Finite and Algorithmic Model Theory

Lecture 1 (Dresden 12.10.22, Long version)

Lecturer: Bartosz "Bart" Bednarczyk

Technische Universität Dresden & Uniwersytet Wrocławski

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Today's agenda

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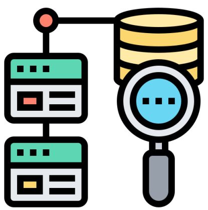

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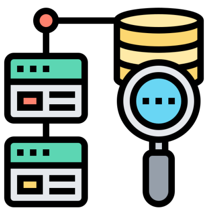

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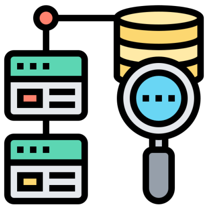

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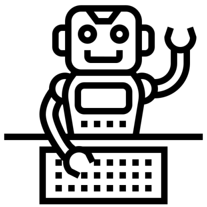

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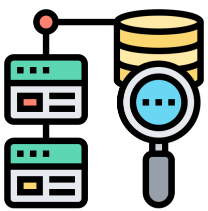

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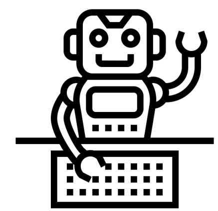

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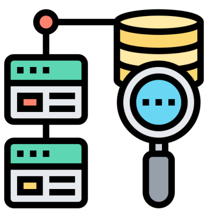

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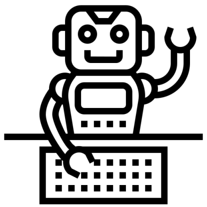

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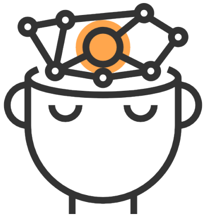

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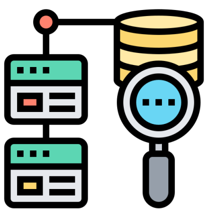

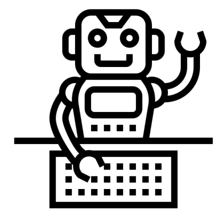


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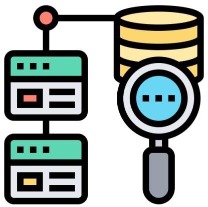
Query languages? 

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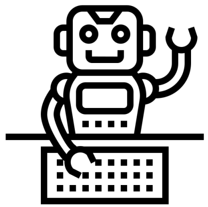
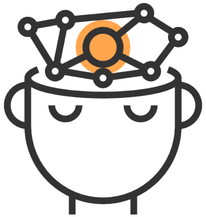
Formal languages? 

Complexity? 

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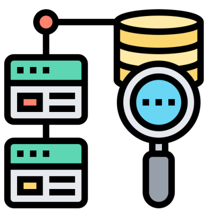

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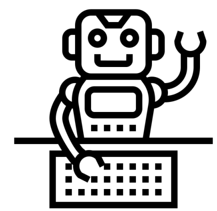

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Feel free to ask questions and interrupt me!

Don't be shy! If needed send me an email (bartosz.bednarczyk@cs.uni.wroc.pl) or approach me after the lecture!

Reminder: this is an advanced lecture. Target: people that had fun learning logic during BSc studies!

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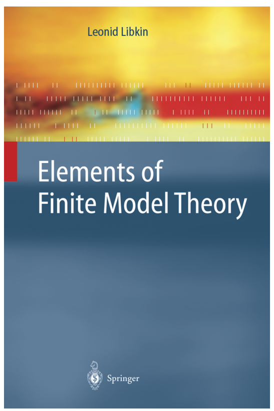

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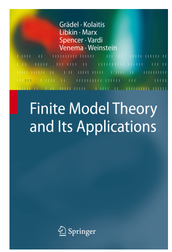

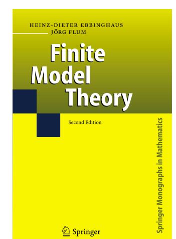

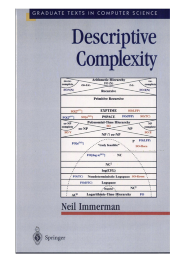

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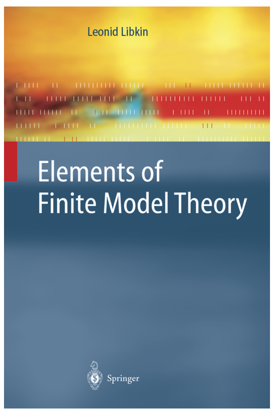

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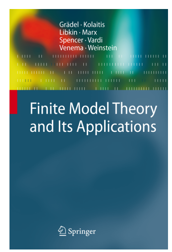

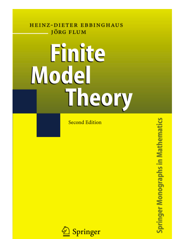

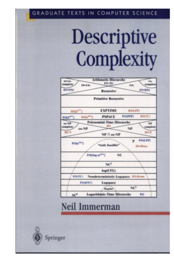

Last but Not Least: I offer MSc/PHD research projects for motivated students!

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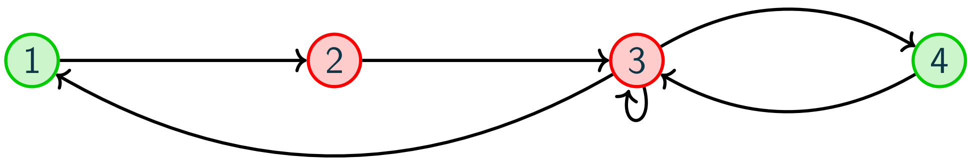

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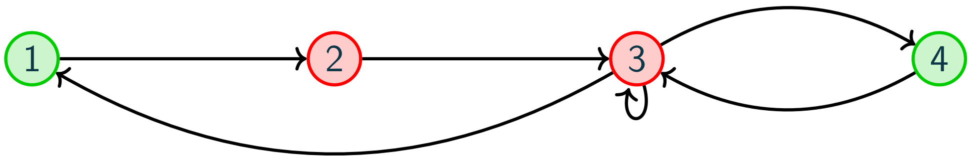
$\mathfrak{A} :=$ 

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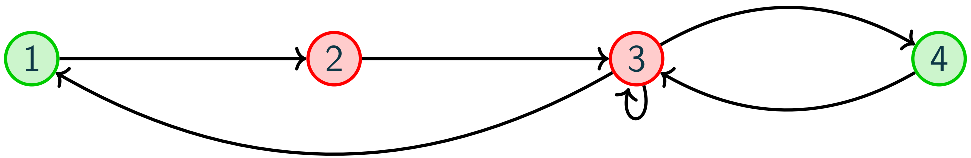
$\mathfrak{A} :=$ 

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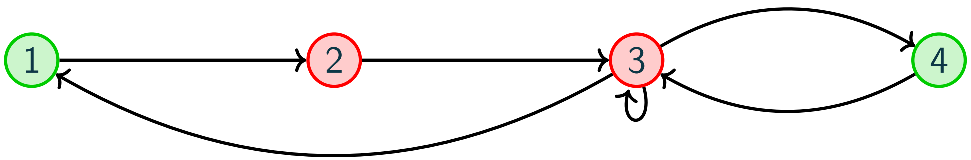
$\mathfrak{A} :=$ 

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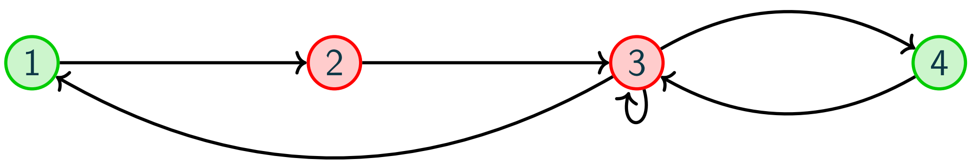
$\mathfrak{A} :=$ 

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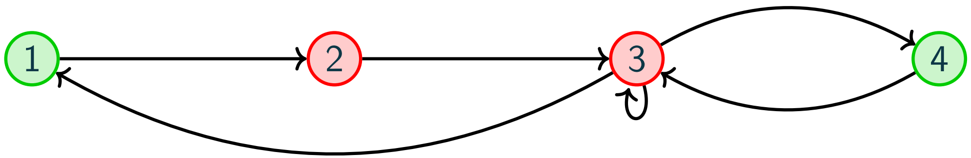
$\mathfrak{A} :=$ 

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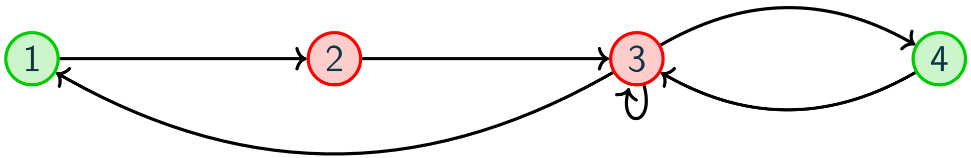
$\mathfrak{A} :=$ 

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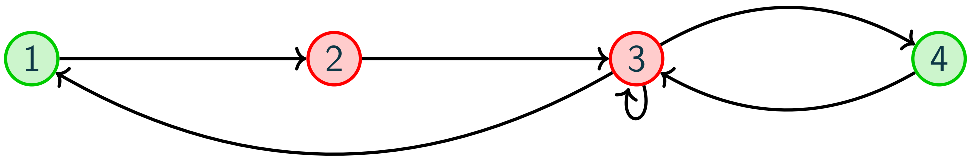
$\mathfrak{A} :=$ 

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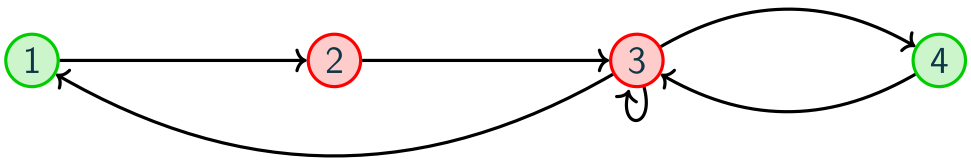
$\mathfrak{A} :=$ 

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Constants $\approx$ elements, unary relations $\approx$ colours, binary (resp. higher-arity) relations $\approx$ (hyper)edges

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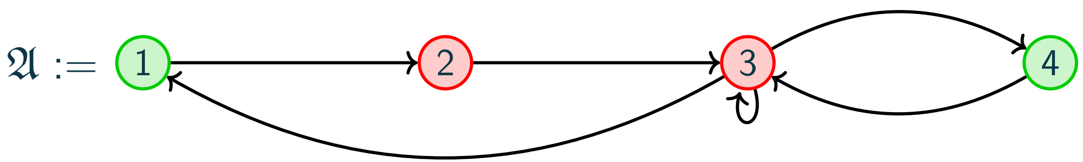

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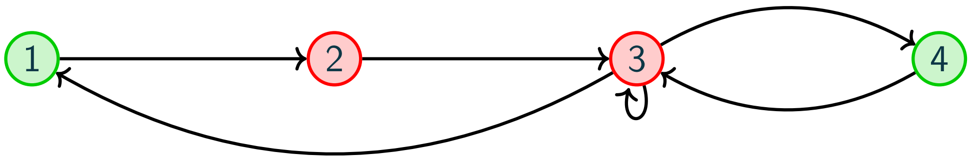

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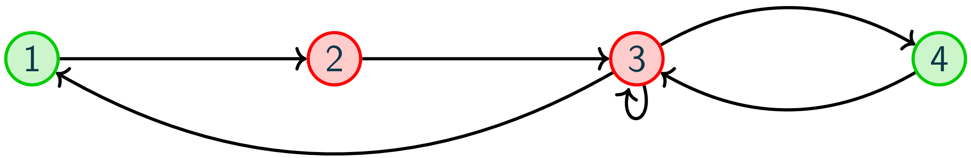
$\mathfrak{A} :=$ 

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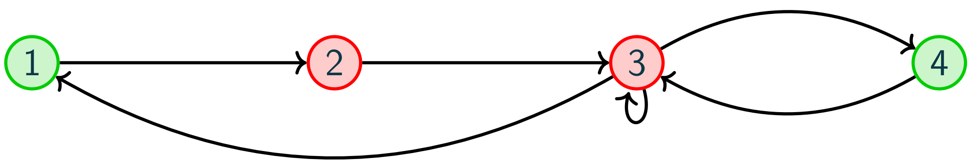
$\mathfrak{A} :=$ 

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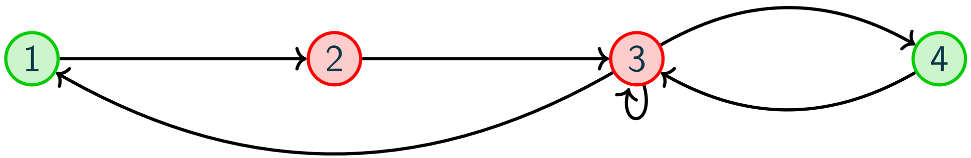

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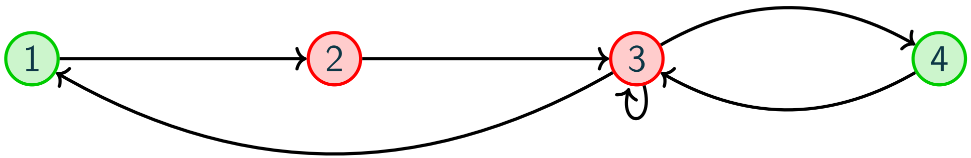

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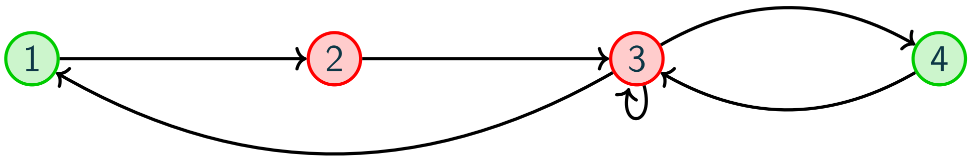

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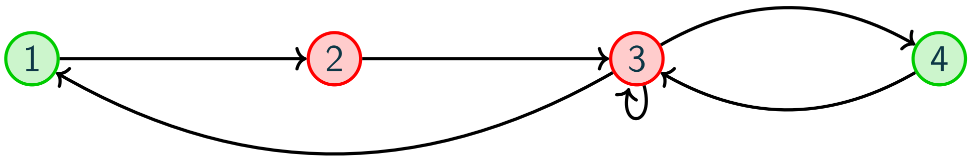
$\mathfrak{A} :=$ 

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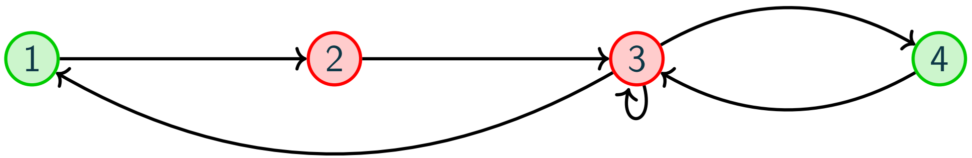
$\mathfrak{A} :=$ 

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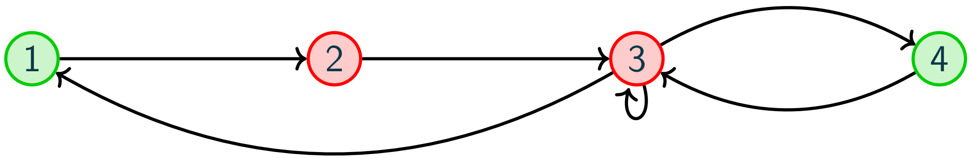
$\mathfrak{A} :=$ 

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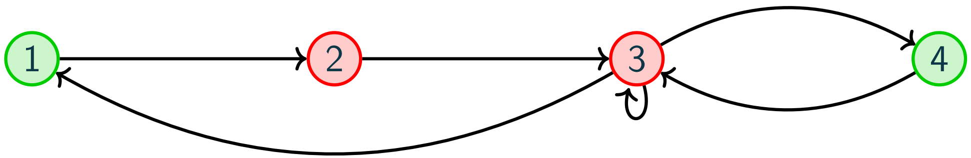

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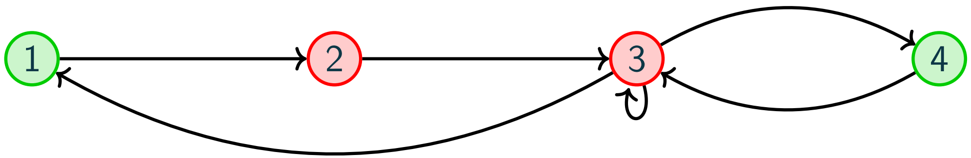
$\mathfrak{A} :=$ 

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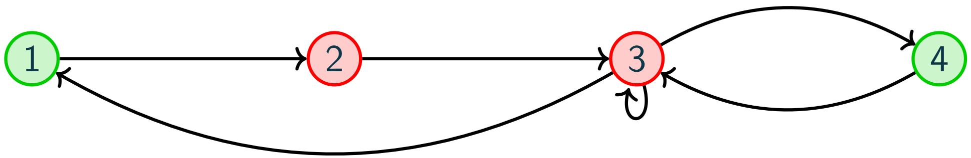
$\mathfrak{A} :=$ 

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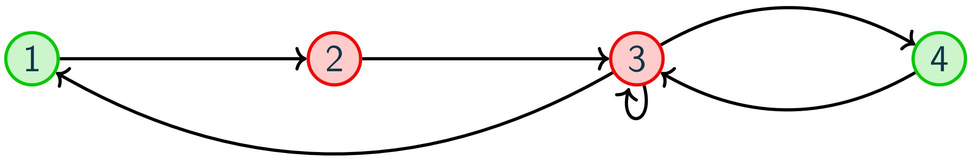
$\mathfrak{A} :=$ 

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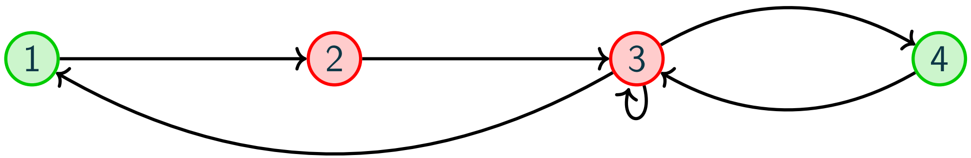
$\mathfrak{A} :=$ 

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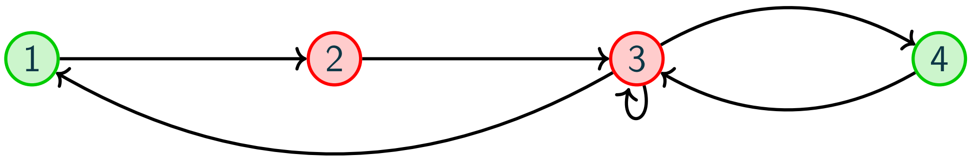
$\mathfrak{A} :=$ 

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Made formal via abstract model theory, c.f. article at ncatlab.org and Lindström's theorems.

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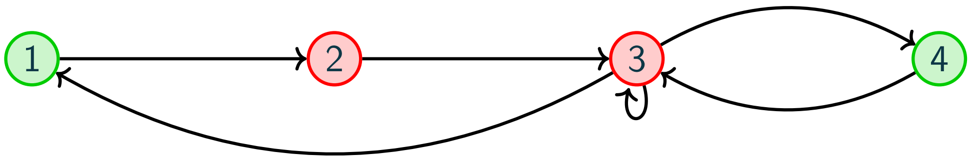

over a signature $\tau := \{\mathrm{G}^{(1)}, \mathrm{R}^{(1)}, \mathrm{E}^{(2)}\}$

$\mathrm{G}^{\mathfrak{A}} := \{1, 4\}, \qquad \mathrm{R}^{\mathfrak{A}} := \{2, 3\}$

$\mathrm{E}^{\mathfrak{A}} := \{(1, 2), (2, 3), (3, 1), (3, 3)(3, 4), (4, 3)\}$

A signature contains (at most countably* many) constant and relation symbols (each with a fixed arity).

Structure = Domain + interpretation of symbols, e.g. $\mathfrak{A} := (A, \cdot^{\mathfrak{A}})$ depicted above,

where $A = \{1, 2, 3, 4\}$ and $\cdot^{\mathfrak{A}}(\mathrm{G}), \cdot^{\mathfrak{A}}(\mathrm{R}), \cdot^{\mathfrak{A}}(\mathrm{E})$ are as above.

Example (of a First-Order Logic (FO) Formula)

(in a coloured graph:) Any node is either green or red.

$\varphi := \forall x\ (\mathrm{G}(x) \vee \mathrm{R}(x)) \wedge (\mathrm{G}(x) \leftrightarrow \neg \mathrm{R}(x))$

We write $\mathfrak{A} \models \varphi$ to indicate that $\mathfrak{A}$ satisfies φ or $\mathfrak{A}$ is a model of φ.

Formulae often employ: Variables: $x, y, z, X, Y, \ldots$ Boolean connectives: $\wedge, \vee, \neg, \leftrightarrow, \bigvee_{i=0}^{\infty}, \ldots$

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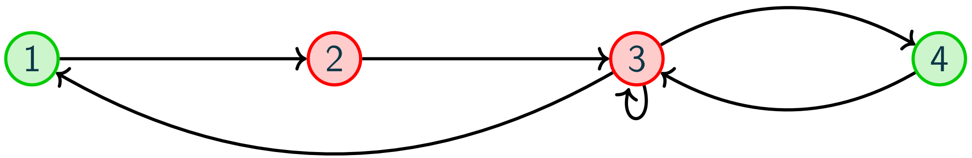
$\mathfrak{A} :=$ 

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More examples I.

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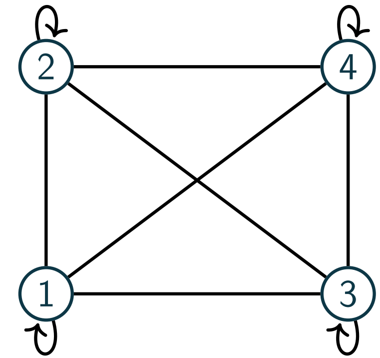

$\models \varphi_{\text{4clique}}$

More examples I.

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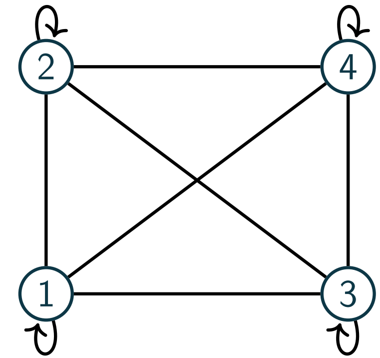

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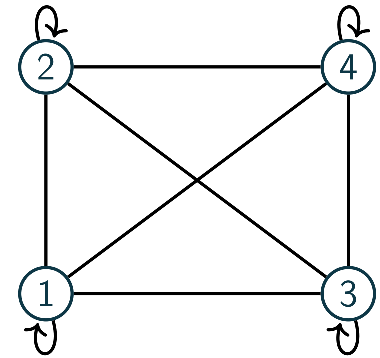

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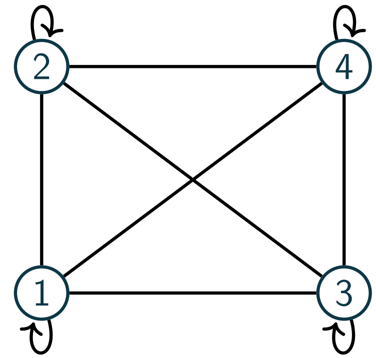

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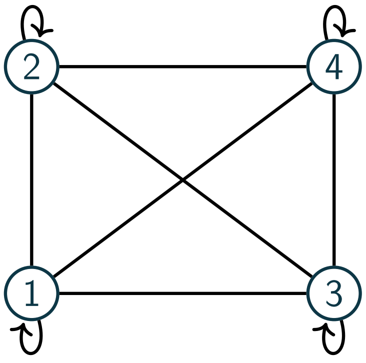

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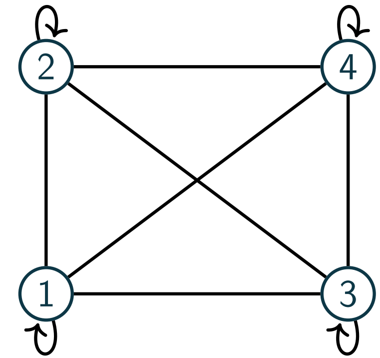

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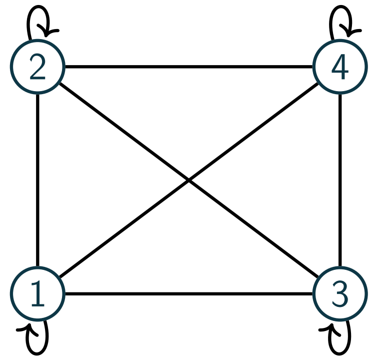

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Exercise (Write a formula over $\{\mathrm{E}^{(2)}\}$ checking if a graph is two-colorable.)

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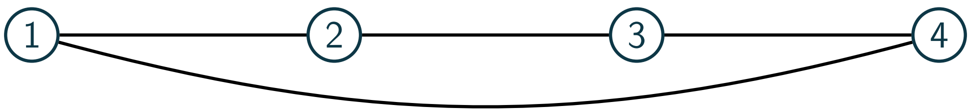

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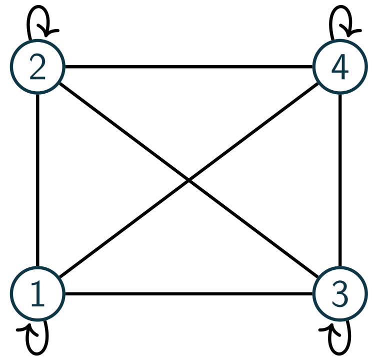

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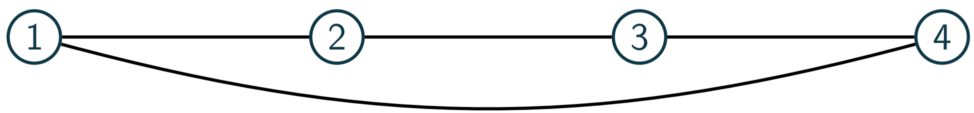

$$\varphi_{2COL} \; = \; \exists \mathrm{G} \exists \mathrm{R} \; (x \in \mathrm{G} \vee x \in \mathrm{R}) \wedge (x \in \mathrm{G} \leftrightarrow x \notin \mathrm{R}) \; \wedge \; \varphi_{ok}$$

More examples I.

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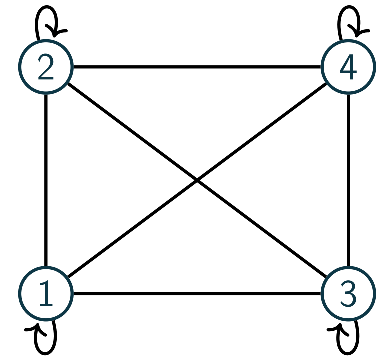

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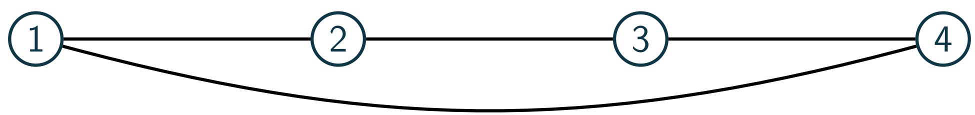
$\mathfrak{G} :=$ 

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More examples I.

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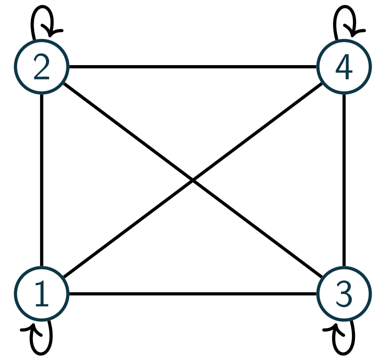

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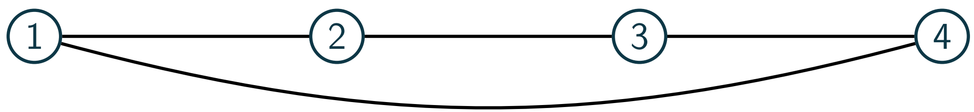
$\mathfrak{G} :=$ 

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Quantification over sets:

More examples I.

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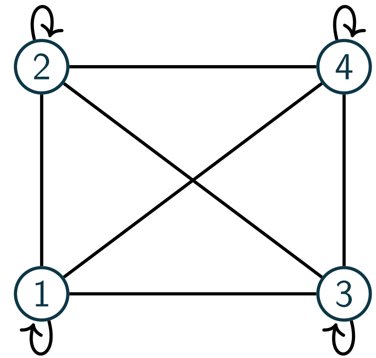

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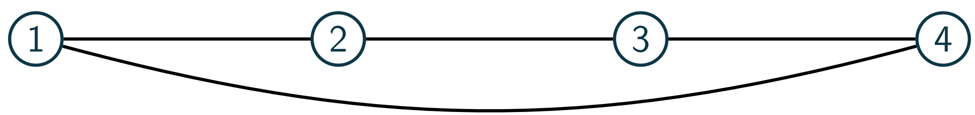

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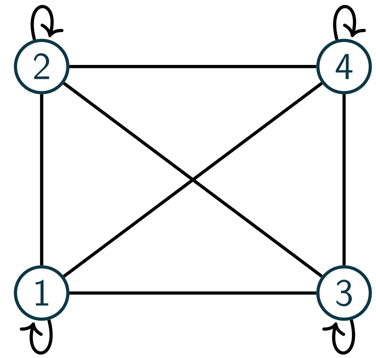

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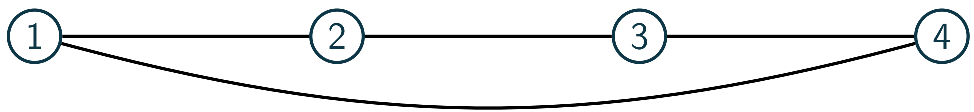

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There exists a colouring with G and R and it is correct

More examples I.

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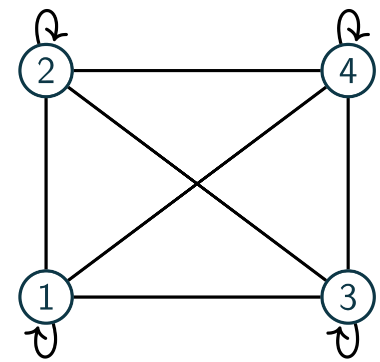

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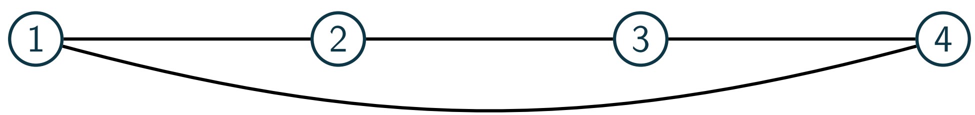

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More examples I.

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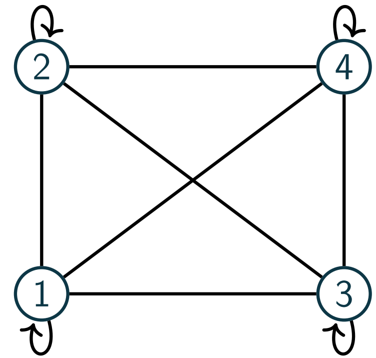

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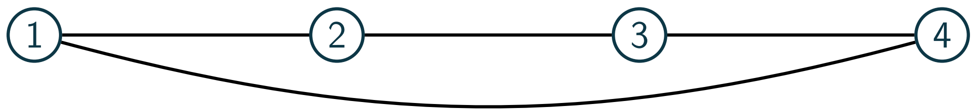

$\models \varphi_{2COL}$

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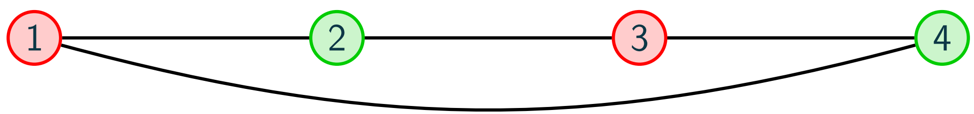
$\mathfrak{G}' :=$



$\models \varphi_{ok}$

More examples II.

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Exercise (Write an $\mathrm{FO}[\{\mathrm{E}^{(2)}, \mathtt{a}, \mathtt{b}\}]$ formula $\varphi_k^{\mathsf{reach}(\mathtt{a},\mathtt{b})}$ testing if there is a path from $\mathtt{a}$ to $\mathtt{b}$ of length k.)

More examples II.

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1. Case $k = 0$ is trivial:

More examples II.

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More examples II.

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No. And we will show it today!

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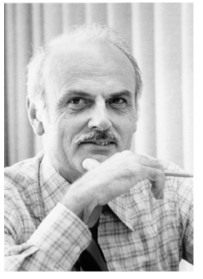

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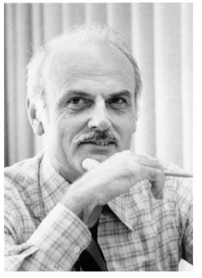

Other useful logic: Datalog ≈ SQL + recursion

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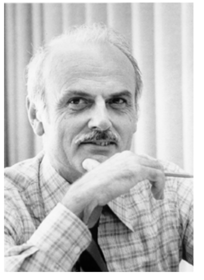

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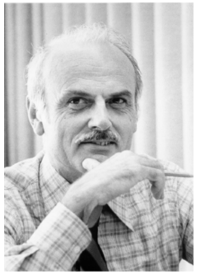

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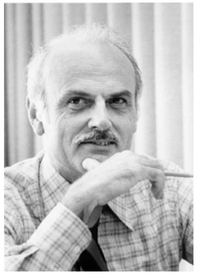

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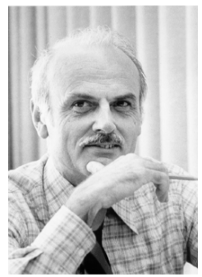

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Description logics: a family of logics for knowledge representation.

W3C® SNOMED CT The global language of healthcare protégé Dublin Core Metadata Initiative Making it easier to find information

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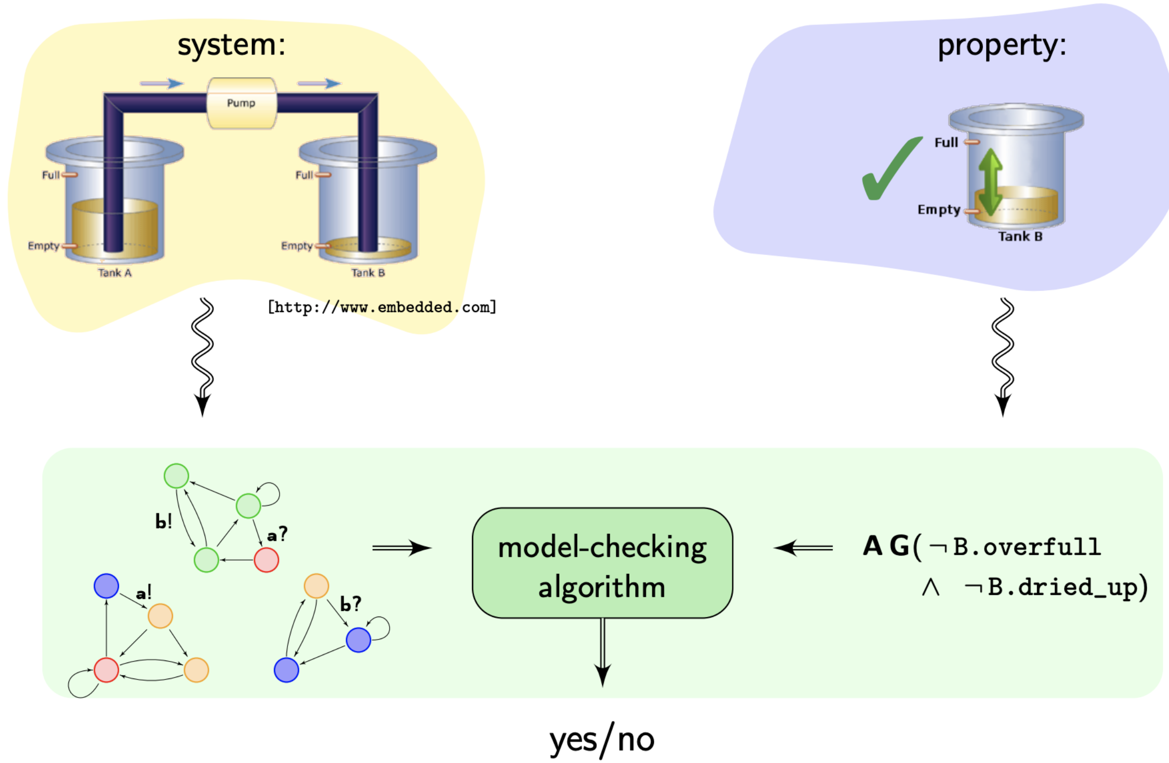

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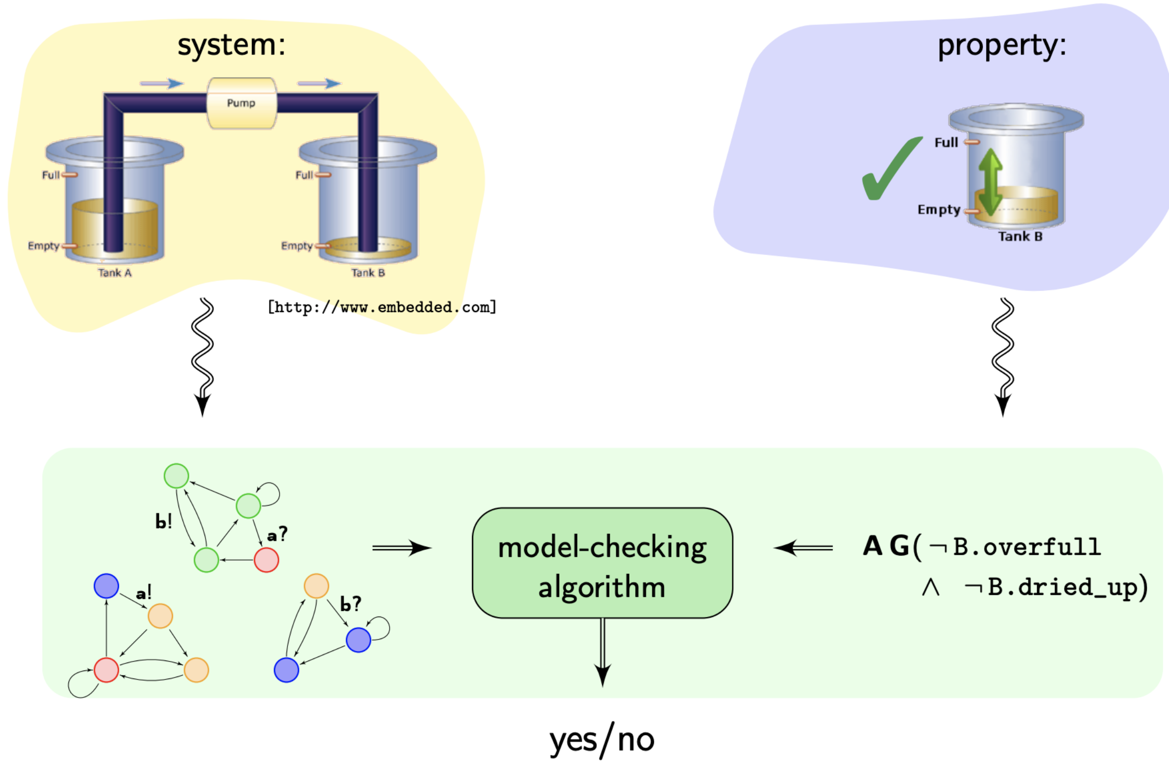

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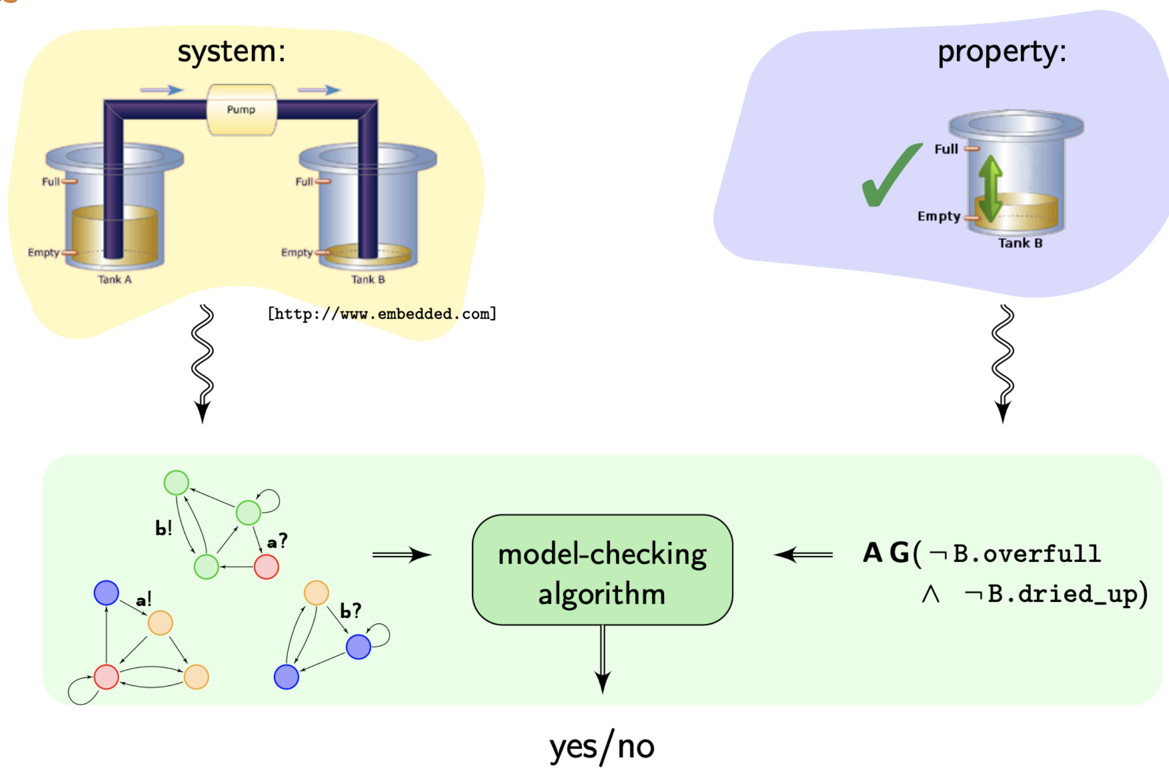

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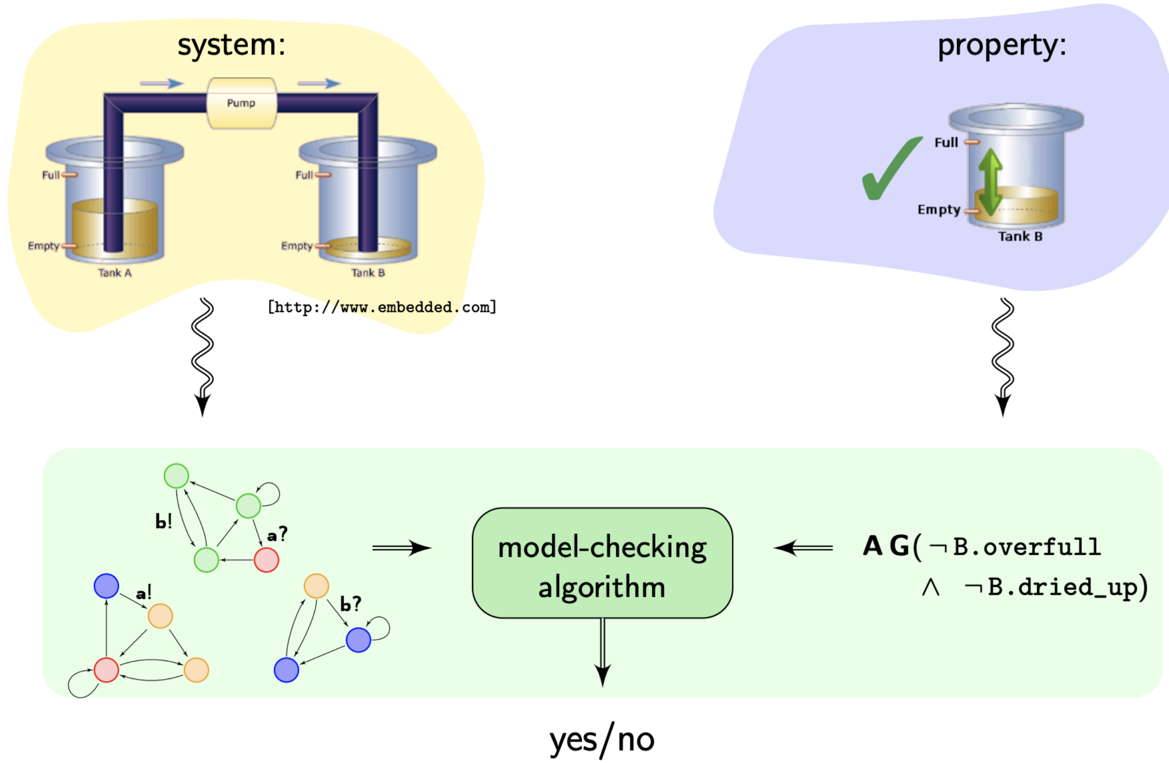

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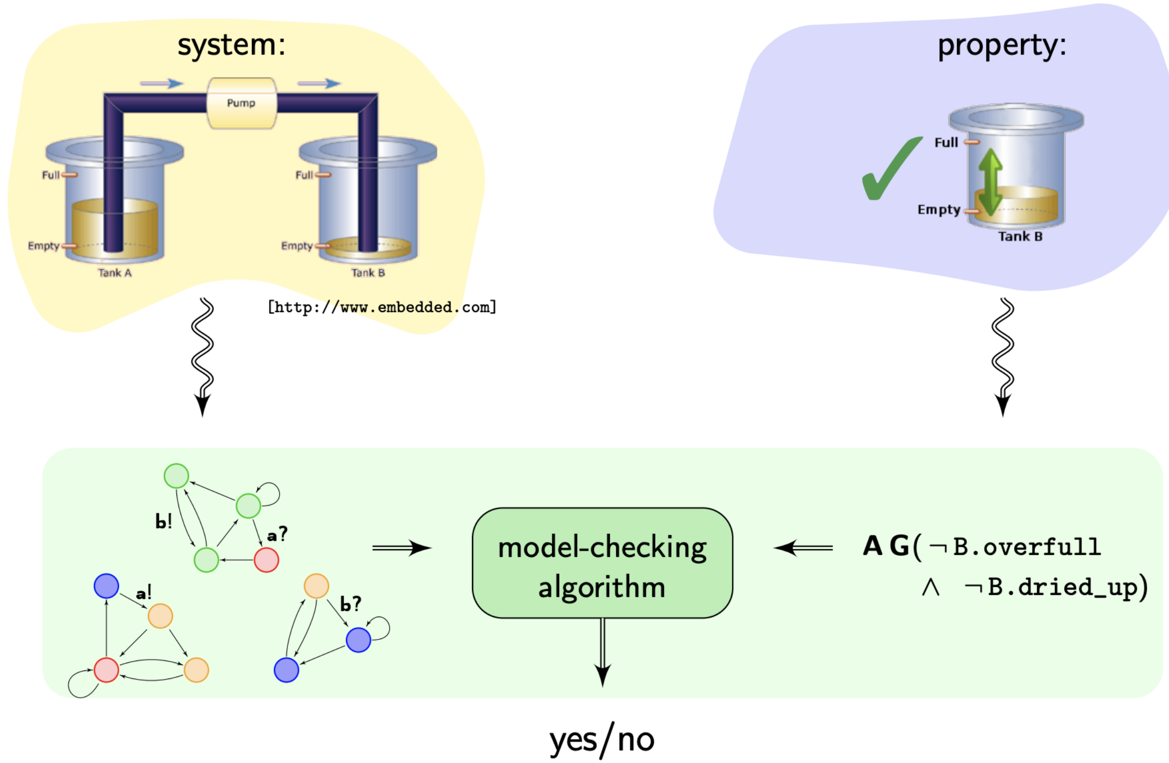

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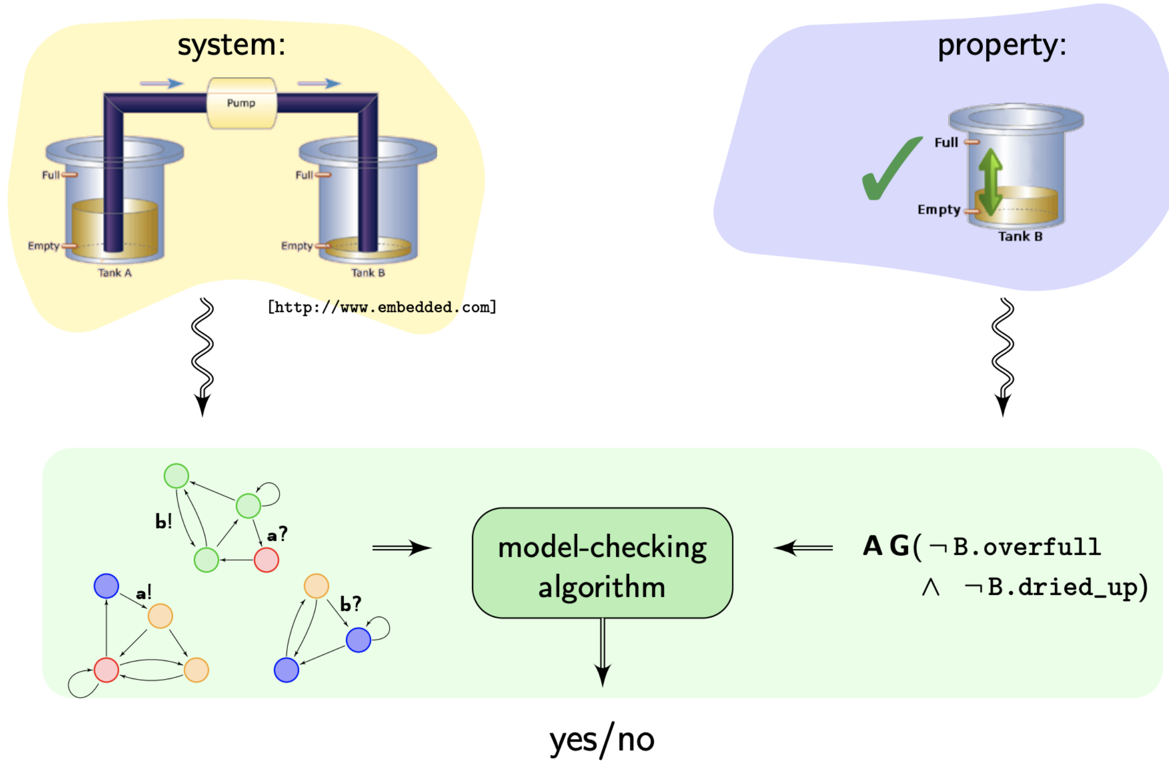

Motivations II: why do we care about logic?

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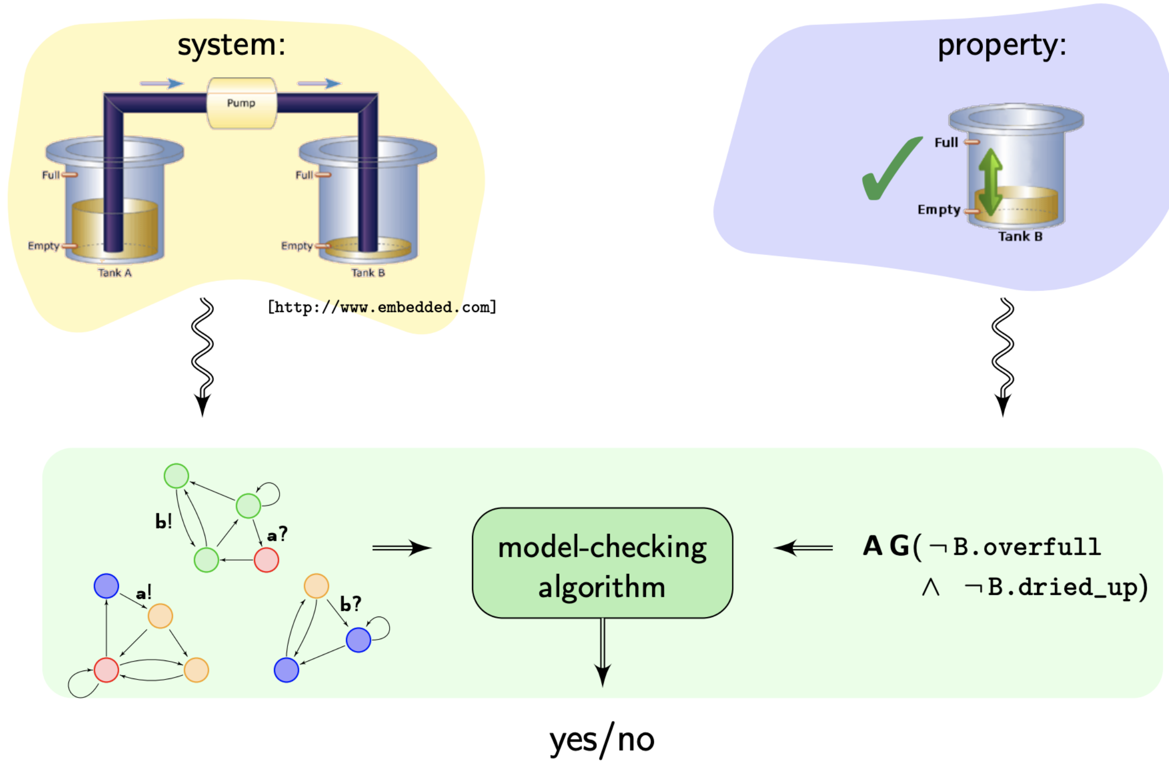

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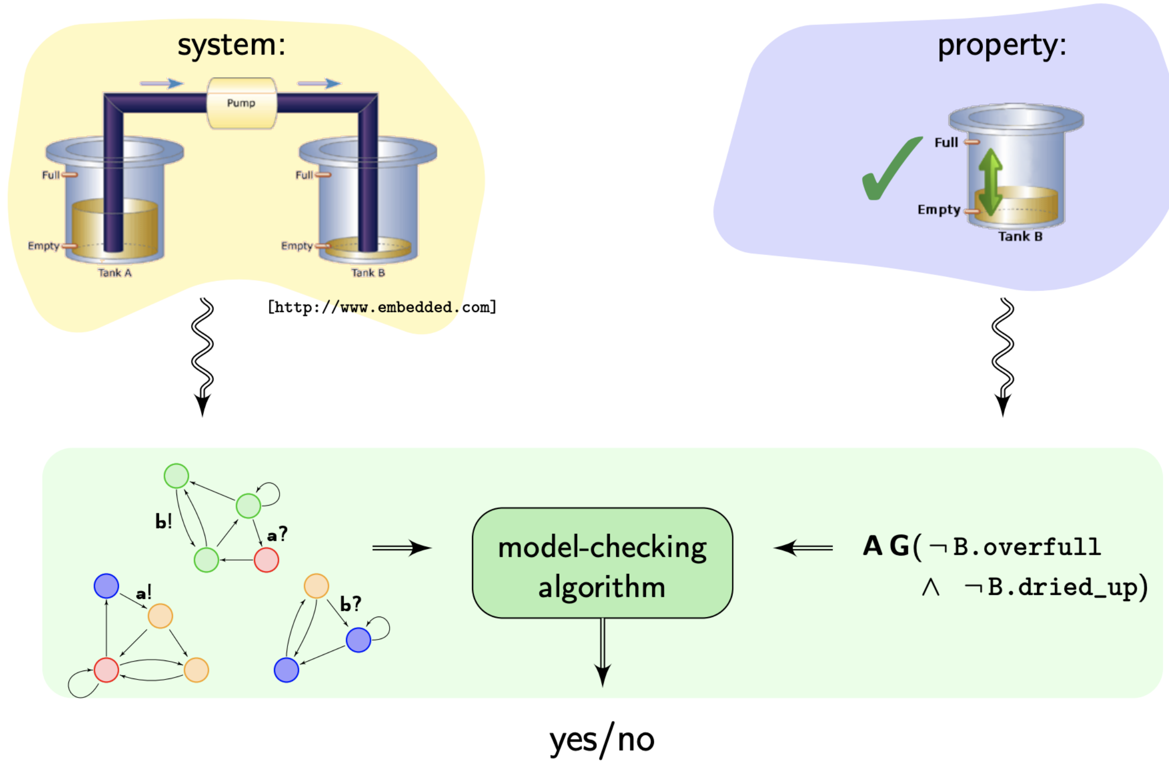
```
// hello.c
#include <stdlib.h>

void test() {
  int *s = NULL;
  *s = 42;
}
```



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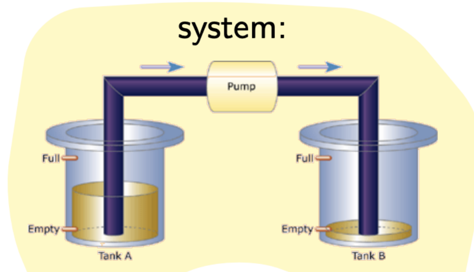

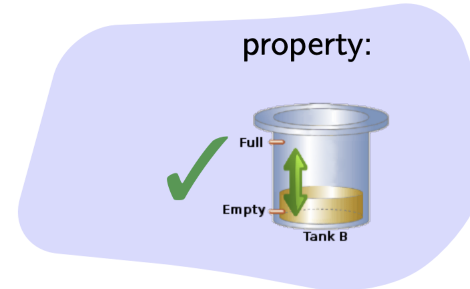


```
vim hello.c
// hello.c
#include <stdlib.h>

void test() {
  int *s = NULL;
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}
```

```
bartoszbednarczyk@Minsky-Machine: ~/Downloads/Infer
$ infer run -- gcc -c hello.c

Capturing in make/cc mode...
Found 1 source file to analyze in /Users/bartoszbednarczyk/Downloads/Infer/infer-out


Analysis finished in 775mss

Found 1 issue

hello.c:6: error: NULL_DEREFERENCE
  pointer `s` last assigned on line 5 could be null and is dereferenced at line 6, column 3.
  4.   void test() {
  5.     int *s = NULL;
  6. >   *s = 42;
  7.   }


Summary of the reports

  NULL_DEREFERENCE: 1
```

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In "standard" computational complexity we measure resources, e.g. space and time.

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$O(n)$ time

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$\Theta(n \log(n))$ memory?

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solvable in PSPACE?

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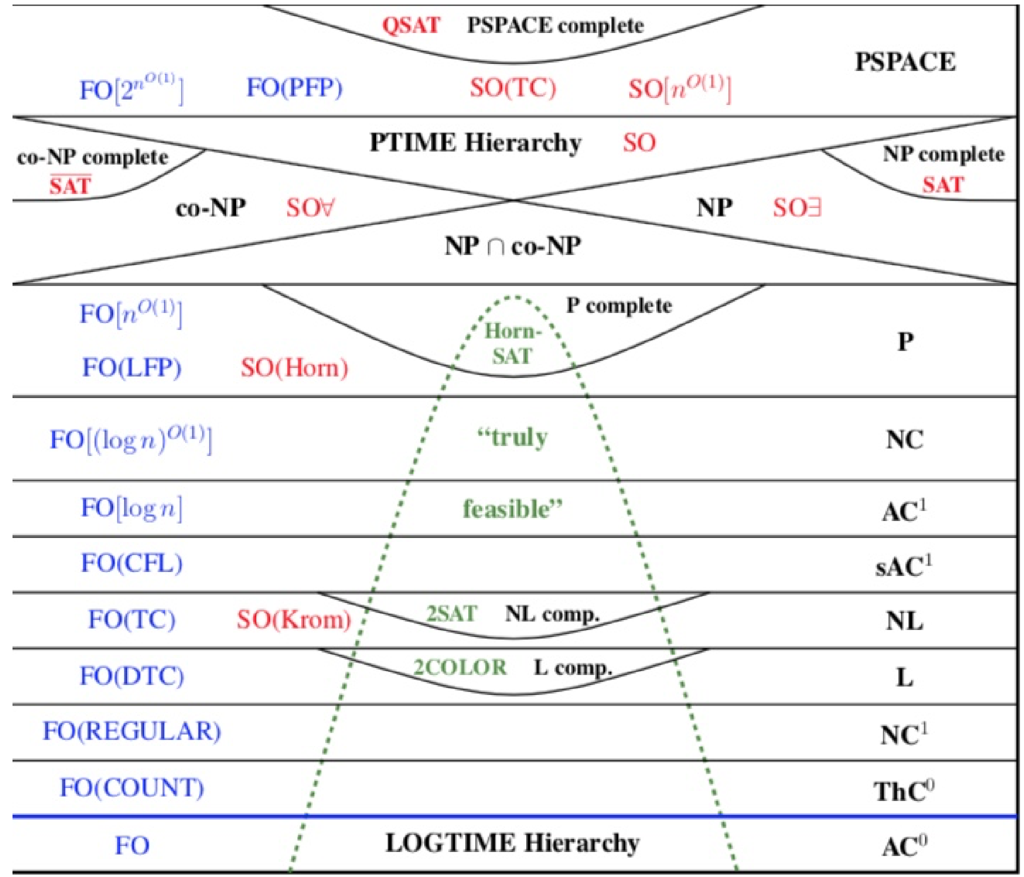

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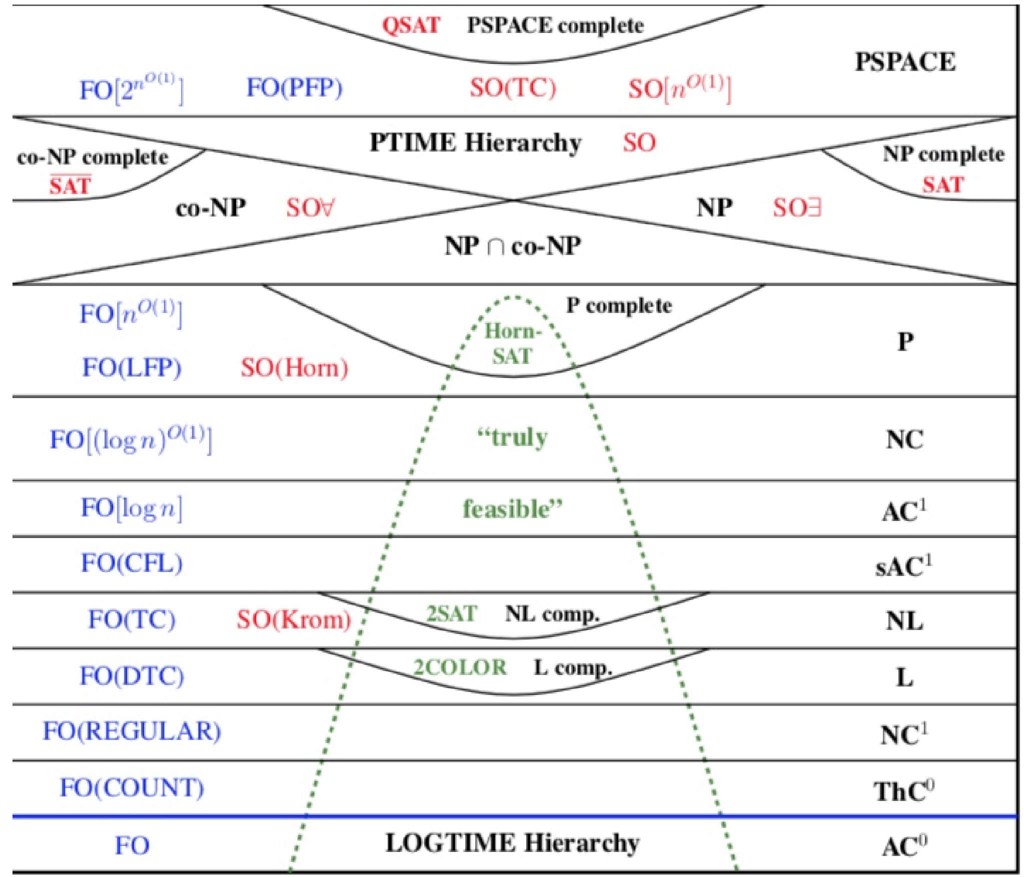

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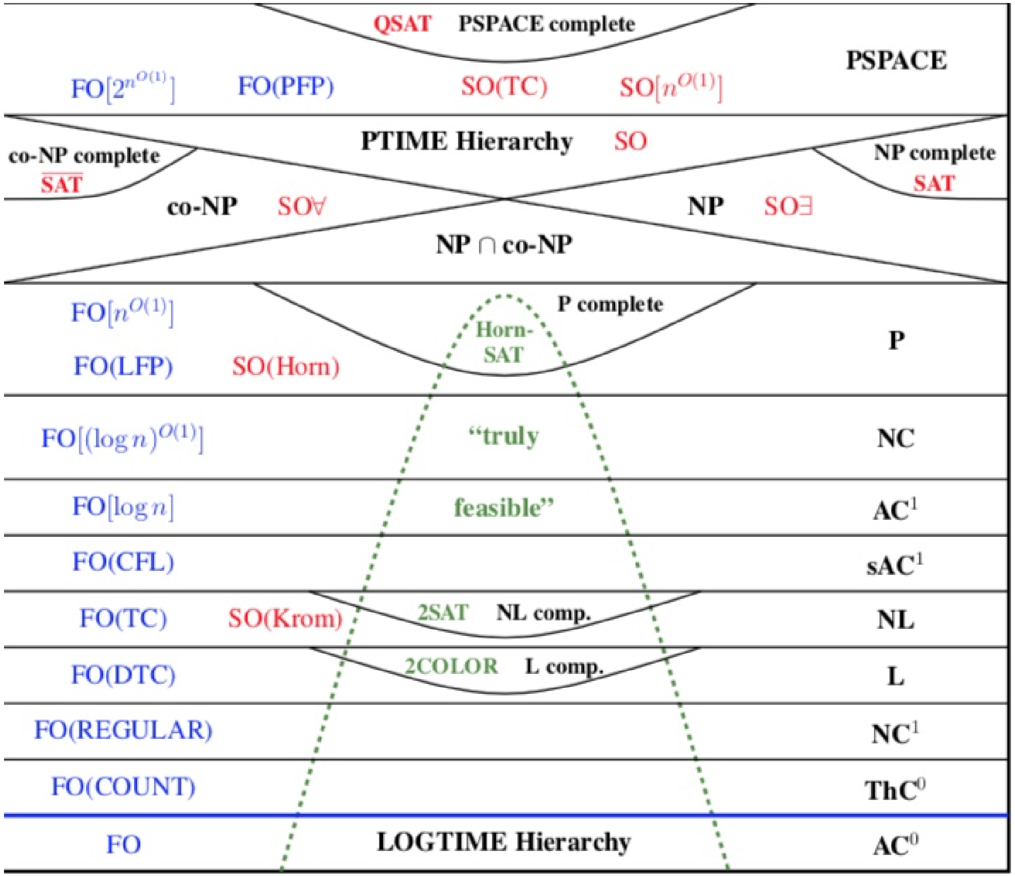

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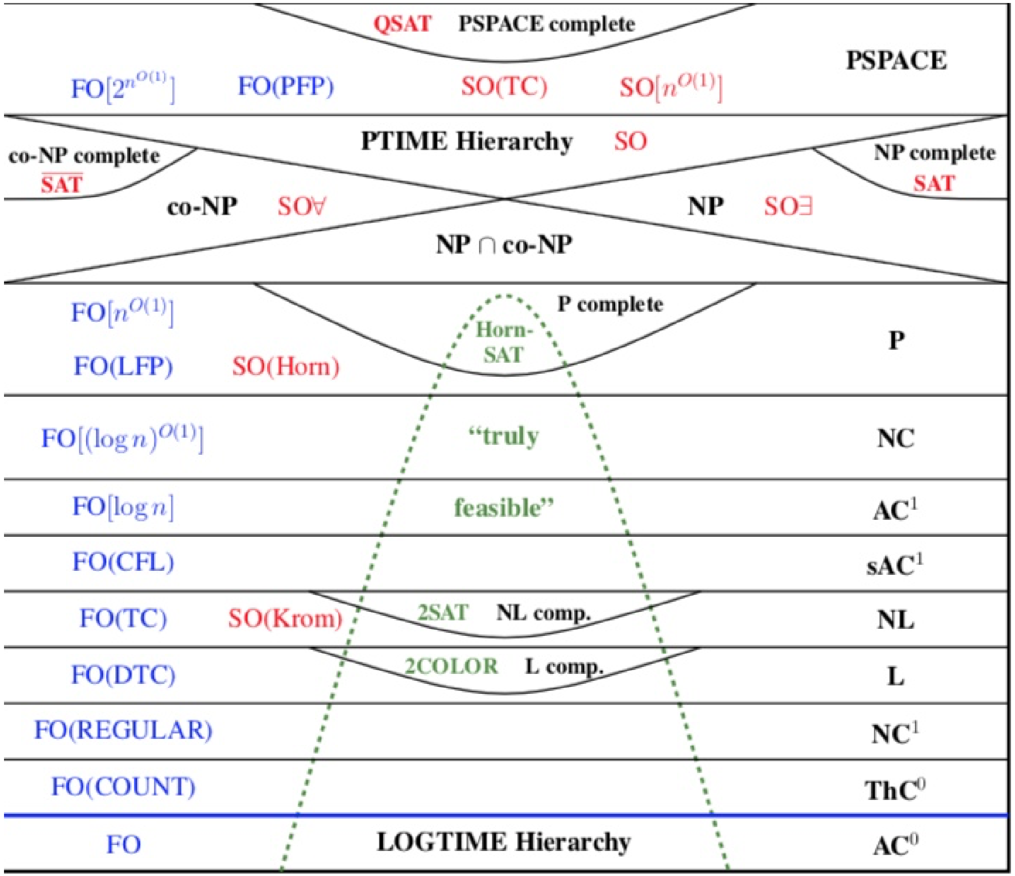

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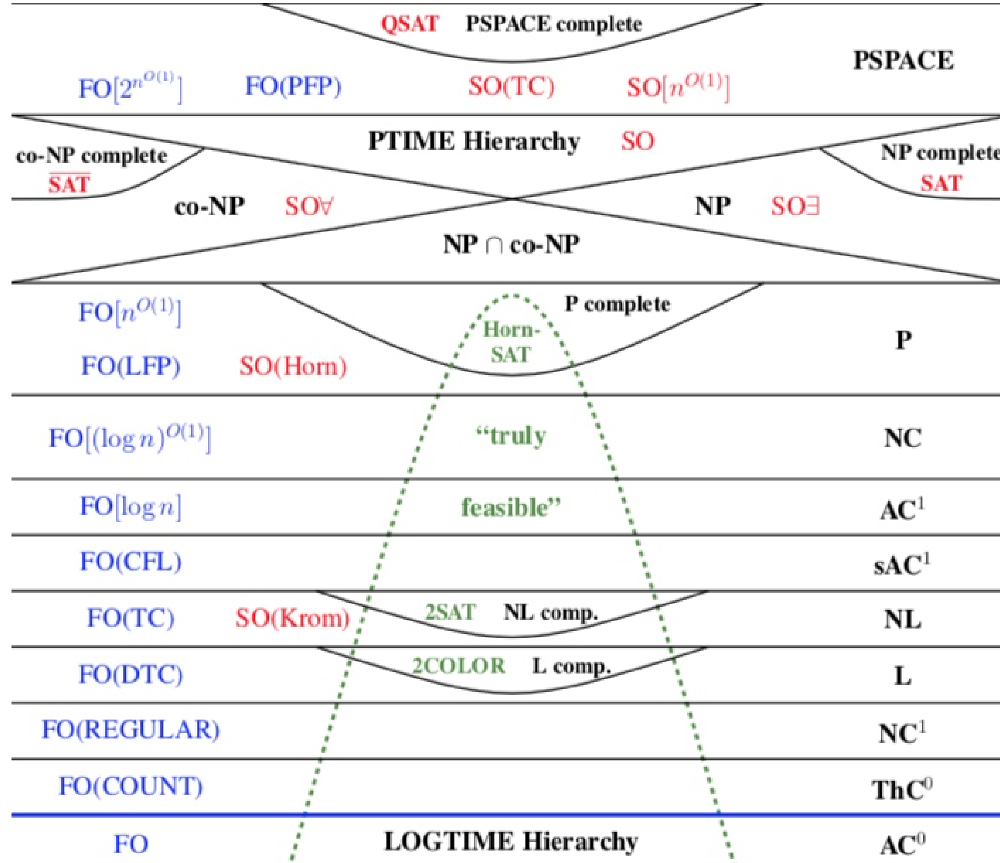

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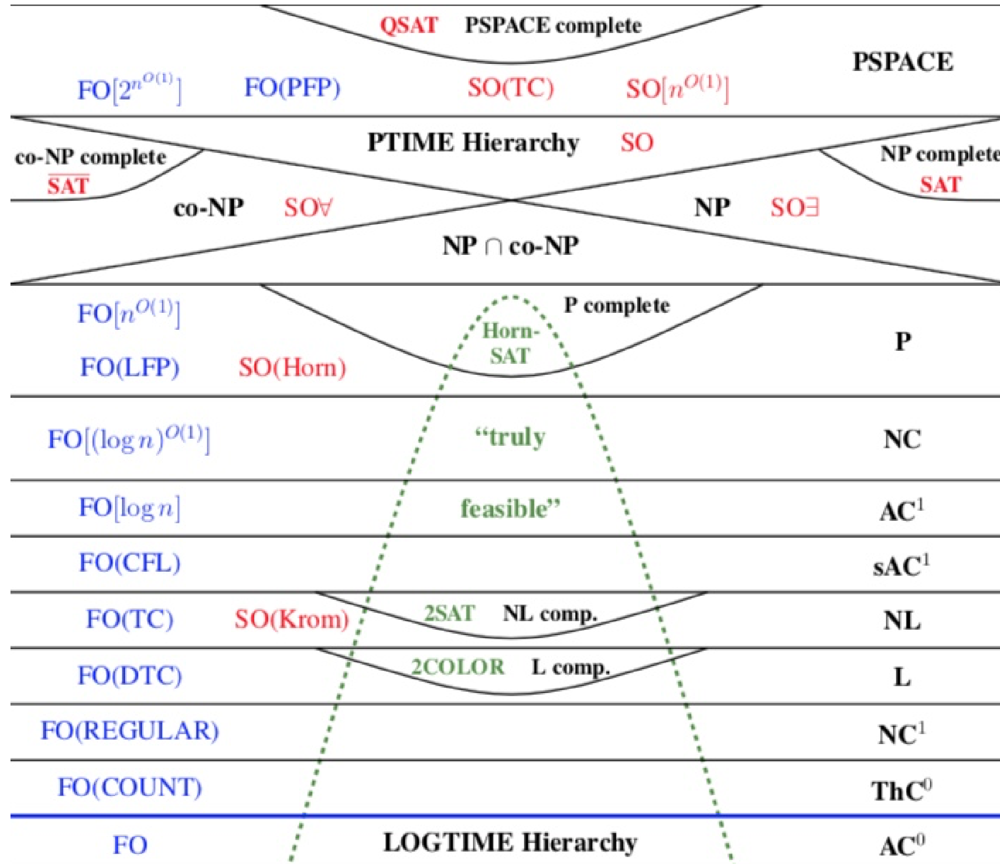

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Is there a logic for $\mathrm{P}\textsc{Time}$?
No idea since 1988.

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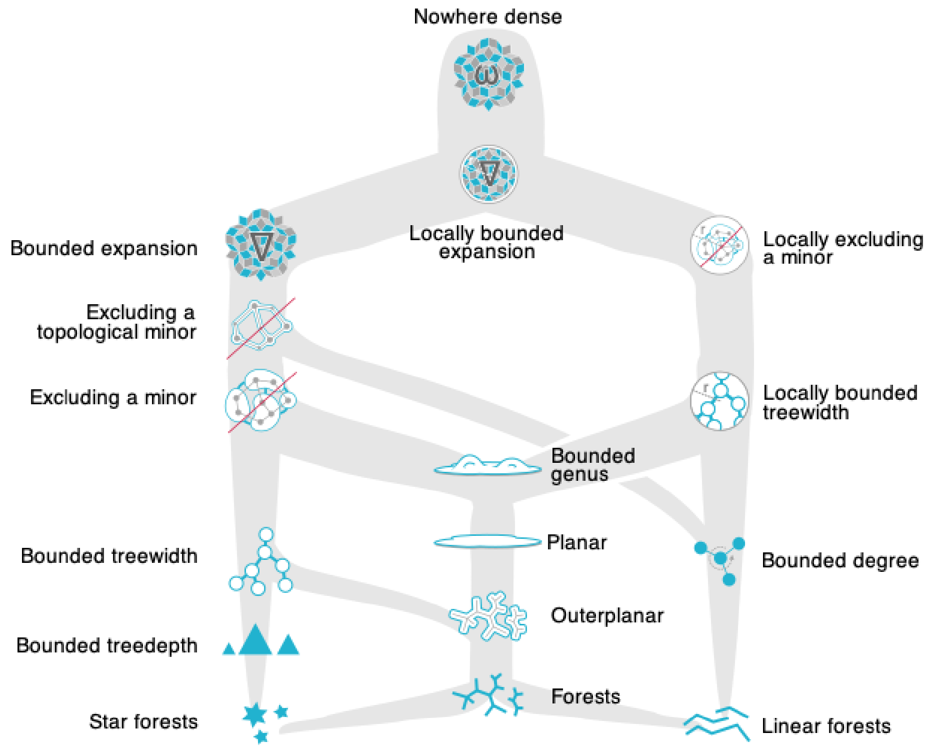

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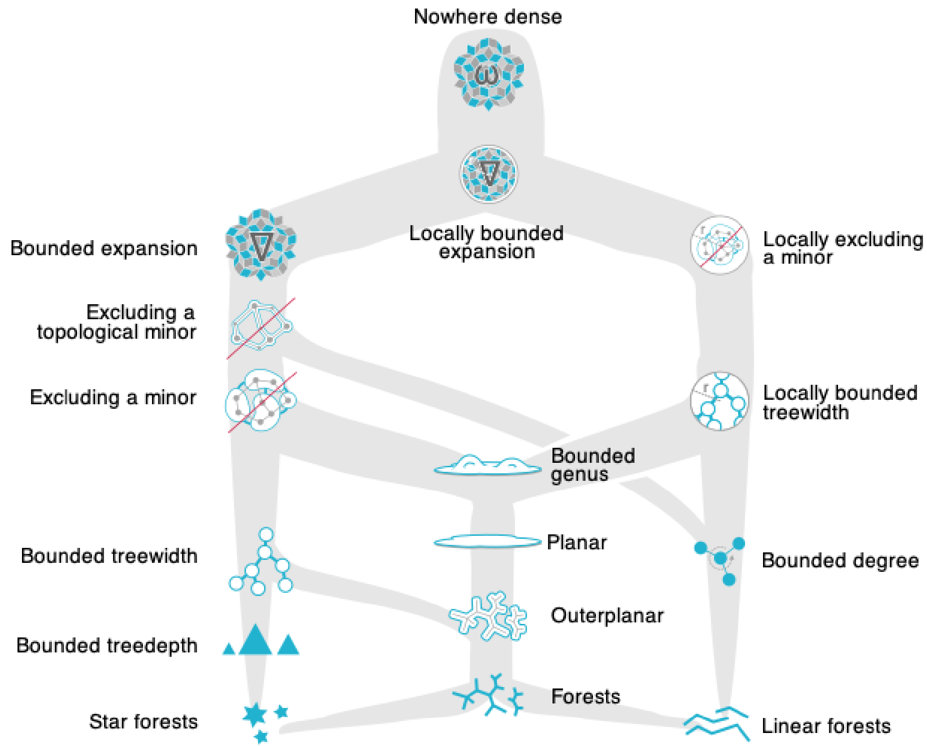

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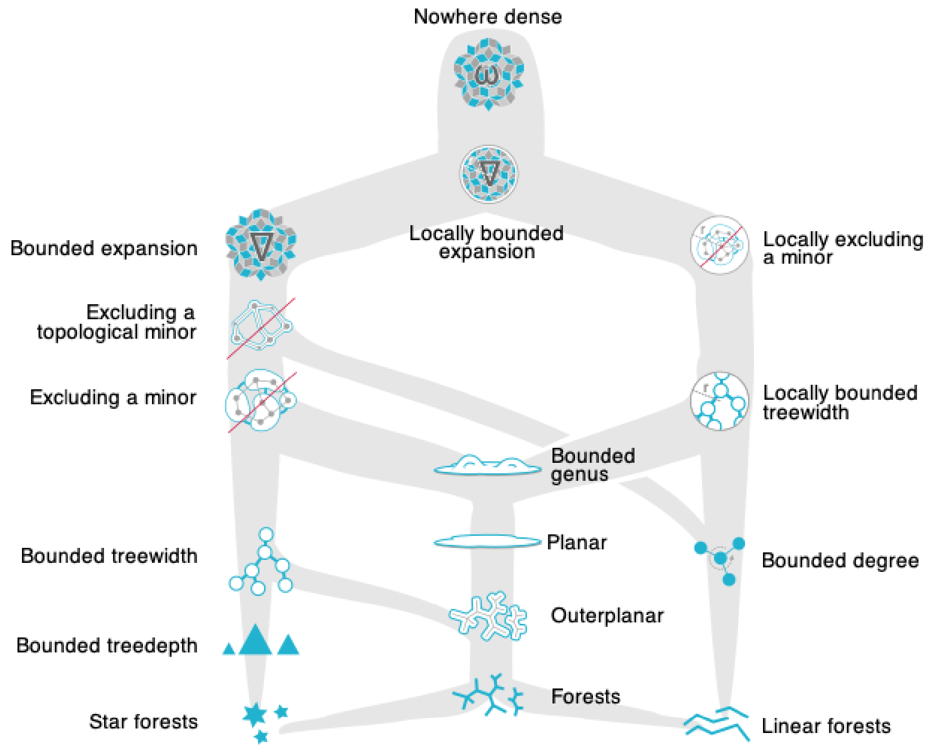

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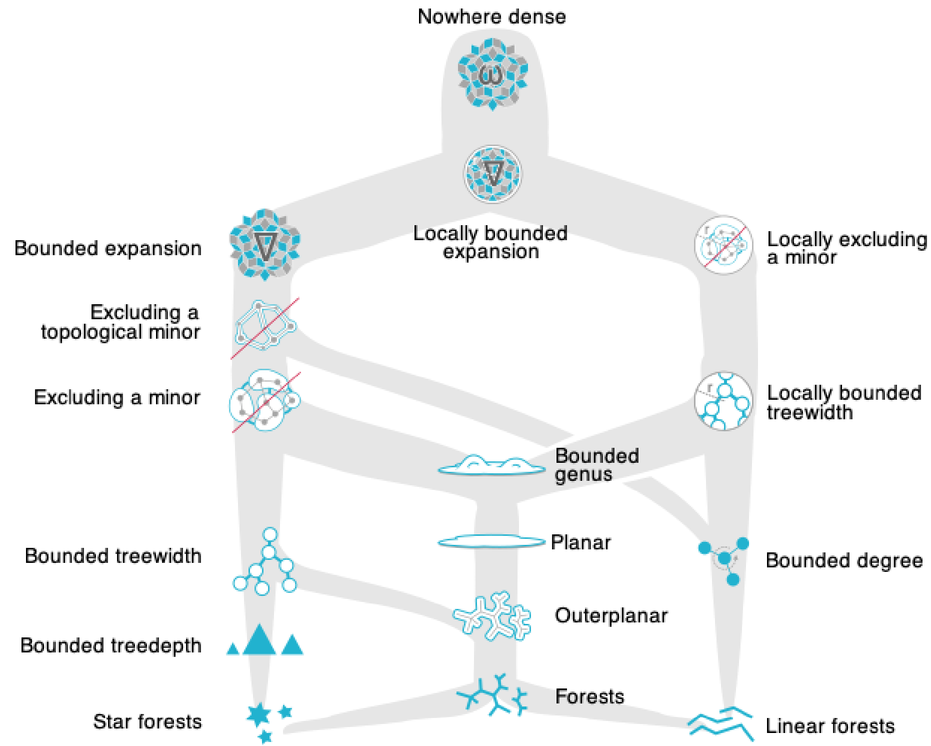

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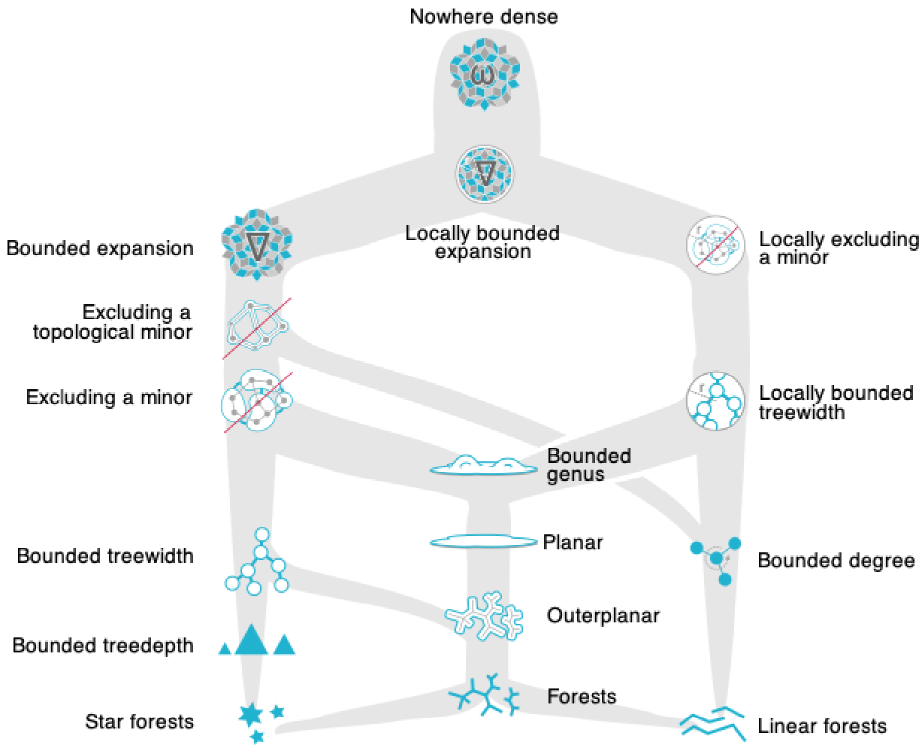

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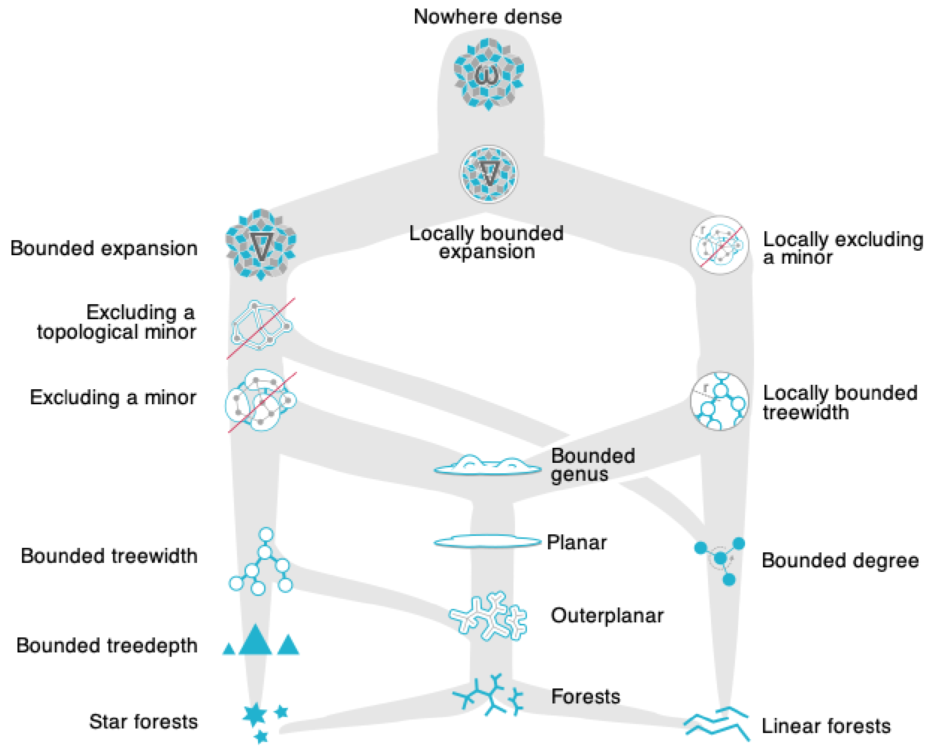
Theorem (Grohe, Kreutzer, Siebertz 2014)

$\mathcal{O}(|\varphi|^{1+\varepsilon})$ for $\mathcal{C}$:= nowhere-dense graphs.



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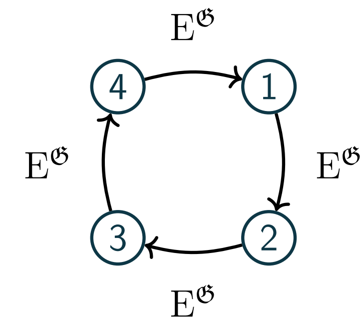

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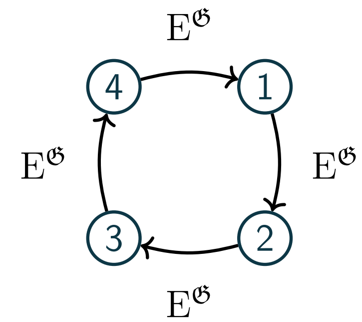

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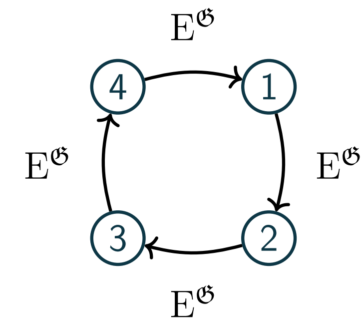

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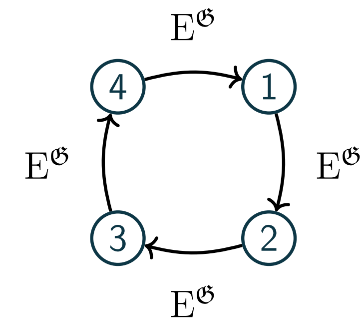

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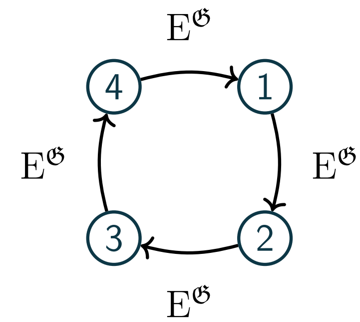

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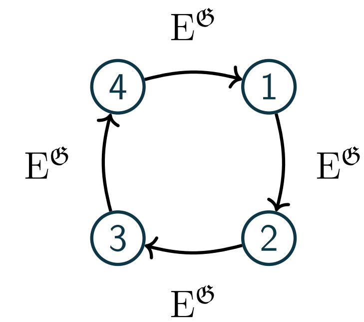
Structures



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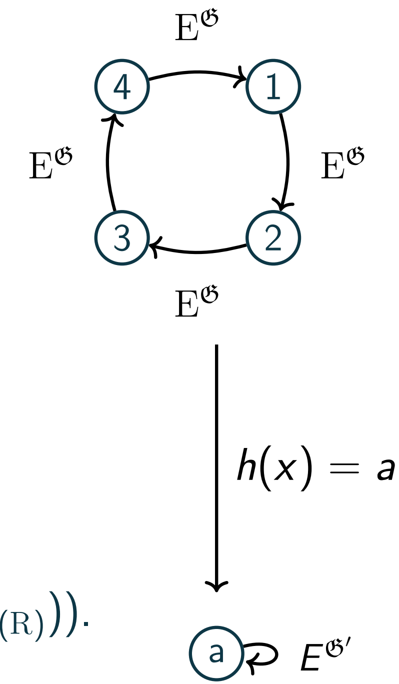

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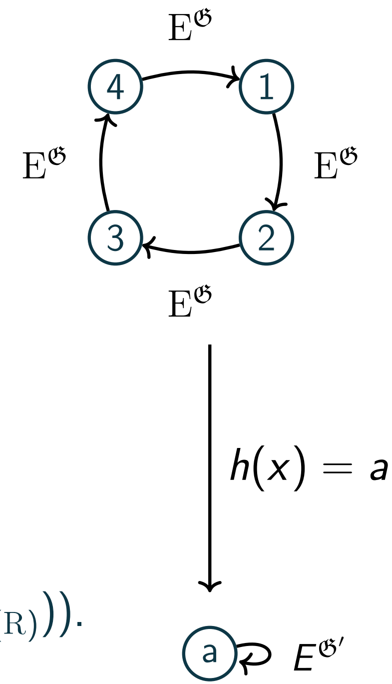

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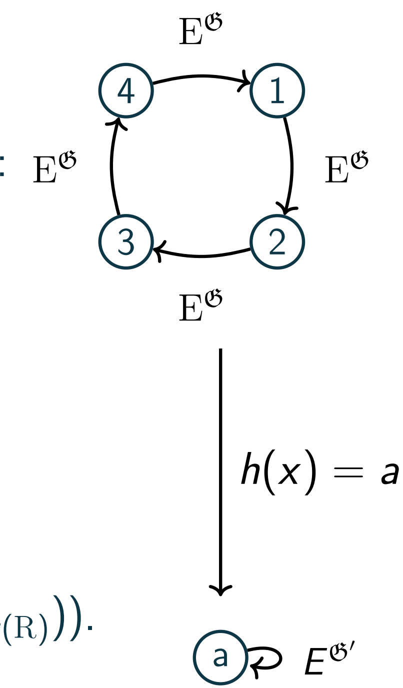

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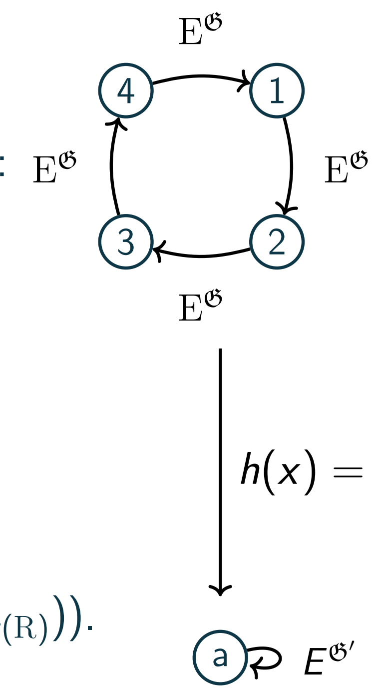

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(we treat $\mathcal{T}$ as extra axioms, note that proofs are finite)

Gödel 1929: $\mathcal{T} \models \varphi$ iff $\mathcal{T} \vdash \varphi$

SAT for FO is Recursively Enumerable

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Ax}{\forall x[P(x)], \forall x[P(x) \to Q(x)], Q(a) \vdash Q(a)} \quad \dfrac{\dfrac{Ax}{\forall x[P(x)], \forall x[P(x) \to Q(x)] \vdash P(a), Q(a)}\,[\neg\vdash]}{\forall x[P(x)], \forall x[P(x) \to Q(x)], \neg P(a) \vdash Q(a)}}{\forall x[P(x)], \forall x[P(x) \to Q(x)], \neg P(a) \lor Q(a) \vdash Q(a)}\,[\lor\vdash]}{\forall x[P(x)], \forall x[P(x) \to Q(x)], P(a) \to Q(a) \vdash Q(a)}\,[\to\vdash\ r.w.]}{\forall x[P(x)], \forall x[P(x) \to Q(x)] \vdash Q(a)}\,[\forall\vdash]}{\forall x[P(x)], \forall x[P(x) \to Q(x)] \vdash \forall x[Q(x)}\,[\vdash\forall]$$

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Use case: Showing inexpressivity

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Towards a contradiction suppose $\mathcal{T}$ is unsatisfiable. So $\mathcal{T} \models \bot$. By (1) there is a finite $\mathcal{T}_0 \subseteq \mathcal{T}$ such that $\mathcal{T}_0 \models \bot$.

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Towards a contradiction suppose $\mathcal{T}$ is unsatisfiable. So $\mathcal{T} \models \bot$.

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Thus $\mathcal{T}$ has an unsatisfiable finite subset ($\mathcal{T}_0$). A contradiction!

Employing compactness I: Reachability in $\{E\}$-structures

Employing compactness I: Reachability in $\{\mathrm{E}\}$-structures

The general proof scheme to show that the property $\mathcal{P}$ is not FO-definable.

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Ad absurdum suppose that φ defines $\mathcal{P}$.

Employing compactness I: Reachability in $\{E\}$-structures

The general proof scheme to show that the property $\mathcal{P}$ is not FO-definable.

Ad absurdum suppose that φ defines $\mathcal{P}$. $\rightsquigarrow$ Manufacture a theory $\mathcal{T}$ containing φ. $\rightsquigarrow$

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$$\varphi_0^{\mathrm{reach(a,b)}} := \mathrm{a} = \mathrm{b},\ \varphi_1^{\mathrm{reach(a,b)}} := \mathrm{E}(\mathrm{a}, \mathrm{b}), \varphi_k^{\mathrm{reach(a,b)}} := \exists x_1 \ldots \exists x_{k-1}\ \mathrm{E}(\mathrm{a}, x_1) \land \land_{i=1}^{k-2} \mathrm{E}(x_i, x_{i+1}) \land \mathrm{E}(x_{k-1}, \mathrm{b})$$

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Assume that there is such φ, and let $\mathcal{T}$ be

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The general proof scheme to show that the property $\mathcal{P}$ is not FO-definable.

Ad absurdum suppose that φ defines $\mathcal{P}$. $\rightsquigarrow$ Manufacture a theory $\mathcal{T}$ containing φ. $\rightsquigarrow$

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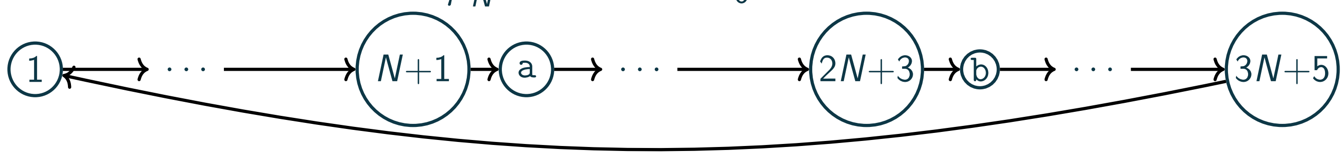

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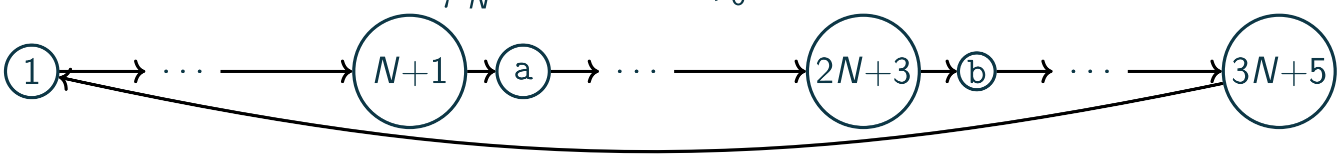

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No info about the finite models!

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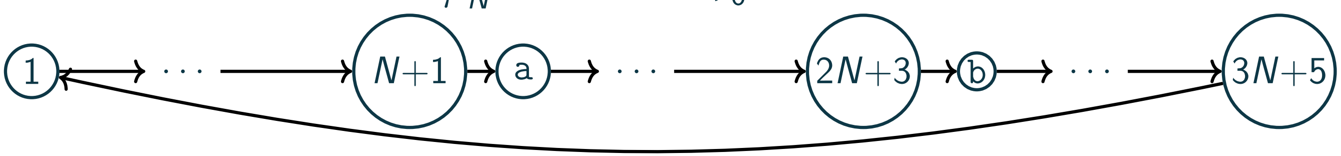

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By $\mathfrak{A} \models \mathcal{T}_1$ we get $\mathfrak{A} \models \varphi$. A contradiction (with the semantics of $\models$)!

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