Complexity Theory

Turing Machines and Languages

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11 Oct 2017

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Deterministic Turing Machines

A Model for Computation

Clear

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Answer

With Turing machines.

Turing Machines

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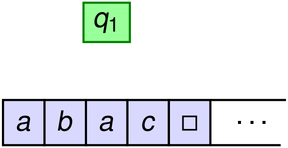
Definition 2.2

A (deterministic) **Turing Machine** $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$ consists of

- a finite set Q of **states**,
- an **input alphabet** Σ not containing $\square$,
- a **tape alphabet** Γ such that $\Gamma \supseteq \Sigma \cup \{\square\}$.
- a **transition function** $\delta\colon Q \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}\}$
- an **initial state** $q_0 \in Q$,
- an **accepting state** $q_{\text{accept}} \in Q$, and
- an **rejecting state** $q_{\text{reject}} \in Q$ such that $q_{\text{accept}} \neq q_{\text{reject}}$.

Turing Machines

Example 2.3



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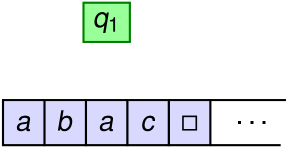

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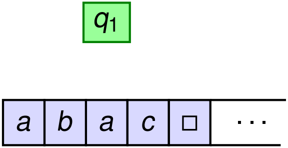

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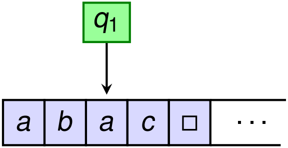

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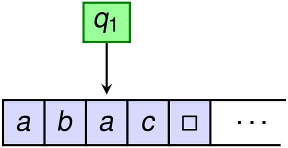

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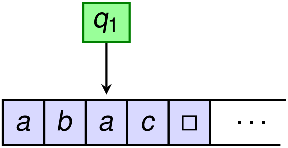

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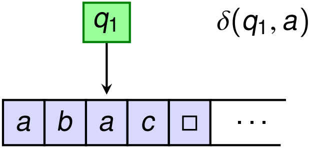

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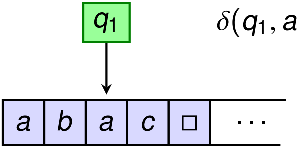

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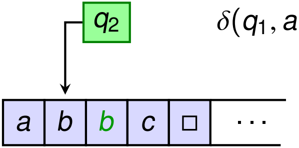

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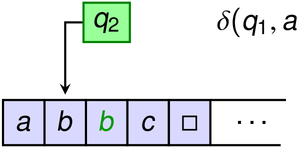

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- The head moves and writes according to the transition function δ; the current state also changes accordingly
- The head will stay put when attempting to cross the left tape end

Configurations

Observation: to describe the current step of a computation of a TM it is enough to know

- the content of the tape,
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Some special configurations:

- The **start configuration** for some input word $w \in \Sigma^*$ is the configuration q_0w
- A configuration uqv is **accepting** if $q = q_{\text{accept}}$.
- A configuration uqv is **rejecting** if $q = q_{\text{reject}}$.

Computation

We write

- $\mathcal{C} \vdash_{\mathcal{M}} \mathcal{C}'$ only if $\mathcal{C}'$ can be reached from $\mathcal{C}$ by one computation step of $\mathcal{M}$;
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We say that $\mathcal{M}$ **halts** on input w if and only if there is a finite sequence of configurations

$$\mathcal{C}_0 \vdash_{\mathcal{M}} \mathcal{C}_1 \vdash_{\mathcal{M}} \cdots \vdash_{\mathcal{M}} \mathcal{C}_\ell$$

such that $\mathcal{C}_0$ is the start configuration of $\mathcal{M}$ on input w and $\mathcal{C}_\ell$ is an accepting or rejecting configuration. Otherwise $\mathcal{M}$ **loops** on input w.

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Recognisability and Decidability

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Definition 2.5

Let $\mathcal{M}$ be a Turing machine with input alphabet Σ. The **language accepted by $\mathcal{M}$** is the set

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A language $\mathcal{L} \subseteq \Sigma^*$ is called **Turing-decidable** (**decidable**, **recursive**) if and only if there exists a Turing machine $\mathcal{M}$ such that $\mathcal{L} = \mathcal{L}(\mathcal{M})$ and $\mathcal{M}$ halts on every input. In this case we say that $\mathcal{M}$ **decides** $\mathcal{L}$.

Example

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The language $\mathcal{L} := \{ \mathrm{a}^{2^n} \mid n \geq 0 \}$ is decidable.

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Proof

A Turing machine $\mathcal{M}$ that decides $\mathcal{L}$ is

$\mathcal{M}$:= On input w, where w is a string

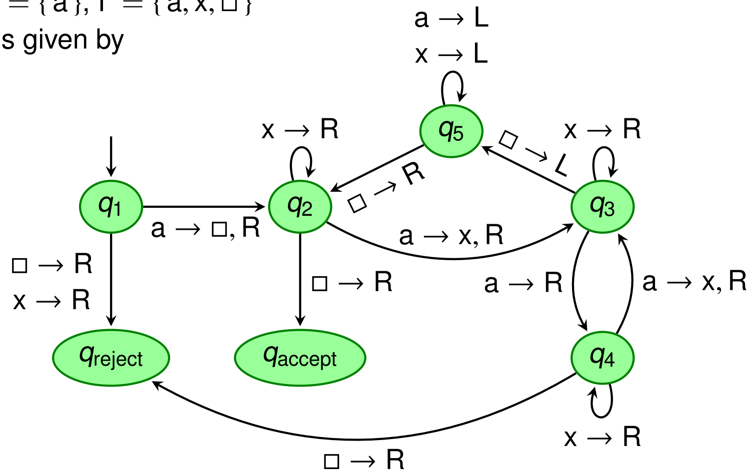
- Go from left to right over the tape and cross off every other 0
- If in the first step the tape contained a single 0, *accept*
- If in the first step the number of 0s on the tape was odd, *reject*
- Return the head the beginning of the tape
- Go to the first step

Example (cont'd)

Formally, $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_1, q_{\text{accept}}, q_{\text{reject}})$, where

- $Q = \{ q_1, q_2, q_3, q_4, q_5, q_{\text{accept}}, q_{\text{reject}} \}$
- $\Sigma = \{ \text{a} \}$, $\Gamma = \{ \text{a}, \text{x}, \square \}$

and δ is given by



Problems as Languages

Observation

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The question whether a graph is connected or not can be seen as the **word problem** of the following language

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Notation

The encoding of objects $\mathcal{O}_1, \ldots, \mathcal{O}_n$ we denote by $\langle \mathcal{O}_1, \ldots, \mathcal{O}_n \rangle$.

The Church-Turing Thesis

It turns out that Turing-machines are **equivalent** to a number of formalisations of the intuitive notion of an **algorithm**

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Because of this it is believed that Turing-machines completely capture the intuitive notion of an algorithm. $\leadsto$ **Church-Turing Thesis**:

> *"A function on the natural numbers is intuitively computable if and only if it can be computed by a Turing machine."*

($\rightarrow$ Wikipedia: Church-Turing Thesis)

Variants of Turing Machines

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It has also been shown that deterministic, single-tape Turing machines are equivalent to a wide range of other forms of Turing machines:

- Multi-tape Turing machines
- Nondeterministic Turing machines
- Turing machines with doubly-infinite tape
- Multi-head Turing machines
- Two-dimensional Turing machines
- Write-once Turing machines
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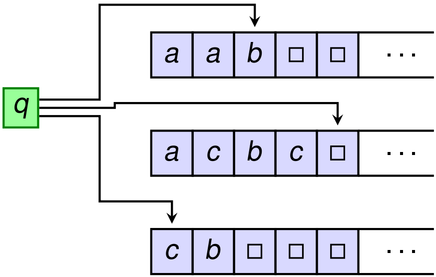

Multi-Tape Turing Machines

Definition 2.7

Let $k \in \mathbb{N}$. Then a (deterministic) ***k*-tape Turing machine** is a tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where

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The notions of a **configuration** and of the **language accepted by M** are defined analogously to the single-tape case.

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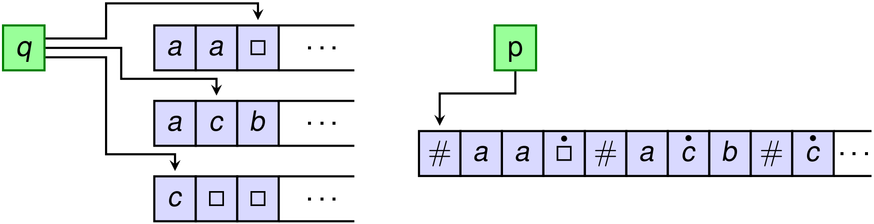

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The notions of **accepting** and **rejecting computations** are defined accordingly. Note: there may be more than one or no computation of a nondeterministic TM on a given input.
A nondeterministic TM M **accepts** an input w if and only if **there exists** some accepting computation of M on input w.

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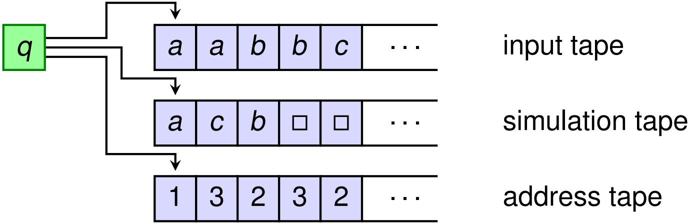
Idea

- D deterministically traverses in breath-first order the tree of configuration of N, where each branch represents a different possibility for N to continue.
- For this, successively try out all possible choices of transitions allowed by N.

Nondeterministic Turing Machines

Sketch of D:



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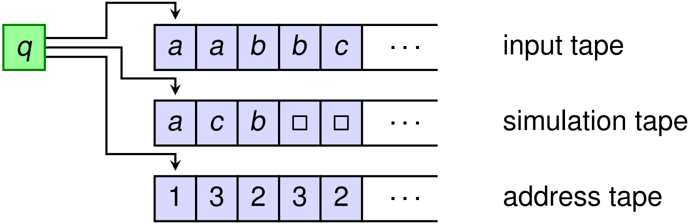

Let b be the maximal number of choices in δ, i.e.,

$$b := \max\left\{ |\delta(q,x)| \;\middle|\; q \in Q, x \in \Gamma \right\}.$$

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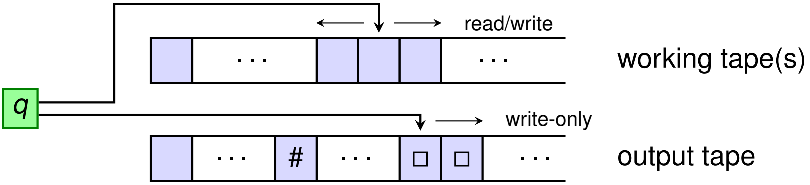

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Proof.

Let E be an enumerator for $\mathcal{L}$. Then the following TM accepts $\mathcal{L}$:

$\mathcal{M} :=$ On input w

- Simulate E on the empty input. Compare every string output by E with w
- If w appears in the output of E, *accept*

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$\mathcal{E}$:= Ignore the input.

- Repeat for $i = 1, 2, 3, \ldots$
  - Run M for i steps on each input $s_1, s_2, \ldots, s_i$
  - If any computation accepts, print the corresponding s_j followed by #

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Theorem 2.12

If $\mathcal{L}$ is Turing-recognisable, then there exists an enumerator for $\mathcal{L}$ that prints each word of $\mathcal{L}$ exactly once.

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A language $\mathcal{L}$ is decidable if and only if there exists an enumerator for $\mathcal{L}$ that outputs exactly the words of $\mathcal{L}$ in some order of non-decreasing length.

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Then M' enumerates exactly the words of $\mathcal{L}$ in some order of non-decreasing length.

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Summary and Outlook

Turing Machines are a simple model of computation

Recognisable (semi-decidable) = recursively enumerable

Decidable = computable = recursive

Many variants of TMs exist – they normally recognise/decide the same languages

What's next?

- A short look into undecidability
- Recursion and self-referentiality
- Actual complexity classes