

Restricted Chase Termination

You Want More than Fairness

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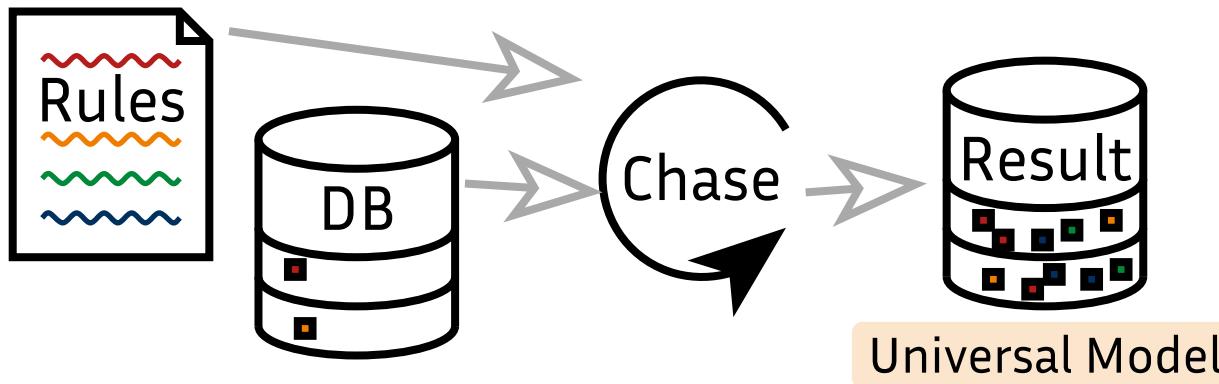
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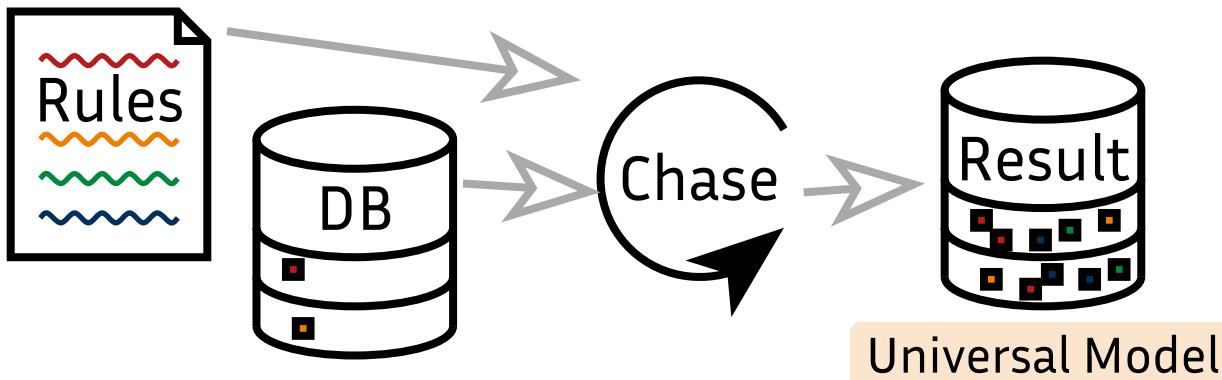
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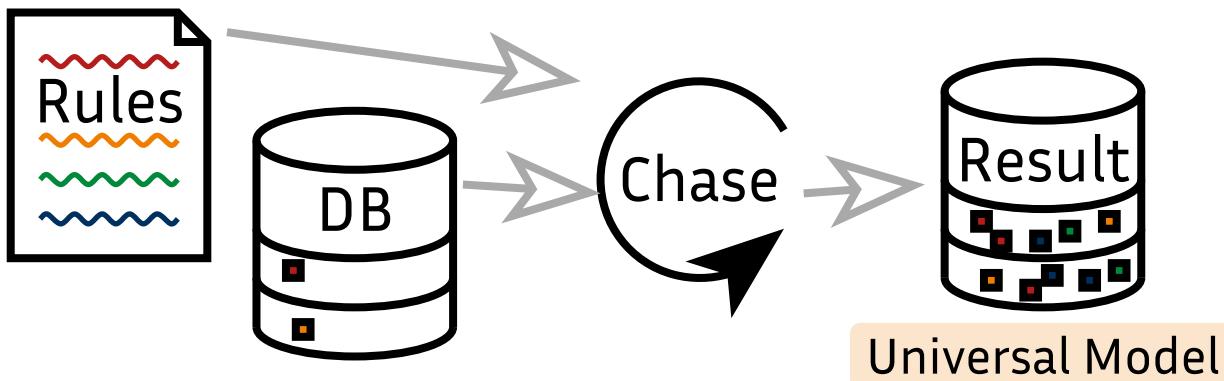


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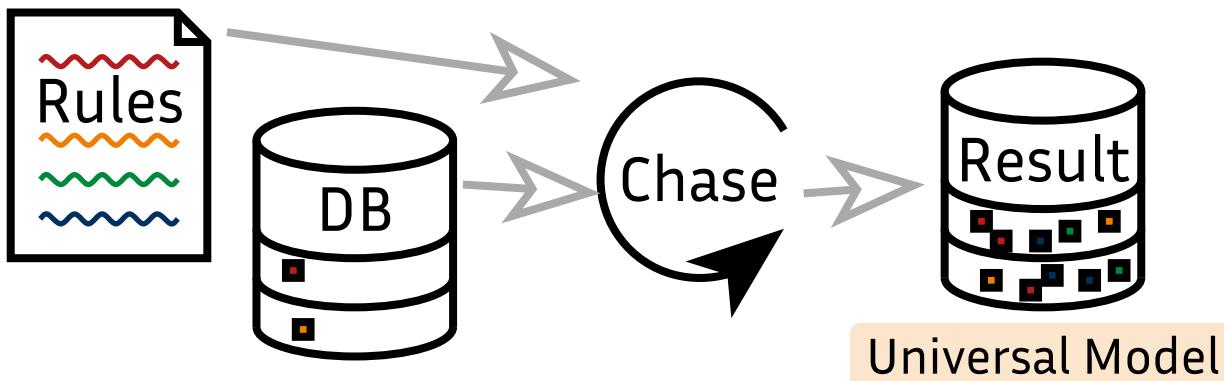
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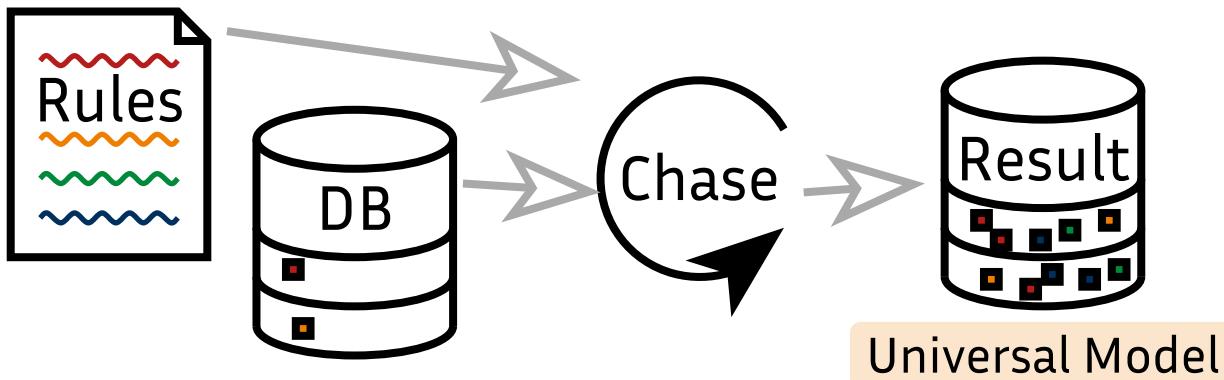
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Step 1: $R(c, n_1), B(n_1)$

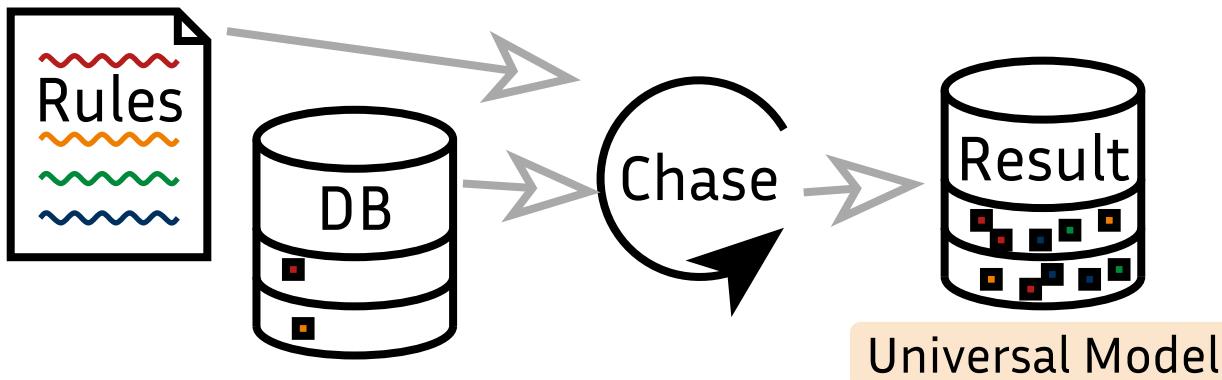
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Step 2: $A(n_1)$; Step 3: $R(n_1, c)$

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Universal Models are “most general” and can answer conjunctive queries.

Chase Termination is Undecidable

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Step 1-2: $R(c, n_1), B(n_1), A(n_1)$

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Step 3: $R(n_1, n_2), B(n_2)$

Chase Termination is Undecidable

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A(x) → ∃z. R(x, z) ∧ B(z)  
B(x) → A(x)  
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Step 1-2: R(c, n₁), B(n₁), A(n₁)
Step 3: R(n₁, n₂), B(n₂)
Step N: R(n₁, c) (due to Fairness)

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Chase Termination is Undecidable (2)

CTK_{\forall}^r is the set of all knowledge bases, i.e. pairs of rule sets, and databases, on which every (\forall) restricted (r) chase sequence terminates.

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Overview from (Grahne and Onet 2018)

CTX_{\exists}^r	\exists	\forall
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Σ_1^0 - Semi-Decidable Languages
(e.g. Halting Problem)

Π_2^0 - Co-Semi-Decidable with
Semi-Decision Oracle
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Membership: Run all Chase Sequences in Parallel

Hardness: TM can be simulated with Existential Rules

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Membership: Use CTK_{\exists}^r oracle.

Hardness: (more involved)
see (Grahne and Onet 2018)

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This does not work because of *fairness*!

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The upper bound was still unknown! (Until slide 6)

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Possible Fix: We could demand something stronger, e.g. "breadth-first".

Fairness for NTMs - based on (Harel 1986)

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This problem is Σ_1^1 -complete - first analytical hierarchy level - beyond infinitely many Turing jumps. (Harel 1986)

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Run visiting the desig. state infinitely often iff infinite fair chase sequence.
Hence: Complement of CTK_{\forall}^r is in Σ_1^1 ; therefore CTK_{\forall}^r is in Π_1^1 .

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Idea: With **emergency brakes**, we can force the chase to terminate after finitely many steps. If the designated state is visited, we create a new brake.

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K	Σ_1^0 -complete	Π_1^1 -complete
R	Π_2^0 -complete	Π_1^1 -complete

Summing up...

Fairness makes Chase Termination
highly Undecidable.

Stricter, finitely verifiable conditions
could solve this (e.g. breadth-first).

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📢 *Talk to me about Lean and Typst 😎*

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