TECHNISCHE UNIVERSITÄT DRESDEN

Artificial Intelligence, Computational Logic

PROBLEM SOLVING AND SEARCH IN ARTIFICIAL INTELLIGENCE

Lecture 6 ASP II *slides adapted from Torsten Schaub [Gebser et al.(2012)]

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Dresden, 2nd June 2015

Agenda

1. Introduction
2. Uninformed Search versus Informed Search (Best First Search, A* Search, Heuristics)
3. Local Search, Stochastic Hill Climbing, Simulated Annealing
4. Tabu Search
5. Answer-set Programming (ASP)
6. Constraint Satisfaction (CSP)
7. Structural Decomposition Techniques (Tree/Hypertree Decompositions)
8. Evolutionary Algorithms/ Genetic Algorithms

Overview ASP II

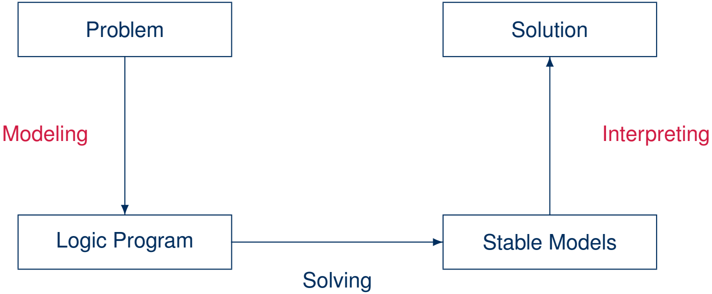
- Modeling
  1. Basic Modeling
  2. Methodology
- Language
  3. Motivation
  4. Core language
  5. Extended language
- Language Extensions
  6. Two kinds of negation
  7. Disjunctive logic programs
  8. Aggregates
- Computational Aspects
  9. Complexity

Modeling: Overview

1. Basic Modeling
2. Methodology

Outline

Modeling and Interpreting



Modeling

- For solving a problem class **C** for a problem instance **I**, encode
  1. the problem instance **I** as a set $P_{\mathbf{I}}$ of facts and
  2. the problem class **C** as a set $P_{\mathbf{C}}$ of rules

  such that the solutions to **C** for **I** can be (polynomially) extracted from the stable models of $P_{\mathbf{I}} \cup P_{\mathbf{C}}$

- $P_{\mathbf{I}}$ is (still) called problem instance
- $P_{\mathbf{C}}$ is often called the problem encoding

- An encoding $P_{\mathbf{C}}$ is uniform, if it can be used to solve all its problem instances
  That is, $P_{\mathbf{C}}$ encodes the solutions to **C** for any set $P_{\mathbf{I}}$ of facts

Outline

Basic methodology

Methodology

Generate and Test (or: Guess and Check)

Generator	Generate potential stable model candidates (typically through non-deterministic constructs)
Tester	Eliminate invalid candidates (typically through integrity constraints)

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Nutshell

Logic program = Data + Generator + Tester (+ Optimizer)

Outline

Satisfiability testing

- Problem Instance: A propositional formula ϕ in CNF
- Problem Class: Is there an assignment of propositional variables to true and false such that a given formula ϕ is true

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- Example: Consider formula

$$(a \vee \neg b) \wedge (\neg a \vee b)$$

- Logic Program:

Generator	**Tester**	**Stable models**
$\{a, b\} \leftarrow$	$\leftarrow \quad not\ a, b$	$X_1 = \{a, b\}$
	$\leftarrow \quad a, not\ b$	$X_2 = \{\}$

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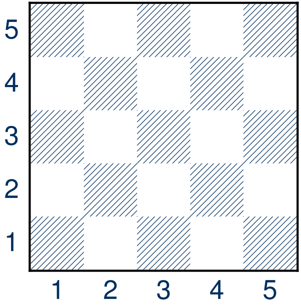
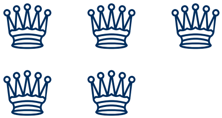
Outline

The n-Queens Problem



- Place n queens on an $n \times n$ chess board
- Queens must not attack one another



Defining the Field

queens.lp

```
row(1..n).
col(1..n).
```

- Create file `queens.lp`
- Define the field
  - n rows
  - n columns

Defining the Field

Running . . .

```
$ gringo queens.lp --const n=5 | clasp
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5)
SATISFIABLE

Models       : 1
Time         : 0.000
  Prepare    : 0.000
  Prepro.    : 0.000
  Solving    : 0.000
```

Placing some Queens

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I), col(J) }.
```

- Guess a solution candidate
  by placing some queens on the board

Placing some Queens

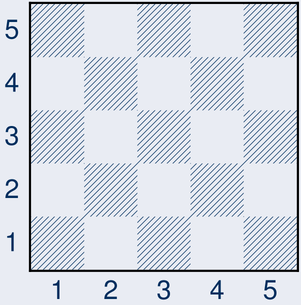
Running ...

```
$ gringo queens.lp --const n=5 | clasp 3
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5)
Answer: 2
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) queen(1,1)
Answer: 3
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) queen(2,1)
SATISFIABLE

Models      : 3+
...
```

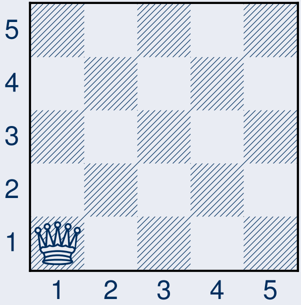
Placing some Queens: Answer 1

Answer 1



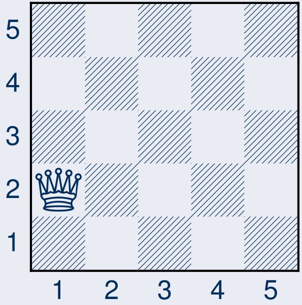
Placing some Queens: Answer 2

Answer 2



Placing some Queens: Answer 3

Answer 3



Placing n Queens

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I), col(J) }.
:- not n { queen(I,J) } n.
```

- Place exactly n queens on the board

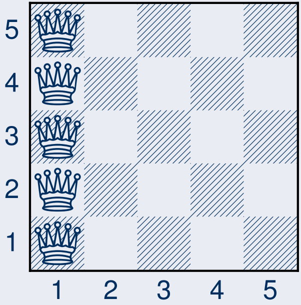
Placing n Queens

Running . . .

```
$ gringo queens.lp --const n=5 | clasp 2
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(5,1) queen(4,1) queen(3,1) \
queen(2,1) queen(1,1)
Answer: 2
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(1,2) queen(4,1) queen(3,1) \
queen(2,1) queen(1,1)
...
```

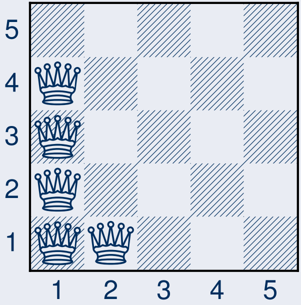
Placing n Queens: Answer 1

Answer 1



Placing n Queens: Answer 2

Answer 2



Horizontal and Vertical Attack

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I), col(J) }.
:- not n { queen(I,J) } n.
:- queen(I,J), queen(I,J'), J != J'.
```

- Forbid horizontal attacks

Horizontal and Vertical Attack

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I), col(J) }.
:- not n { queen(I,J) } n.
:- queen(I,J), queen(I,J'), J != J'.
:- queen(I,J), queen(I',J), I != I'.
```

- Forbid horizontal attacks
- Forbid vertical attacks

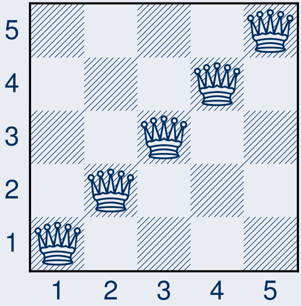
Horizontal and Vertical Attack

Running . . .

```
$ gringo queens.lp --const n=5 | clasp
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(5,5) queen(4,4) queen(3,3) \
queen(2,2) queen(1,1)
...
```

Horizontal and Vertical Attack: Answer 1

Answer 1



Diagonal Attack

queens.lp

```
row(1..n).
col(1..n).
{ queen(I,J) : row(I), col(J) }.
:- not n { queen(I,J) } n.
:- queen(I,J), queen(I,J'), J != J'.
:- queen(I,J), queen(I',J), I != I'.
:- queen(I,J), queen(I',J'), (I,J) != (I',J'), I-J == I'-J'.
:- queen(I,J), queen(I',J'), (I,J) != (I',J'), I+J == I'+J'.
```

- Forbid diagonal attacks

Diagonal Attack

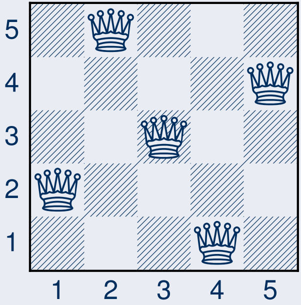
Running . . .

```
$ gringo queens.lp --const n=5 | clasp
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(4,5) queen(1,4) queen(3,3) queen(5,2) queen(2,1)
SATISFIABLE

Models      : 1+
Time        : 0.000
  Prepare   : 0.000
  Prepro.   : 0.000
  Solving   : 0.000
```

Diagonal Attack: Answer 1

Answer 1



Optimizing

queens-opt.lp

```
1 { queen(I,1..n) } 1 :- I = 1..n.
1 { queen(1..n,J) } 1 :- J = 1..n.
 :- 2 { queen(D-J,J) }, D =   2..2*n.
 :- 2 { queen(D+J,J) }, D = 1-n..n-1.
```

- Encoding can be optimized
- Much faster to solve

And sometimes it rocks

```
$ clingo -c n=5000 queens-opt-diag.lp -config=jumpy -q -stats=3
clingo version 4.1.0
Solving...
SATISFIABLE

Models       : 1+
Time         : 3758.143s (Solving: 1905.22s 1st Model: 1896.20s Unsat: 0.00s)
CPU Time     : 3758.320s

Choices      : 288594554
Conflicts    : 3442   (Analyzed: 3442)
Restarts     : 17     (Average: 202.47 Last: 3442)
Model-Level  : 7594728.0
Problems     : 1      (Average Length: 0.00 Splits: 0)
Lemmas       : 3442   (Deleted: 0)
  Binary     : 0      (Ratio:   0.00%)
  Ternary    : 0      (Ratio:   0.00%)
  Conflict   : 3442   (Average Length: 229056.5 Ratio: 100.00%)
  Loop       : 0      (Average Length:    0.0 Ratio:   0.00%)
  Other      : 0      (Average Length:    0.0 Ratio:   0.00%)

Atoms        : 75084857 (Original: 75069989 Auxiliary: 14868)
Rules        : 100129956 (1: 50059992/100090100 2: 39990/29856 3: 10000/10000)
Bodies       : 25090103
Equivalences : 125029999 (Atom=Atom: 50009999 Body=Body: 0 Other: 75020000)
Tight        : Yes
Variables    : 25024868 (Eliminated: 11781 Frozen: 25000000)
Constraints  : 66664  (Binary: 35.6% Ternary:  0.0% Other: 64.4%)

Backjumps    : 3442   (Average: 681.19 Max: 169512 Sum: 2344658)
  Executed   : 3442   (Average: 681.19 Max: 169512 Sum: 2344658 Ratio: 100.00%)
  Bounded    : 0      (Average:  0.00 Max:    0 Sum:      0 Ratio:   0.00%)
```

Outline

Traveling Salesperson

Traveling Salesperson

```
node(1..6).

edge(1,(2;3;4)).  edge(2,(4;5;6)).  edge(3,(1;4;5)).
edge(4,(1;2)).    edge(5,(3;4;6)).  edge(6,(2;3;5)).
```

Traveling Salesperson

```
node(1..6).

edge(1,(2;3;4)).  edge(2,(4;5;6)).  edge(3,(1;4;5)).
edge(4,(1;2)).    edge(5,(3;4;6)).  edge(6,(2;3;5)).

cost(1,2,2).  cost(1,3,3).  cost(1,4,1).
cost(2,4,2).  cost(2,5,2).  cost(2,6,4).
cost(3,1,3).  cost(3,4,2).  cost(3,5,2).
cost(4,1,1).  cost(4,2,2).
cost(5,3,2).  cost(5,4,2).  cost(5,6,1).
cost(6,2,4).  cost(6,3,3).  cost(6,5,1).
```

Traveling Salesperson

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node(1..6).




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cost(2,4,2).  cost(2,5,2).  cost(2,6,4).
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cost(4,1,1).  cost(4,2,2).
cost(5,3,2).  cost(5,4,2).  cost(5,6,1).
cost(6,2,4).  cost(6,3,3).  cost(6,5,1).

edge(X,Y) :- cost(X,Y,_).
```

Traveling Salesperson

```
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(X).
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(Y).
```

Traveling Salesperson

```
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(X).
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(Y).

reached(Y) :- cycle(1,Y).
reached(Y) :- cycle(X,Y), reached(X).
```

Traveling Salesperson

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1 { cycle(X,Y) : edge(X,Y) } 1 :- node(X).
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(Y).

reached(Y) :- cycle(1,Y).
reached(Y) :- cycle(X,Y), reached(X).

:- node(Y), not reached(Y).
```

Traveling Salesperson

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1 { cycle(X,Y) : edge(X,Y) } 1 :- node(X).
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(Y).

reached(Y) :- cycle(1,Y).
reached(Y) :- cycle(X,Y), reached(X).

:- node(Y), not reached(Y).

#minimize { C,X,Y : cycle(X,Y), cost(X,Y,C) }.
```

Language: Overview

Outline

Basic language extensions

- The expressiveness of a language can be enhanced by introducing new constructs
- To this end, we must address the following issues:
  - What is the syntax of the new language construct?
  - What is the semantics of the new language construct?
  - How to implement the new language construct?

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  - What is the semantics of the new language construct?
  - How to implement the new language construct?
- A way of providing semantics is to furnish a translation removing the new constructs, eg. classical negation
- This translation might also be used for implementing the language extension

Outline

Outline

Integrity constraint

- Idea Eliminate unwanted solution candidates
- Syntax An integrity constraint is of the form

$$\leftarrow a_1, \ldots, a_m, \mathit{not}\ a_{m+1}, \ldots, \mathit{not}\ a_n$$

  where $0 \leq m \leq n$ and each a_i is an atom for $1 \leq i \leq n$
- Example `:- edge(3,7), color(3,red), color(7,red).`

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- Example `:- edge(3,7), color(3,red), color(7,red).`
- Embedding The above integrity constraint can be turned into the normal rule

$$x \leftarrow a_1, \ldots, a_m, \mathit{not}\ a_{m+1}, \ldots, \mathit{not}\ a_n, \mathit{not}\ x$$

  where x is a new symbol, that is, $x \notin \mathcal{A}$.

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  where x is a new symbol, that is, $x \notin \mathcal{A}$.
- Another example $P = \{a \leftarrow \mathit{not}\ b,\ b \leftarrow \mathit{not}\ a\}$
  versus $P' = P \cup \{\leftarrow a\}$ and $P'' = P \cup \{\leftarrow \mathit{not}\ a\}$

Outline

Choice rule

- Idea Choices over subsets
- Syntax A choice rule is of the form

$$\{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, \mathit{not}\ a_{n+1}, \ldots, \mathit{not}\ a_o$$

where $0 \leq m \leq n \leq o$ and each a_i is an atom for $1 \leq i \leq o$

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- Example `{ buy(pizza); buy(wine); buy(corn) } :- at(grocery).`

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- Informal meaning If the body is satisfied by the stable model at hand, then any subset of $\{a_1, \dots, a_m\}$ can be included in the stable model
- Example `{ buy(pizza); buy(wine); buy(corn) } :- at(grocery).`
- Another Example $P = \{\{a\} \leftarrow b,\ b \leftarrow\}$ has two stable models: $\{b\}$ and $\{a, b\}$

Embedding in normal rules

- A choice rule of form

$$\{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, \mathit{not}\ a_{n+1}, \ldots, \mathit{not}\ a_o$$

can be translated into $2m + 1$ normal rules

$$\begin{array}{rcl} b & \leftarrow & a_{m+1}, \ldots, a_n, \mathit{not}\ a_{n+1}, \ldots, \mathit{not}\ a_o \end{array}$$

$$\begin{array}{rclcrcl} a_1 & \leftarrow & b, \mathit{not}\ a'_1 & \ldots & a_m & \leftarrow & b, \mathit{not}\ a'_m \\ a'_1 & \leftarrow & \mathit{not}\ a_1 & \ldots & a'_m & \leftarrow & \mathit{not}\ a_m \end{array}$$

by introducing new atoms $b, a'_1, \ldots, a'_m$.

Embedding in normal rules

- A choice rule of form

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can be translated into $2m + 1$ normal rules

$$\begin{array}{lcl} b & \leftarrow & a_{m+1}, \ldots, a_n, \mathit{not}\ a_{n+1}, \ldots, \mathit{not}\ a_o \\ a_1 & \leftarrow & b, \mathit{not}\ a'_1 \quad \ldots \quad a_m \leftarrow b, \mathit{not}\ a'_m \\ a'_1 & \leftarrow & \mathit{not}\ a_1 \quad\ \ \ldots \quad a'_m \leftarrow \mathit{not}\ a_m \end{array}$$

by introducing new atoms $b, a'_1, \ldots, a'_m$.

Embedding in normal rules

- A choice rule of form

$$\{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, \mathit{not}\ a_{n+1}, \ldots, \mathit{not}\ a_o$$

can be translated into $2m + 1$ normal rules

$$\begin{array}{rcllrcl} b & \leftarrow & \multicolumn{5}{l}{a_{m+1}, \ldots, a_n, \mathit{not}\ a_{n+1}, \ldots, \mathit{not}\ a_o} \\ a_1 & \leftarrow & b, \mathit{not}\ a'_1 & \ldots & a_m & \leftarrow & b, \mathit{not}\ a'_m \\ a'_1 & \leftarrow & \mathit{not}\ a_1 & \ldots & a'_m & \leftarrow & \mathit{not}\ a_m \end{array}$$

by introducing new atoms $b, a'_1, \ldots, a'_m$.

Outline

Cardinality rule

- Idea Control (lower) cardinality of subsets
- Syntax A cardinality rule is the form

$$a_0 \leftarrow l \; \{ \; a_1, \ldots, a_m, not \; a_{m+1}, \ldots, not \; a_n \; \}$$

where $0 \leq m \leq n$ and each a_i is an atom for $1 \leq i \leq n$;
l is a non-negative integer.

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  where $0 \leq m \leq n$ and each a_i is an atom for $1 \leq i \leq n$;
  l is a non-negative integer.

- Informal meaning The head atom belongs to the stable model, if at least l elements of the body are included in the stable model
- Note l acts as a lower bound on the body

Cardinality rule

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- Syntax A cardinality rule is the form

$$a_0 \leftarrow l \; \{ \; a_1, \ldots, a_m, not \; a_{m+1}, \ldots, not \; a_n \; \}$$

  where $0 \leq m \leq n$ and each a_i is an atom for $1 \leq i \leq n$;
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- Another Example $P = \{a \leftarrow 1\{b, c\}, \; b \leftarrow\}$ has stable model $\{a, b\}$

Embedding in normal rules

- Replace each cardinality rule

$$a_0 \leftarrow l \; \{ \; a_1, \ldots, a_m, not \; a_{m+1}, \ldots, not \; a_n \; \}$$

by $a_0 \leftarrow ctr(1, l)$

where atom $ctr(i, j)$ represents the fact that at least j of the literals having an equal or greater index than i, are in a stable model

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- Program $\{a \leftarrow,\ c \leftarrow 1\ \{a, b\}\}$ has the stable model $\{a, c\}$
- Translating the cardinality rule yields the rules

$$
\begin{array}{rcl}
a & \leftarrow &
\end{array}
\qquad
\begin{array}{rcl}
c & \leftarrow & ctr(1,1) \\
ctr(1,2) & \leftarrow & ctr(2,1), a \\
ctr(1,1) & \leftarrow & ctr(2,1) \\
ctr(2,2) & \leftarrow & ctr(3,1), b \\
ctr(2,1) & \leftarrow & ctr(3,1) \\
ctr(1,1) & \leftarrow & ctr(2,0), a \\
ctr(1,0) & \leftarrow & ctr(2,0) \\
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ctr(3,0) & \leftarrow &
\end{array}
$$

having stable model $\{a, ctr(3,0), ctr(2,0), ctr(1,0), ctr(1,1), c\}$

. . . and vice versa

- A normal rule

$$a_0 \leftarrow a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n$$

can be represented by the cardinality rule

$$a_0 \leftarrow n\ \{a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n\}$$

Cardinality rules with upper bounds

- A rule of the form

$$a_0 \leftarrow l \; \{ \; a_1, \ldots, a_m, \mathit{not} \; a_{m+1}, \ldots, \mathit{not} \; a_n \; \} \; u \tag{1}$$

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$$\begin{array}{rcl} a_0 & \leftarrow & b, \mathit{not} \; c \\ b & \leftarrow & l \; \{ \; a_1, \ldots, a_m, \mathit{not} \; a_{m+1}, \ldots, \mathit{not} \; a_n \; \} \\ c & \leftarrow & u{+}1 \; \{ \; a_1, \ldots, a_m, \mathit{not} \; a_{m+1}, \ldots, \mathit{not} \; a_n \; \} \end{array}$$

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- Note The single constraint in the body of the cardinality rule (1) is referred to as a cardinality constraint

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- Syntax A cardinality constraint is of the form

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- In other words, if

$$l \leq |\,(\{a_1, \ldots, a_m\} \cap X) \cup (\{a_{m+1}, \ldots, a_n\} \setminus X)\,| \leq u$$

Cardinality constraints as heads

- A rule of the form

  $$l\ \{a_1, \ldots, a_m, not\ a_{m+1}, \ldots, not\ a_n\}\ u \leftarrow a_{n+1}, \ldots, a_o, not\ a_{o+1}, \ldots, not\ a_p$$

  where $0 \leq m \leq n \leq o \leq p$ and each a_i is an atom for $1 \leq i \leq p$;
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- Example `1{ color(v42,red); color(v42,green); color(v42,blue) }1.`

Full-fledged cardinality rules

- A rule of the form

$$l_0 \; S_0 \; u_0 \leftarrow l_1 \; S_1 \; u_1, \ldots, l_n \; S_n \; u_n$$

where for $0 \leq i \leq n$ each $l_i \; S_i \; u_i$

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where a, b_i, c_i are new symbols (and $S_0^+ = \{p \in \mathcal{A} \mid p \in S_0\}$)

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Outline

Weight rule

- Syntax A weight rule is the form

$$a_0 \leftarrow l \; \{ \; w_1 : a_1, \ldots, w_m : a_m, w_{m+1} : not \; a_{m+1}, \ldots, w_n : not \; a_n \; \}$$

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- A weighted literal $w_i : \ell_i$ associates each literal ℓ_i with a weight w_i

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- Note A cardinality rule is a weight rule where $w_i = 1$ for $0 \leq i \leq n$

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- Example

```
10 { 4:course(db); 6:course(ai); 8:course(project); 3:course(xml) } 20
```

Outline

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3 Motivation

4 Core language
- Integrity constraint
- Choice rule
- Cardinality rule
- Weight rule

5 Extended language
- Conditional literal
- Optimization statement

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- Syntax A conditional literal is of the form

$$\ell : \ell_1, \ldots, \ell_n$$

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- Informal meaning A conditional literal can be regarded as the list of elements in the set $\{\ell \mid \ell_1, \ldots, \ell_n\}$
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Outline

Optimization statement

- Idea Express (multiple) cost functions subject to minimization and/or maximization
- Syntax A minimize statement is of the form

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  Priority levels, p_i, allow for representing lexicographically ordered minimization objectives
- Meaning A minimize statement is a directive that instructs the ASP solver to compute optimal stable models by minimizing a weighted sum of elements

Optimization statement

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- Example When configuring a computer, we may want to maximize hard disk capacity, while minimizing price

```
#maximize { 250@1:hd(1), 500@1:hd(2), 750@1:hd(3), 1000@1:hd(4) }.
#minimize { 30@2:hd(1), 40@2:hd(2), 60@2:hd(3), 80@2:hd(4) }.
```

  The priority levels indicate that (minimizing) price is more important than (maximizing) capacity

Language Extensions: Overview

6. Two kinds of negation
7. Disjunctive logic programs
8. Aggregates

Outline

Motivation

- Classical versus default negation
  - Symbol $\neg$ and *not*

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    - $cross \leftarrow \neg train$
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$$P^{\neg} = \{a \leftarrow b, \neg b \mid a \in (\mathcal{A} \cup \overline{\mathcal{A}}), b \in \mathcal{A}\}$$

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- A set X of atoms is a stable model of a program P over $\mathcal{A} \cup \overline{\mathcal{A}}$, if X is a stable model of $P \cup P^{\neg}$

An example

- The program

$$P = \{a \leftarrow not\ b,\ b \leftarrow not\ a\} \cup \{c \leftarrow b,\ \neg c \leftarrow b\}$$

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- The stable models of P are given by the ones of $P \cup P^{\neg}$, viz $\{a\}$

Properties

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- Note Strictly speaking, an inconsistemt set like $\mathcal{A} \cup \overline{\mathcal{A}}$ is not a model
- For a logic program P over $\mathcal{A} \cup \overline{\mathcal{A}}$, exactly one of the following two cases applies:
  1. All stable models of P are consistent or
  2. $X = \mathcal{A} \cup \overline{\mathcal{A}}$ is the only stable model of P

Train spotting

- $P_1 = \{cross \leftarrow not\ train\}$
- $P_2 = \{cross \leftarrow \neg train\}$
- $P_3 = \{cross \leftarrow \neg train,\ \neg train \leftarrow\}$
- $P_4 = \{cross \leftarrow \neg train,\ \neg train \leftarrow,\ \neg cross \leftarrow\}$
- $P_5 = \{cross \leftarrow \neg train,\ \neg train \leftarrow not\ train\}$
- $P_6 = \{cross \leftarrow \neg train,\ \neg train \leftarrow not\ train,\ \neg cross \leftarrow\}$

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  - stable model: $\{cross\}$

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Train spotting

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Train spotting

- $P_3 = \{cross \leftarrow \neg train,\ \neg train \leftarrow\}$
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Train spotting

- $P_4 = \{cross \leftarrow \neg train,\ \neg train \leftarrow,\ \neg cross \leftarrow\}$

Train spotting

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  - stable model: $\{cross, \neg cross, train, \neg train\}$

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- $P_5 = \{cross \leftarrow \neg train,\ \neg train \leftarrow not\ train\}$

Train spotting

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Train spotting

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Train spotting

- $P_6 = \{cross \leftarrow \neg train,\ \neg train \leftarrow not\ train,\ \neg cross \leftarrow\}$
  - no stable model

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  - stable model: $\{cross, \neg train\}$
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- Given a program P over $\mathcal{A}$, consider the program

$$
\begin{aligned}
\widetilde{P} \quad = \quad & \{r \in P \mid head(r) \neq not\ a\} \\
& \cup \{\leftarrow body(r) \cup \{not\ \widetilde{a}\} \mid r \in P \text{ and } head(r) = not\ a\} \\
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- A set X of atoms is a stable model of a program P (with default negation in rule heads) over $\mathcal{A}$,
  if $X = Y \cap \mathcal{A}$ for some stable model Y of $\widetilde{P}$ over $\mathcal{A} \cup \widetilde{\mathcal{A}}$

Outline

Disjunctive logic programs

- A disjunctive rule, r, is of the form

$$a_1\ ; \ldots\ ; a_m \leftarrow a_{m+1}, \ldots, a_n, not\ a_{n+1}, \ldots, not\ a_o$$

  where $0 \leq m \leq n \leq o$ and each a_i is an atom for $0 \leq i \leq o$
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- A disjunctive logic program is a finite set of disjunctive rules
- Notation

$$\begin{array}{rcl} \mathit{head}(r) & = & \{a_1, \ldots, a_m\} \\ \mathit{body}(r) & = & \{a_{m+1}, \ldots, a_n, \mathit{not}\ a_{n+1}, \ldots, \mathit{not}\ a_o\} \\ \mathit{body}(r)^+ & = & \{a_{m+1}, \ldots, a_n\} \\ \mathit{body}(r)^- & = & \{a_{n+1}, \ldots, a_o\} \\ \mathit{atom}(P) & = & \bigcup_{r \in P} \left(\mathit{head}(r) \cup \mathit{body}(r)^+ \cup \mathit{body}(r)^-\right) \\ \mathit{body}(P) & = & \{\mathit{body}(r) \mid r \in P\} \end{array}$$

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- A program is called positive if $\mathit{body}(r)^- = \emptyset$ for all its rules

Stable models

- Positive programs
  - A set X of atoms is closed under a positive program P iff for any $r \in P$, $head(r) \cap X \neq \emptyset$ whenever $body(r)^+ \subseteq X$
    - X corresponds to a model of P (seen as a formula)
  - The set of all $\subseteq$-minimal sets of atoms being closed under a positive program P is denoted by $\min_{\subseteq}(P)$
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A "positive" example

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- The sets $\{a, b\}$, $\{a, c\}$, and $\{a, b, c\}$ are closed under P
- We have $\min_{\subseteq}(P) = \{\{a, b\}, \{a, c\}\}$

Graph coloring (reloaded)

```
node(1..6).

edge(1,(2;3;4)).  edge(2,(4;5;6)).  edge(3,(1;4;5)).
edge(4,(1;2)).    edge(5,(3;4;6)).  edge(6,(2;3;5)).


color(X,r) ; color(X,b) ; color(X,g) :- node(X).


:- edge(X,Y), color(X,C), color(Y,C).
```

Graph coloring (reloaded)

```
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col(r).   col(b).   col(g).

color(X,C) : col(C) :- node(X).

:- edge(X,Y), color(X,C), color(Y,C).
```

More Examples

- $P_1 = \{a \,; b \,; c \leftarrow\}$

More Examples

- $P_1 = \{a \,; b \,; c \leftarrow\}$
  - stable models $\{a\}$, $\{b\}$, and $\{c\}$

More Examples

- $P_2 = \{a\,; b\,; c \leftarrow\,,\ \leftarrow a\}$

More Examples

- $P_2 = \{a\ ; b\ ; c \leftarrow\ ,\ \leftarrow a\}$
  - stable models $\{b\}$ and $\{c\}$

More Examples

- $P_3 = \{a \,; b \,; c \leftarrow \,,\ \leftarrow a \,,\ b \leftarrow c \,,\ c \leftarrow b\}$

More Examples

- $P_3 = \{a\ ; b\ ; c \leftarrow\ ,\ \leftarrow a\ ,\ b \leftarrow c\ ,\ c \leftarrow b\}$
  - stable model $\{b, c\}$

More Examples

- $P_4 = \{a \,; b \leftarrow c \;,\; b \leftarrow not\ a, not\ c \;,\; a \,; c \leftarrow not\ b\}$

More Examples

- $P_4 = \{a\ ; b \leftarrow c\ ,\ b \leftarrow not\ a, not\ c\ ,\ a\ ; c \leftarrow not\ b\}$
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More Examples

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- $P_2 = \{a \,;b \,;c \leftarrow \,,\, \leftarrow a\}$
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- $P_4 = \{a \,;b \leftarrow c \,,\, b \leftarrow \mathit{not}\ a, \mathit{not}\ c \,,\, a \,;c \leftarrow \mathit{not}\ b\}$
  - stable models $\{a\}$ and $\{b\}$

Some properties

- A disjunctive logic program may have zero, one, or multiple stable models
- If X is a stable model of a disjunctive logic program P, then X is a model of P (seen as a formula)
- If X and Y are stable models of a disjunctive logic program P, then $X \not\subset Y$

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- If X is a stable model of a disjunctive logic program P, then X is a model of P (seen as a formula)
- If X and Y are stable models of a disjunctive logic program P, then $X \not\subset Y$
- If $A \in X$ for some stable model X of a disjunctive logic program P, then there is a rule $r \in P$ such that $body(r)^+ \subseteq X$, $body(r)^- \cap X = \emptyset$, and $head(r) \cap X = \{A\}$

An example with variables

$$P \quad = \quad \left\{ \begin{array}{lcl} a(1,2) & \leftarrow & \\ b(X) \,;\, c(Y) & \leftarrow & a(X,Y), not\ c(Y) \end{array} \right\}$$

An example with variables

$$P \quad = \quad \left\{ \begin{array}{lll} a(1,2) & \leftarrow & \\ b(X)\ ; c(Y) & \leftarrow & a(X,Y), \mathit{not}\ c(Y) \end{array} \right\}$$

$$\mathit{ground}(P) \quad = \quad \left\{ \begin{array}{lll} a(1,2) & \leftarrow & \\ b(1)\ ; c(1) & \leftarrow & a(1,1), \mathit{not}\ c(1) \\ b(1)\ ; c(2) & \leftarrow & a(1,2), \mathit{not}\ c(2) \\ b(2)\ ; c(1) & \leftarrow & a(2,1), \mathit{not}\ c(1) \\ b(2)\ ; c(2) & \leftarrow & a(2,2), \mathit{not}\ c(2) \end{array} \right\}$$

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For every stable model X of P, we have

- $a(1,2) \in X$ and
- $\{a(1,1), a(2,1), a(2,2)\} \cap X = \emptyset$

An example with variables

$$
ground(P) \quad = \quad \left\{ \begin{array}{lcl} a(1,2) & \leftarrow & \\ b(1) \,;\, c(1) & \leftarrow & a(1,1), \mathit{not}\ c(1) \\ b(1) \,;\, c(2) & \leftarrow & a(1,2), \mathit{not}\ c(2) \\ b(2) \,;\, c(1) & \leftarrow & a(2,1), \mathit{not}\ c(1) \\ b(2) \,;\, c(2) & \leftarrow & a(2,2), \mathit{not}\ c(2) \end{array} \right\}
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- Consider $X = \{a(1,2), b(1)\}$

An example with variables

$$
ground(P)^X \quad = \quad \left\{ \begin{array}{lcl} a(1,2) & \leftarrow & \\ b(1)\;;c(1) & \leftarrow & a(1,1) \\ b(1)\;;c(2) & \leftarrow & a(1,2) \\ b(2)\;;c(1) & \leftarrow & a(2,1) \\ b(2)\;;c(2) & \leftarrow & a(2,2) \end{array} \right\}
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- Consider $X = \{a(1,2), b(1)\}$

An example with variables

$$ground(P)^X \quad = \quad \left\{ \begin{array}{lcl} a(1,2) & \leftarrow & \\ b(1)\,;c(1) & \leftarrow & a(1,1) \\ b(1)\,;c(2) & \leftarrow & a(1,2) \\ b(2)\,;c(1) & \leftarrow & a(2,1) \\ b(2)\,;c(2) & \leftarrow & a(2,2) \end{array} \right\}$$

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- We get $\min_{\subseteq}(ground(P)^X) = \{\ \{a(1,2), b(1)\},\ \{a(1,2), c(2)\}\ \}$

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- We get $\min_{\subseteq}(ground(P)^X) = \{\ \{a(1,2), b(1)\},\ \{a(1,2), c(2)\}\ \}$
- X is a stable model of P because $X \in \min_{\subseteq}(ground(P)^X)$

An example with variables

$$ground(P) \quad = \quad \left\{ \begin{array}{lcl} a(1,2) & \leftarrow & \\ b(1)\ ; c(1) & \leftarrow & a(1,1), \mathit{not}\ c(1) \\ b(1)\ ; c(2) & \leftarrow & a(1,2), \mathit{not}\ c(2) \\ b(2)\ ; c(1) & \leftarrow & a(2,1), \mathit{not}\ c(1) \\ b(2)\ ; c(2) & \leftarrow & a(2,2), \mathit{not}\ c(2) \end{array} \right\}$$

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- Consider $X = \{a(1,2), c(2)\}$
- We get $\min_{\subseteq}(ground(P)^X) = \{\ \{a(1,2)\}\ \}$
- X is no stable model of P because $X \notin \min_{\subseteq}(ground(P)^X)$

Default negation in rule heads

- Consider disjunctive rules of the form

$$a_1 \; ; \ldots \; ; a_m \; ; not \; a_{m+1} \; ; \ldots \; ; not \; a_n \leftarrow a_{n+1}, \ldots, a_o, not \; a_{o+1}, \ldots, not \; a_p$$

  where $0 \leq m \leq n \leq o \leq p$ and each a_i is an atom for $0 \leq i \leq p$

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- Given a program P over $\mathcal{A}$, consider the program

$$\begin{aligned}\widetilde{P} \quad = \quad & \{head(r)^+ \leftarrow body(r) \cup \{not\ \widetilde{a} \mid a \in head(r)^-\} \mid r \in P\} \\ & \cup \{\widetilde{a} \leftarrow not\ a \mid r \in P \text{ and } a \in head(r)^-\}\end{aligned}$$

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- A set X of atoms is a stable model of a disjunctive program P (with default negation in rule heads) over $\mathcal{A}$, if $X = Y \cap \mathcal{A}$ for some stable model Y of $\widetilde{P}$ over $\mathcal{A} \cup \widetilde{\mathcal{A}}$

An example

- The program

$$P = \{a \; ; \; not \; a \leftarrow\}$$

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- $\widetilde{P}$ has two stable models, $\{a\}$ and $\{\widetilde{a}\}$
- This induces the stable models $\{a\}$ and $\emptyset$ of P

Outline

Motivation

- Aggregates provide a general way to obtain a single value from a collection of input values
- Popular aggregate (functions)
  - average
  - count
  - maximum
  - minimum
  - sum
- Cardinality and weight constraints rely on count and sum aggregates

Syntax

- An aggregate has the form:

$$\alpha \; \{w_1 : a_1, \ldots, w_m : a_m, w_{m+1} : not\ a_{m+1}, \ldots, w_n : not\ a_n\} \prec k$$

  where for $1 \leq i \leq n$
  - α stands for a function mapping multisets over $\mathbb{Z}$ to $\mathbb{Z} \cup \{+\infty, -\infty\}$
  - $\prec$ stands for a relation between $\mathbb{Z} \cup \{+\infty, -\infty\}$ and $\mathbb{Z}$
  - $k \in \mathbb{Z}$
  - a_i are atoms and
  - w_i are integers
- Example $sum \; \{30 : hd(a), \ldots, 50 : hd(m)\} \leq 300$

Semantics

- A (positive) aggregate $\alpha \{w_1 : a_1, \ldots, w_n : a_n\} \prec k$ can be represented by the formula:

$$\bigwedge_{I \subseteq \{1,\ldots,n\}, \alpha\{w_i | i \in I\} \not\prec k} \left(\bigwedge_{i \in I} a_i \rightarrow \bigvee_{i \in \bar{I}} a_i \right)$$

  where $\bar{I} = \{1, \ldots, n\} \setminus I$ and $\not\prec$ is the complement of $\prec$
- Then, $\alpha \{w_1 : a_1, \ldots, w_n : a_n\} \prec k$ is true in X iff the above formula is true in X

Example

- Consider $sum\{1 : p, 1 : q\} \neq 1$
  That is, $a_1 = p$, $a_2 = q$ and $w_1 = 1$, $w_2 = 1$
- Calculemus!

I	$\{w_i \mid i \in I\}$	$\sum\{w_i \mid i \in I\}$	$\sum\{w_i \mid i \in I\} = 1$
$\emptyset$	$\{\}$	0	*false*
$\{1\}$	$\{1\}$	1	*true*
$\{2\}$	$\{1\}$	1	*true*
$\{1, 2\}$	$\{1, 1\}$	2	*false*

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- We get $(p \to q) \wedge (q \to p)$
- Analogously, we obtain $(p \vee q) \wedge \neg(p \wedge q)$ for $sum\{1 : p, 1 : q\} = 1$.

Monotonicity

- Monotone aggregates
  - For instance,
    - $body(r)^+$
    - $sum\{1 : p, 1 : q\} > 1$
  - We get a simpler characterization: $\bigwedge_{I \subseteq \{1,\ldots,n\}, \alpha\{w_i | i \in I\} \not\prec k} \bigvee_{i \in \bar{I}} a_i$
- Anti-monotone aggregates
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- Non-monotone aggregates
  - For instance, $sum\{1 : p, 1 : q\} \neq 1$ is non-monotone.

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- Monotone aggregates
  - For instance,
    - $body(r)^+$
    - $sum\{1:p, 1:q\} > 1$ amounts to $p \wedge q$
  - We get a simpler characterization: $\bigwedge_{I \subseteq \{1,\dots,n\}, \alpha\{w_i \mid i \in I\} \not\prec k} \bigvee_{i \in \bar{I}} a_i$
- Anti-monotone aggregates
  - For instance,
    - $body(r)^-$
    - $sum\{1:p, 1:q\} < 1$ amounts to $\neg p \wedge \neg q$
  - We get a simpler characterization: $\bigwedge_{I \subseteq \{1,\dots,n\}, \alpha\{w_i \mid i \in I\} \not\prec k} \neg \bigwedge_{i \in I} a_i$
- Non-monotone aggregates
  - For instance, $sum\{1:p, 1:q\} \neq 1$ is non-monotone.

Computational Aspects: Overview

9 Complexity

Outline

Complexity

Let a be an atom and X be a set of atoms

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- For a positive normal logic program P:
  - Deciding whether X is the stable model of P is P-complete
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- For a normal logic program P:
  - Deciding whether X is a stable model of P is P-complete
  - Deciding whether a is in a stable model of P is NP-complete
- For a normal logic program P with optimization statements:
  - Deciding whether X is an optimal stable model of P is co-NP-complete
  - Deciding whether a is in an optimal stable model of P is Δ_2^p-complete

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Let a be an atom and X be a set of atoms

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- For a disjunctive logic program P with optimization statements:
  - Deciding whether X is an optimal stable model of P is co-NP^{NP}-complete
  - Deciding whether a is in an optimal stable model of P is Δ_3^p-complete
- For a propositional theory Φ:
  - Deciding whether X is a stable model of Φ is co-NP-complete
  - Deciding whether a is in a stable model of Φ is NP^{NP}-complete

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- See also: `http://potassco.sourceforge.net`