TU Dresden, Fakultät Informatik
Stephan Mennicke, Markus Krötzsch

Winter Term 2024/2025

Complexity Theory

Exercise 1: Mathematical Foundations, and Decidability and Recognisability

22nd October 2024

Exercise 1.1. Consider a non-empty set M and a function $f : M \to 2^M$. Show that f is not surjective.

Exercise 1.2. Show the following claims.

1. $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.
2. $|\mathbb{N}| = |\mathbb{Q}|$.
3. $|\mathbb{N}| \neq |\mathbb{R}|$.

Exercise 1.3. Show the following claims.

1. There exist non-regular languages.
2. There exist undecidable languages.
3. There exist non-Turing-recognizable languages.

Exercise 1.4. Let $G = \{V, E\}$ be a simple undirected graph such that $|V| \geq 2$. Show that G contains two or more nodes that have equal degree. That is, show that that there is a pair of nodes that occur in the same number of edges.

Exercise 1.5. Let $A = \{ s \}$, where

$$s := \begin{cases} 0 & \text{if life will never be found on Mars,} \\ 1 & \text{if life will be found on Mars someday.} \end{cases}$$

Is A decidable? (For the purpose of this problem, assume that the question whether life will be found on Mars has an unambiguous "yes" or "no" answer.)

Exercise 1.6. Show that the class of Turing-decidable languages is closed under (1) union, (2) concatenation, (3) intersection, and (4) (Kleene) star.

∗ **Exercise 1.7.** Show that the class of Turing-recognizable languages is closed under homomorphism. A function $h : \Sigma^* \to \Delta^*$ is called a *homomorphism on* $\Sigma^\star$ if for all $u, v \in \Sigma^*$, $h(uv) = h(u)h(v)$.

Exercise 1.8. We consider two extensions of Turing machines.

- A *Turing machine with two-sided unbounded tape* is a single-tape Turing machine where the tape is unbounded on both sides.
- A *Turing machine with two-dimensional unbounded tape* is a Turing machine where each tape cell does not only a left and a right neighbour but also a neighbour above and one below. Accordingly, the head of a Turing machine can move left-, right-, up-, and downwards.

Argue that such machines can be simulated by ordinary Turing machines.

Exercise 1.9. Let $\mathsf{ALL}_{\mathsf{DFA}} = \{\, \langle A \rangle \mid A \text{ is a DFA that accepts every word} \,\}$. Show that $\mathsf{ALL}_{\mathsf{DFA}}$ is decidable.

Exercise 1.10. Let $\mathsf{E}_{\mathsf{TM}} = \{\, \langle M \rangle \mid M \text{ is a TM such that } \mathcal{L}(M) = \emptyset \,\}$. Show that $\overline{\mathsf{E}_{\mathsf{TM}}}$ is Turing-recognizable.

Exercise 1.11. Let C be a language. Prove that C is Turing-recognizable if and only if a decidable language D exists such that $C = \{\, x \mid \exists y. \langle x, y \rangle \in D \,\}$.