{Pr}$, $p \mathrel{\mathcal{R}} q$ implies

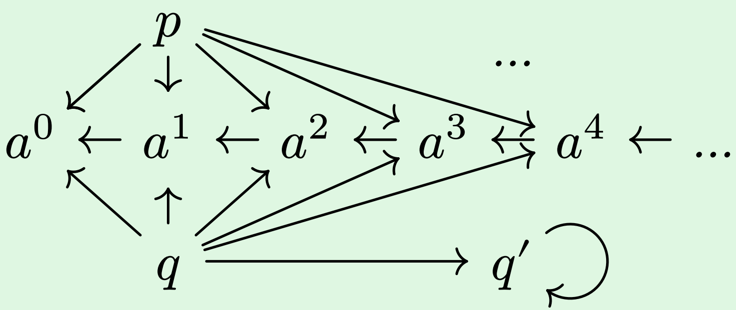
1. for all p' with $p \xrightarrow{a} p'$, there is a q' with $q \xrightarrow{a} q'$ and $p' \mathrel{\mathcal{R}} q'$, and
2. for all q' with $q \xrightarrow{a} q'$, there is a p' with $p \xrightarrow{a} p'$ and $p' \mathrel{\mathcal{R}} q'$

for all $a \in \mathsf{Act}$. We call p and q *bisimilar*, denoted $p \simeq q$, if there is a bisimulation $\mathcal{R}$ such that $p \mathrel{\mathcal{R}} q$. $\simeq$ is called *bisimilarity*.

Does $p \simeq q$ hold in the previous example?

$p \simeq q$?

Example.



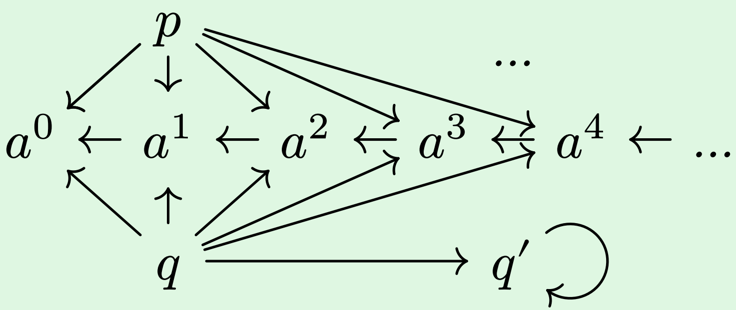
Assume, there is a bisimulation $\mathcal{R}$ such that $p \mathrel{\mathcal{R}} q$. Then for $q \longrightarrow q'$, there is some $m \in \mathbb{N}$, so that $p \longrightarrow a^m$ and $a^m \mathrel{\mathcal{R}} q'$.

Claim: *For all $n \in \mathbb{N}$, $a^m \not\simeq q'$.*

1. $n = 0$, $a^n \not\longrightarrow$ *whereas* $q' \longrightarrow q'$.
2. $n \to n+1$, $a^{n+1} \longrightarrow a^n$. *Thus,* $a^{n+1} \simeq q'$ *if and only if* $a^n \simeq q'$. *By induction hypothesis,* $a^n \not\simeq q'$. *In conclusion,* $q^{n+1} \not\simeq q'$.

What is Wrong with $\simeq_\omega$?

Example.



1. p is
   - acyclic,
   - infinite-state,
   - infinitely branching, and
   - **not** even image-finite
2. q is cyclic, ..., and **not** even image-finite

What is Wrong with $\simeq_\omega$?

Theorem 21 $\simeq$ and $\simeq_\omega$ coincide on *image-finite* LTSs.

Proof: We prove both directions separately. Consider all processes and, in fact, the underlying LTS to be *image-finite*.

$\simeq \subseteq \simeq_\omega$ For each $n \in \mathbb{N}$, we show that $p \simeq q$ implies $p \simeq_n q$.

$n = 0$ Since $\simeq_n = \simeq_0 = \mathsf{Pr} \times \mathsf{Pr}$, $p \simeq_n q$ holds trivially.

Hypothesis For $n \in \mathbb{N}$, $p \simeq q$ implies $p \simeq_n q$.

$n \to n + 1$ If $p \simeq q$ holds, we show that $p \simeq_{n+1} q$. For each $a \in \mathsf{Act}$

1. if $p \xrightarrow{a} p'$, there is a q' with $q \xrightarrow{a} q'$ and $p' \simeq q'$. By induction hypothesis, $p' \simeq q'$ implies $p' \simeq_n q'$.
2. if $q \xrightarrow{a} q'$, there is a p' with $p \xrightarrow{a} p'$ and $p' \simeq q'$. By induction hypothesis, $p' \simeq q'$ implies $p' \simeq_n q'$.

Thus, every step of p (q, resp.) can be answered such that their successors are related by $\simeq_n$, proving that $p \simeq_{n+1} q$ holds.

What is Wrong with $\simeq_\omega$?

$\simeq_\omega \subseteq \simeq$ We show that $\mathcal{R} = \{(p,q) \mid p \simeq_\omega q\}$ is a bisimulation. Consider a pair $(p,q) \in \mathcal{R}$.

- Suppose, $p \xrightarrow{a} p'$.
- For all $n \in \mathbb{N}$,
  as $p \simeq_{n+1} q$, there is some q_n such that $q \xrightarrow{a} q_n$ and $p' \simeq_n q_n$;
- Since q is image-finite, the set $Q = \left\{ q' \,\middle|\, q \xrightarrow{a} q' \right\}$ is finite;
  thus, there must be one $q' \in Q$ such that $p' \simeq_n q'$ for each $n \in \mathbb{N} \Rightarrow p' \simeq_\omega q'$

∎

Outline

1. Algebraic Properties of Bisimilarity
2. Algorithmics of Bisimilarity from Different Points of View
   a. Bisimilarity is Decidable 😀
   b. Bisimilarity is P-complete 😁
   c. Bisimilarity is Undecidable 😲
3. Everything you always wanted to know about **Petri nets**
   a. Decidability Results
   b. Complexity Results: Immerman–Szelepcsényi on Steroids
4. If time allows
   a. Mobile processes: the π-calculus
   b. Relative expressive power
   c. Foundations of data-aware processes