Agenda

1. Introduction
2. Uninformed Search versus Informed Search (Best First Search, A* Search, Heuristics)
3. Local Search, Stochastic Hill Climbing, Simulated Annealing
4. Tabu Search
5. Answer-set Programming (ASP)
6. Constraint Satisfaction (CSP)
7. Structural Decomposition Techniques (Tree/Hypertree Decompositions)
8. Evolutionary Algorithms/ Genetic Algorithms
Overview ASP II

- Modeling
  - 1 Basic Modeling
  - 2 Methodology
- Language
  - 3 Motivation
  - 4 Core language
  - 5 Extended language
- Language Extensions
  - 6 Two kinds of negation
  - 7 Disjunctive logic programs
  - 8 Aggregates
- Computational Aspects
  - 9 Complexity
Modeling: Overview

1. Basic Modeling
2. Methodology
Outline

1. Basic Modeling
2. Methodology
Modeling and Interpreting

Problem

Modeling

Logic Program

Solving

Solution

Interpreting

Stable Models
Modeling

- For solving a problem class $C$ for a problem instance $I$, encode
  
  1. the problem instance $I$ as a set $P_I$ of facts and
  2. the problem class $C$ as a set $P_C$ of rules

  such that the solutions to $C$ for $I$ can be (polynomially) extracted from the stable models of $P_I \cup P_C$

- $P_I$ is (still) called problem instance
- $P_C$ is often called the problem encoding

- An encoding $P_C$ is uniform, if it can be used to solve all its problem instances
  That is, $P_C$ encodes the solutions to $C$ for any set $P_I$ of facts
Outline

1. Basic Modeling
2. Methodology
Basic methodology

Methodology

Generate and Test (or: Guess and Check)

Generator
Generate potential stable model candidates
(typically through non-deterministic constructs)

Tester
Eliminate invalid candidates
(typically through integrity constraints)
## Basic methodology

<table>
<thead>
<tr>
<th>Methodology</th>
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## Nutshell

Logic program = Data + Generator + Tester ( + Optimizer)
Outline

1. Basic Modeling

2. Methodology
   - Satisfiability
   - Queens
   - Traveling Salesperson
Satisfiability testing

- **Problem Instance**: A propositional formula $\phi$ in CNF
- **Problem Class**: Is there an assignment of propositional variables to true and false such that a given formula $\phi$ is true

Example: Consider formula $(a \lor \neg b) \land (\neg a \lor b)$

Logic Program:

```
X_1 = {a, b}
X_2 = {}
```

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Satisfiability testing

• **Problem Instance:** A propositional formula $\phi$ in CNF

• **Problem Class:** Is there an assignment of propositional variables to true and false such that a given formula $\phi$ is true

• **Example:** Consider formula

$$ (a \lor \neg b) \land (\neg a \lor b) $$

• **Logic Program:**

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<tr>
<td>${a, b}$</td>
<td>$\leftarrow , not , a, b$</td>
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</tr>
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Satisfiability testing

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- **Example**: Consider formula

  $$(a \lor \neg b) \land (\neg a \lor b)$$

- **Logic Program**:

  $$\begin{align*}
  \text{Generator} & \quad \{a, b\} \leftarrow \\
  \text{Tester} & \quad \leftarrow \ not \ a, b \\
  & \quad \leftarrow \ a, \ not \ b \\
  \text{Stable models} : & \quad X_1 = \{a, b\} \\
  & \quad X_2 = \{\} 
  \end{align*}$$
Satisfiability testing

- **Problem Instance**: A propositional formula $\phi$ in CNF
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</tr>
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Satisfiability testing

- **Problem Instance**: A propositional formula $\phi$ in CNF
- **Problem Class**: Is there an assignment of propositional variables to true and false such that a given formula $\phi$ is true

- **Example**: Consider formula

  \[(a \lor \neg b) \land (\neg a \lor b)\]

- **Logic Program**:

  Generator: $\{ a, b \} \leftarrow$
  
  Tester: $\leftarrow \neg a, b$
  $\leftarrow a, \neg b$

  Stable models:

  $X_1 = \{ a, b \}$
  $X_2 = \{ \}$
Outline

1. Basic Modeling

2. Methodology
   - Satisfiability
   - Queens
   - Traveling Salesperson
The n-Queens Problem

- Place $n$ queens on an $n \times n$ chess board
- Queens must not attack one another
Defining the Field

queens.lp

\begin{verbatim}
row(1..n).
col(1..n).
\end{verbatim}

- Create file `queens.lp`
- Define the field
  - \( n \) rows
  - \( n \) columns
Running ...

$ gringo queens.lp --const n=5 | clasp
Answer: 1
row(1) row(2) row(3) row(4) row(5) \ncol(1) col(2) col(3) col(4) col(5)
SATISFIABLE

Models : 1
Time   : 0.000
  Prepare : 0.000
  Prepro. : 0.000
  Solving : 0.000
Guess a solution candidate by placing some queens on the board
Placing some Queens

Running ...

```
$ gringo queens.lp --const n=5 | clasp 3
Answer: 1
row(1) row(2) row(3) row(4) row(5) \ 
col(1) col(2) col(3) col(4) col(5)
Answer: 2
row(1) row(2) row(3) row(4) row(5) \ 
col(1) col(2) col(3) col(4) col(5) queen(1,1)
Answer: 3
row(1) row(2) row(3) row(4) row(5) \ 
col(1) col(2) col(3) col(4) col(5) queen(2,1)
SATISFIABLE

Models : 3+
...
```
Placing some Queens: Answer 1

<table>
<thead>
<tr>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
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</table>

Answer 1
Placing some Queens: Answer 2

Answer 2

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Placing some Queens: Answer 3

Answer 3

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Placing $n$ Queens

```lp
row(1..n).
col(1..n).
{ queen(I,J) : row(I), col(J) }.
:- not n { queen(I,J) } n.
```

- Place exactly $n$ queens on the board
$ gringo queens.lp --const n=5 | clasp 2
Answer: 1
row(1) row(2) row(3) row(4) row(5) \ncol(1) col(2) col(3) col(4) col(5) \nqueen(5,1) queen(4,1) queen(3,1) \nqueen(2,1) queen(1,1)
Answer: 2
row(1) row(2) row(3) row(4) row(5) \ncol(1) col(2) col(3) col(4) col(5) \nqueen(1,2) queen(4,1) queen(3,1) \nqueen(2,1) queen(1,1)
...
Placing $n$ Queens: Answer 1
Placing $n$ Queens: Answer 2

Answer 2

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Horizontal and Vertical Attack

```lp
row(1..n).
col(1..n).
{ queen(I,J) : row(I), col(J) }.
:- not n { queen(I,J) } n.
:- queen(I,J), queen(I,J'), J != J'.
```

- Forbid horizontal attacks
Horizontal and Vertical Attack

queens.lp

```prolog
row(1..n).
col(1..n).
\{ queen(I,J) : row(I), col(J) \}.
:- not n \{ queen(I,J) \} n.
:- queen(I,J), queen(I,J'), J != J'.
:- queen(I,J), queen(I',J), I != I'.
```

- Forbid horizontal attacks
- Forbid vertical attacks
$ gringo queens.lp --const n=5 | clasp
Answer: 1
row(1) row(2) row(3) row(4) row(5) \ncol(1) col(2) col(3) col(4) col(5) \nqueen(5,5) queen(4,4) queen(3,3) \nqueen(2,2) queen(1,1)
...
Horizontal and Vertical Attack: Answer 1

Answer 1

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Diagonal Attack

```
queens.lp

row(1..n).
col(1..n).
{ queen(I,J) : row(I), col(J) }.
:- not n { queen(I,J) } n.
:- queen(I,J), queen(I,J'), J != J'.
:- queen(I,J), queen(I',J), I != I'.
:- queen(I,J), queen(I',J'), (I,J) != (I',J'), I-J == I'-J'.
:- queen(I,J), queen(I',J'), (I,J) != (I',J'), I+J == I'+J'.

• Forbid diagonal attacks
```
Diagonal Attack

Running ...

$ gringo queens.lp --const n=5 | clasp
Answer: 1
row(1) row(2) row(3) row(4) row(5) \
col(1) col(2) col(3) col(4) col(5) \
queen(4,5) queen(1,4) queen(3,3) queen(5,2) queen(2,1)
SATISFIABLE

Models : 1+
Time : 0.000
  Prepare : 0.000
  Prepro. : 0.000
  Solving : 0.000
Diagonal Attack: Answer 1

Answer 1

A diagram of a chessboard with crowns on specific squares, indicating a diagonal attack.
Encoding can be optimized

Much faster to solve
And sometimes it rocks

$ clingo -c n=5000 queens-opt-diag.lp -config=jumpy -q -stats=3
clingo version 4.1.0
Solving...
SATISFIABLE

Models : 1+
Time : 3758.143s (Solving: 1905.22s 1st Model: 1896.20s Unsat: 0.00s)
CPU Time : 3758.320s

Choices : 288594554
Conflicts : 3442 (Analyzed: 3442)
Restarts : 17 (Average: 202.47 Last: 3442)
Model-Level : 7594728.0
Problems : 1 (Average Length: 0.00 Splits: 0)
Lemmas : 3442 (Deleted: 0)
  Binary : 0 (Ratio: 0.00%)
  Ternary : 0 (Ratio: 0.00%)
Conflict : 3442 (Average Length: 229056.5 Ratio: 100.00%)
Loop : 0 (Average Length: 0.0 Ratio: 0.00%)
Other : 0 (Average Length: 0.0 Ratio: 0.00%)

Atoms : 75084857 (Original: 75069989 Auxiliary: 14868)
Bodies : 25090103
Equivalences : 125029999 (Atom=Atom: 50009999 Body=Body: 0 Other: 75020000)
Tight : Yes
Variables : 25024868 (Eliminated: 11781 Frozen: 25000000)
Constraints : 66664 (Binary: 35.6% Ternary: 0.0% Other: 64.4%)

Backjumps : 3442 (Average: 681.19 Max: 169512 Sum: 2344658)
  Executed : 3442 (Average: 681.19 Max: 169512 Sum: 2344658 Ratio: 100.00%)
  Bounded : 0 (Average: 0.00 Max: 0 Sum: 0 Ratio: 0.00%)

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Outline

1 Basic Modeling

2 Methodology
   - Satisfiability
   - Queens
   - Traveling Salesperson
Traveling Salesperson
Traveling Salesperson

\begin{verbatim}
node(1..6).

dge(1, (2;3;4)).  edge(2, (4;5;6)).  edge(3, (1;4;5)).
dge(4, (1;2)).  edge(5, (3;4;6)).  edge(6, (2;3;5)).
\end{verbatim}
Traveling Salesperson

node(1..6).

edge(1, (2; 3; 4)). edge(2, (4; 5; 6)). edge(3, (1; 4; 5)).
edge(4, (1; 2)). edge(5, (3; 4; 6)). edge(6, (2; 3; 5)).

cost(1, 2, 2). cost(1, 3, 3). cost(1, 4, 1).
cost(2, 4, 2). cost(2, 5, 2). cost(2, 6, 4).
cost(3, 1, 3). cost(3, 4, 2). cost(3, 5, 2).
cost(4, 1, 1). cost(4, 2, 2).
cost(5, 3, 2). cost(5, 4, 2). cost(5, 6, 1).
cost(6, 2, 4). cost(6, 3, 3). cost(6, 5, 1).
Traveling Salesperson

node(1..6).

cost(1,2,2).  cost(1,3,3).  cost(1,4,1).
cost(2,4,2).  cost(2,5,2).  cost(2,6,4).
cost(3,1,3).  cost(3,4,2).  cost(3,5,2).
cost(4,1,1).  cost(4,2,2).
cost(5,3,2).  cost(5,4,2).  cost(5,6,1).
cost(6,2,4).  cost(6,3,3).  cost(6,5,1).

dge(X,Y) :- cost(X,Y,_).
Traveling Salesperson

1 { cycle(X,Y) : edge(X,Y) } 1 :- node(X).
1 { cycle(X,Y) : edge(X,Y) } 1 :- node(Y).

reached(Y) :- cycle(1,Y).
reached(Y) :- cycle(X,Y), reached(X).
:- node(Y), not reached(Y).

#minimize { C,X,Y : cycle(X,Y), cost(X,Y,C) }.
Traveling Salesperson

1 \{ \text{cycle}(X, Y) : \text{edge}(X, Y) \} 1 :- \text{node}(X).
1 \{ \text{cycle}(X, Y) : \text{edge}(X, Y) \} 1 :- \text{node}(Y).

\text{reached}(Y) :- \text{cycle}(1, Y).
\text{reached}(Y) :- \text{cycle}(X, Y), \text{reached}(X).
Traveling Salesperson

1 \{ \text{cycle}(X, Y) : \text{edge}(X, Y) \} 1 :- \text{node}(X).
1 \{ \text{cycle}(X, Y) : \text{edge}(X, Y) \} 1 :- \text{node}(Y).

\text{reached}(Y) :- \text{cycle}(1, Y).
\text{reached}(Y) :- \text{cycle}(X, Y), \text{reached}(X).

:- \text{node}(Y), \text{not reached}(Y).
Traveling Salesperson

1 \{ \text{cycle}(X,Y) : \text{edge}(X,Y) \} 1 \:- \text{node}(X).
1 \{ \text{cycle}(X,Y) : \text{edge}(X,Y) \} 1 \:- \text{node}(Y).

\text{reached}(Y) \:- \text{cycle}(1,Y).
\text{reached}(Y) \:- \text{cycle}(X,Y), \text{reached}(X).

\:- \text{node}(Y), \text{not reached}(Y).

\#\text{minimize} \{ C,X,Y : \text{cycle}(X,Y), \text{cost}(X,Y,C) \}.
Language: Overview

3 Motivation
4 Core language
5 Extended language
Basic language extensions

- The expressiveness of a language can be enhanced by introducing new constructs.
- To this end, we must address the following issues:
  - What is the syntax of the new language construct?
  - What is the semantics of the new language construct?
  - How to implement the new language construct?

A way of providing semantics is to furnish a translation removing the new constructs, eg. classical negation.
This translation might also be used for implementing the language extension.
• The expressiveness of a language can be enhanced by introducing new constructs

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  – What is the syntax of the new language construct?
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The expressiveness of a language can be enhanced by introducing new constructs.

To this end, we must address the following issues:

- What is the syntax of the new language construct?
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A way of providing semantics is to furnish a translation removing the new constructs, e.g., classical negation.

This translation might also be used for implementing the language extension.
Outline

3 Motivation

4 Core language

5 Extended language
Outline

3 Motivation

4 Core language
   - Integrity constraint
   - Choice rule
   - Cardinality rule
   - Weight rule

5 Extended language
   - Conditional literal
   - Optimization statement
Integrity constraint

- **Idea** Eliminate unwanted solution candidates
- **Syntax** An integrity constraint is of the form

  \[ \leftarrow a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \]

  where \( 0 \leq m \leq n \) and each \( a_i \) is an atom for \( 1 \leq i \leq n \)

- **Example**  
  \[ :- \text{ edge}(3,7), \text{ color}(3,\text{red}), \text{ color}(7,\text{red}). \]
Integrity constraint

• **Idea** Eliminate unwanted solution candidates

• **Syntax** An integrity constraint is of the form

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• **Example**

\[ :- \text{edge}(3,7), \text{color}(3,\text{red}), \text{color}(7,\text{red}). \]

• **Embedding** The above integrity constraint can be turned into the normal rule

\[ x \leftarrow a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n, \text{not } x \]

where \(x\) is a new symbol, that is, \(x \notin A\).
Integrity constraint

- **Idea** Eliminate unwanted solution candidates
- **Syntax** An integrity constraint is of the form

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- **Example** \(-\) edge(3,7), color(3,red), color(7,red).
- **Embedding** The above integrity constraint can be turned into the normal rule

\[ x \leftarrow a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n, \text{not } x \]

where \(x\) is a new symbol, that is, \(x \not\in A\).

- **Another example** \(P = \{a \leftarrow \text{not } b, b \leftarrow \text{not } a\}\) versus \(P' = P \cup \{\leftarrow a\}\) and \(P'' = P \cup \{\leftarrow \text{not } a\}\)
Choice rule

- **Idea** Choices over subsets
- **Syntax** A choice rule is of the form

\[
\{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_o
\]

where \(0 \leq m \leq n \leq o\) and each \(a_i\) is an atom for \(1 \leq i \leq o\)
Choice rule

- **Idea** Choices over subsets
- **Syntax** A choice rule is of the form

\[
\{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_o
\]

where \(0 \leq m \leq n \leq o\) and each \(a_i\) is an atom for \(1 \leq i \leq o\)

- **Informal meaning** If the body is satisfied by the stable model at hand, then any subset of \(\{a_1, \ldots, a_m\}\) can be included in the stable model
Choice rule

- **Idea** Choices over subsets
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\[ \{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_o \]

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- **Example**

  \[
  \{ \text{buy(pizza); buy(wine); buy(corn)} \} \leftarrow \text{at(grocery)}.\]
Choice rule

- **Idea** Choices over subsets
- **Syntax** A choice rule is of the form

\[ \{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, not\ a_{n+1}, \ldots, not\ a_o \]

where \(0 \leq m \leq n \leq o\) and each \(a_i\) is an atom for \(1 \leq i \leq o\)

- **Informal meaning** If the body is satisfied by the stable model at hand, then any subset of \(\{a_1, \ldots, a_m\}\) can be included in the stable model

- **Example** \(\{\text{buy(pizza)}; \text{buy(wine)}; \text{buy(corn)}\} \leftarrow \text{at(grocery)}\).

- **Another Example** \(P = \{\{a\} \leftarrow b, b \leftarrow\}\) has two stable models: \(\{b\}\) and \(\{a, b\}\)
Embedding in normal rules

• A choice rule of form

\[ \{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_o \]

can be translated into \(2m + 1\) normal rules

\[
\begin{align*}
  b & \quad \leftarrow \quad a_{m+1}, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_o \\
  a_1 & \quad \leftarrow \quad b, \text{not } a'_1 \quad \ldots \quad a_m & \quad \leftarrow \quad b, \text{not } a'_m \\
  a'_1 & \quad \leftarrow \quad \text{not } a_1 \quad \ldots \quad a'_m & \quad \leftarrow \quad \text{not } a_m
\end{align*}
\]

by introducing new atoms \(b, a'_1, \ldots, a'_m\).
A choice rule of form
\[ \{a_1, \ldots, a_m\} \leftarrow a_{m+1}, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_o \]
can be translated into 2m + 1 normal rules
\[ b \leftarrow a_{m+1}, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_o \]
\[ a_1 \leftarrow b, \text{not } a'_1 \ldots a_m \leftarrow b, \text{not } a'_m \]
\[ a'_1 \leftarrow \text{not } a_1 \ldots a'_m \leftarrow \text{not } a_m \]
by introducing new atoms \( b, a'_1, \ldots, a'_m \).
Embedding in normal rules

- A choice rule of form

\[
\{ a_1, \ldots, a_m \} \leftarrow a_{m+1}, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_0
\]

can be translated into \(2m + 1\) normal rules

\[
\begin{align*}
    b & \leftarrow a_{m+1}, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_0 \\
    a_1 & \leftarrow b, \text{not } a'_1, \ldots, a'_m & a_m & \leftarrow b, \text{not } a'_m \\
    a'_1 & \leftarrow \text{not } a_1, \ldots, a'_m & a'_m & \leftarrow \text{not } a_m
\end{align*}
\]

by introducing new atoms \(b, a'_1, \ldots, a'_m\).
Outline

3 Motivation

4 Core language
   - Integrity constraint
   - Choice rule
   - Cardinality rule
   - Weight rule

5 Extended language
   - Conditional literal
   - Optimization statement
Cardinality rule

- **Idea** Control (lower) cardinality of subsets
- **Syntax** A **cardinality rule** is the form

\[ a_0 \leftarrow l \{ a_1, \ldots, a_m, not a_{m+1}, \ldots, not a_n \} \]

where \( 0 \leq m \leq n \) and each \( a_i \) is an atom for \( 1 \leq i \leq n \);
\( l \) is a non-negative integer.
Cardinality rule

- **Idea** Control (lower) cardinality of subsets
- **Syntax** A cardinality rule is the form

\[ a_0 \leftarrow l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} \]

where 0 ≤ m ≤ n and each \( a_i \) is an atom for 1 ≤ i ≤ n; l is a non-negative integer.

- **Informal meaning** The head atom belongs to the stable model, if at least l elements of the body are included in the stable model.
- **Note** l acts as a lower bound on the body.
Cardinality rule

- **Idea** Control (lower) cardinality of subsets
- **Syntax** A cardinality rule is the form

\[
a_0 \leftarrow l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \}
\]

where \(0 \leq m \leq n\) and each \(a_i\) is an atom for \(1 \leq i \leq n\); \(l\) is a non-negative integer.

- **Informal meaning** The head atom belongs to the stable model, if at least \(l\) elements of the body are included in the stable model.
- **Note** \(l\) acts as a **lower bound** on the body.
- **Example** \(\text{pass(c42)} : - 2 \{ \text{pass(a1)}; \text{pass(a2)}; \text{pass(a3)} \} \).
Cardinality rule

- **Idea** Control (lower) cardinality of subsets
- **Syntax** A cardinality rule is the form

\[ a_0 \leftarrow l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} \]

where \( 0 \leq m \leq n \) and each \( a_i \) is an atom for \( 1 \leq i \leq n \);
\( l \) is a non-negative integer.

- **Informal meaning** The head atom belongs to the stable model, if at least \( l \) elements of the body are included in the stable model
- **Note** \( l \) acts as a lower bound on the body

- **Example** \( \text{pass(c42)} : - 2 \{ \text{pass(a1)}; \text{pass(a2)}; \text{pass(a3)} \} \).
- **Another Example** \( P = \{ a \leftarrow 1 \{ b, c \}, b \leftarrow \} \) has stable model \( \{ a, b \} \)
Embedding in normal rules

- Replace each cardinality rule

\[ a_0 \leftarrow l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} \]

by

\[ a_0 \leftarrow \text{ctr}(1, l) \]

where atom \( \text{ctr}(i, j) \) represents the fact that at least \( j \) of the literals having an equal or greater index than \( i \), are in a stable model.
Embedding in normal rules

- Replace each cardinality rule

\[ a_0 \leftarrow l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} \]

by \[ a_0 \leftarrow c(t)(1, l) \]

where atom \( c(t)(i, j) \) represents the fact that at least \( j \) of the literals having an equal or greater index than \( i \), are in a stable model.

- The definition of \( c(t)/2 \) is given for \( 0 \leq k \leq l \) by the rules

\[
\begin{align*}
\text{ctr}(i, k+1) & \leftarrow \text{ctr}(i + 1, k), a_i \\
\text{ctr}(i, k) & \leftarrow \text{ctr}(i + 1, k) & \text{for } 1 \leq i \leq m \\
\text{ctr}(j, k+1) & \leftarrow \text{ctr}(j + 1, k), \text{not } a_j \\
\text{ctr}(j, k) & \leftarrow \text{ctr}(j + 1, k) & \text{for } m + 1 \leq j \leq n \\
\text{ctr}(n + 1, 0) & \leftarrow
\end{align*}
\]
Embedding in normal rules

• Replace each cardinality rule

\[ a_0 \leftarrow l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} \]

by \[ a_0 \leftarrow \text{ctr}(1, l) \]

where atom \(\text{ctr}(i, j)\) represents the fact that at least \(j\) of the literals having an equal or greater index than \(i\), are in a stable model

• The definition of \(\text{ctr}/2\) is given for \(0 \leq k \leq l\) by the rules

\[
\begin{align*}
\text{ctr}(i, k+1) & \leftarrow \text{ctr}(i + 1, k), a_i \\
\text{ctr}(i, k) & \leftarrow \text{ctr}(i + 1, k) \quad \text{for } 1 \leq i \leq m \\
\text{ctr}(j, k+1) & \leftarrow \text{ctr}(j + 1, k), \text{not } a_j \\
\text{ctr}(j, k) & \leftarrow \text{ctr}(j + 1, k) \quad \text{for } m + 1 \leq j \leq n \\
\text{ctr}(n + 1, 0) & \leftarrow
\end{align*}
\]
Embedding in normal rules

• Replace each cardinality rule

\[ a_0 \leftarrow l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} \]

by \( a_0 \leftarrow \text{ctr}(1, l) \)

where atom \( \text{ctr}(i, j) \) represents the fact that at least \( j \) of the literals having an equal or greater index than \( i \), are in a stable model

• The definition of \( \text{ctr}/2 \) is given for \( 0 \leq k \leq l \) by the rules

\[
\begin{align*}
\text{ctr}(i, k+1) & \leftarrow \text{ctr}(i+1, k), a_i \\
\text{ctr}(i, k) & \leftarrow \text{ctr}(i+1, k) \\
\text{ctr}(j, k+1) & \leftarrow \text{ctr}(j+1, k), \text{not } a_j \\
\text{ctr}(j, k) & \leftarrow \text{ctr}(j+1, k) \\
\text{ctr}(n+1, 0) & \leftarrow \\
\end{align*}
\]

for \( 1 \leq i \leq m \)

for \( m + 1 \leq j \leq n \)
Embedding in normal rules

- Replace each cardinality rule

\[ a_0 \leftarrow l \{ a_1, \ldots, a_m, not \ a_{m+1}, \ldots, not \ a_n \} \]

by \[ a_0 \leftarrow \text{ctr}(1, l) \]

where atom \( \text{ctr}(i, j) \) represents the fact that at least \( j \) of the literals having an equal or greater index than \( i \), are in a stable model

- The definition of \( \text{ctr}/2 \) is given for \( 0 \leq k \leq l \) by the rules

\[
\begin{align*}
\text{ctr}(i, k+1) & \leftarrow \text{ctr}(i + 1, k), a_i \\
\text{ctr}(i, k) & \leftarrow \text{ctr}(i + 1, k) & \text{for } 1 \leq i \leq m \\
\text{ctr}(j, k+1) & \leftarrow \text{ctr}(j + 1, k), not \ a_j \\
\text{ctr}(j, k) & \leftarrow \text{ctr}(j + 1, k) & \text{for } m + 1 \leq j \leq n \\
\text{ctr}(n + 1, 0) & \leftarrow \\
\end{align*}
\]
Embedding in normal rules

- Replace each cardinality rule

\[
a_0 \leftarrow l \{ a_1, \ldots, a_m, not a_{m+1}, \ldots, not a_n \}
\]

by \[a_0 \leftarrow ctr(1, l)\]

where atom \(ctr(i,j)\) represents the fact that at least \(j\) of the literals having an equal or greater index than \(i\), are in a stable model

- The definition of \(ctr/2\) is given for \(0 \leq k \leq l\) by the rules

\[
\begin{align*}
ctr(i, k+1) & \leftarrow ctr(i + 1, k), a_i \\
ctr(i, k) & \leftarrow ctr(i + 1, k) \\
ctr(j, k+1) & \leftarrow ctr(j + 1, k), not a_j \\
ctr(j, k) & \leftarrow ctr(j + 1, k) \\
ctr(n + 1, 0) & \leftarrow
\end{align*}
\]

for \(1 \leq i \leq m\) and \(m + 1 \leq j \leq n\)
Embedding in normal rules

- Replace each cardinality rule
  \[ a_0 \leftarrow l \{ a_1, \ldots, a_m, not \ a_{m+1}, \ldots, not \ a_n \} \]
  
  by \[ a_0 \leftarrow ctr(1, l) \]

  where atom \( ctr(i,j) \) represents the fact that at least \( j \) of the literals having an equal or greater index than \( i \), are in a stable model.

- The definition of \( ctr/2 \) is given for \( 0 \leq k \leq l \) by the rules
  \[
  \begin{align*}
  ctr(i, k+1) & \leftarrow ctr(i + 1, k), a_i \\
  ctr(i, k) & \leftarrow ctr(i + 1, k) & \text{for } 1 \leq i \leq m \\
  \\
  ctr(j, k+1) & \leftarrow ctr(j + 1, k), not \ a_j \\
  ctr(j, k) & \leftarrow ctr(j + 1, k) & \text{for } m + 1 \leq j \leq n \\
  \\
  ctr(n + 1, 0) & \leftarrow \\
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An example

- Program \( \{a \leftarrow, \ c \leftarrow 1 \ \{a, b\}\} \) has the stable model \( \{a, c\}\)
An example

- Program \{a \leftarrow, c \leftarrow 1 \{a, b\}\} has the stable model \{a, c\}
- Translating the cardinality rule yields the rules

\[
\begin{align*}
  a & \leftarrow \\
  c & \leftarrow \text{ctr}(1, 1) \\
  \text{ctr}(1, 2) & \leftarrow \text{ctr}(2, 1), a \\
  \text{ctr}(1, 1) & \leftarrow \text{ctr}(2, 1) \\
  \text{ctr}(2, 2) & \leftarrow \text{ctr}(3, 1), b \\
  \text{ctr}(2, 1) & \leftarrow \text{ctr}(3, 1) \\
  \text{ctr}(1, 1) & \leftarrow \text{ctr}(2, 0), a \\
  \text{ctr}(1, 0) & \leftarrow \text{ctr}(2, 0) \\
  \text{ctr}(2, 1) & \leftarrow \text{ctr}(3, 0), b \\
  \text{ctr}(2, 0) & \leftarrow \text{ctr}(3, 0) \\
  \text{ctr}(3, 0) & \leftarrow
\end{align*}
\]

having stable model \{a, \text{ctr}(3, 0), \text{ctr}(2, 0), \text{ctr}(1, 0), \text{ctr}(1, 1), c\}
... and vice versa

- A normal rule

\[ a_0 \leftarrow a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \]

can be represented by the cardinality rule

\[ a_0 \leftarrow n \{a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n\} \]
Cardinality rules with upper bounds

- A rule of the form

\[ a_0 \leftarrow l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} u \]  \hspace{1cm} (1)

where 0 ≤ m ≤ n and each \( a_i \) is an atom for 1 ≤ i ≤ n;
l and u are non-negative integers
Cardinality rules with upper bounds

- A rule of the form

\[
a_0 \leftarrow l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} \ u
\]

where \(0 \leq m \leq n\) and each \(a_i\) is an atom for \(1 \leq i \leq n\);
\(l\) and \(u\) are non-negative integers

stands for

\[
\begin{align*}
a_0 & \leftarrow b, \text{not } c \\
b & \leftarrow l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} \\
c & \leftarrow u+1 \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \}
\end{align*}
\]

where \(b\) and \(c\) are new symbols
Cardinality rules with upper bounds

- A rule of the form

\[ a_0 \leftarrow l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} u \]

where \( 0 \leq m \leq n \) and each \( a_i \) is an atom for \( 1 \leq i \leq n \);
\( l \) and \( u \) are non-negative integers

stands for

\[ a_0 \leftarrow b, \text{not } c \]
\[ b \leftarrow l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} \]
\[ c \leftarrow u+1 \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} \]

where \( b \) and \( c \) are new symbols

- Note The single constraint in the body of the cardinality rule (1) is referred to as a cardinality constraint
Cardinality constraints

- Syntax A cardinality constraint is of the form

\[ l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} \leq u \]

where \( 0 \leq m \leq n \) and each \( a_i \) is an atom for \( 1 \leq i \leq n \);
\( l \) and \( u \) are non-negative integers.
Cardinality constraints

- **Syntax** A cardinality constraint is of the form

\[ l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} u \]

where \( 0 \leq m \leq n \) and each \( a_i \) is an atom for \( 1 \leq i \leq n \);
\( l \) and \( u \) are non-negative integers

- **Informal meaning** A cardinality constraint is satisfied by a stable model \( X \), if the number of its contained literals satisfied by \( X \) is between \( l \) and \( u \) (inclusive)
Cardinality constraints

- **Syntax** A cardinality constraint is of the form

  \[ l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} \leq u \]

  where \( 0 \leq m \leq n \) and each \( a_i \) is an atom for \( 1 \leq i \leq n \);
  \( l \) and \( u \) are non-negative integers

- **Informal meaning** A cardinality constraint is satisfied by a stable model \( X \), if the number of its contained literals satisfied by \( X \) is between \( l \) and \( u \) (inclusive)

- In other words, if

  \[ l \leq | \{a_1, \ldots, a_m \} \cap X) \cup (a_{m+1}, \ldots, a_n \} \setminus X) | \leq u \]
A rule of the form

\[ l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} u \leftarrow a_{n+1}, \ldots, a_o, \text{not } a_{o+1}, \ldots, \text{not } a_p \]

where \( 0 \leq m \leq n \leq o \leq p \) and each \( a_i \) is an atom for \( 1 \leq i \leq p \); \( l \) and \( u \) are non-negative integers
Cardinality constraints as heads

- A rule of the form

\[ l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} \ u \leftarrow a_{n+1}, \ldots, a_o, \text{not } a_{o+1}, \ldots, \text{not } a_p \]

where \( 0 \leq m \leq n \leq o \leq p \) and each \( a_i \) is an atom for \( 1 \leq i \leq p \);
\( l \) and \( u \) are non-negative integers

stands for

\[ b \leftarrow a_{n+1}, \ldots, a_o, \text{not } a_{o+1}, \ldots, \text{not } a_p \]
\[ \{ a_1, \ldots, a_m \} \leftarrow b \]
\[ c \leftarrow l \{ a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \} \ u \leftarrow b, \text{not } c \]

where \( b \) and \( c \) are new symbols
Cardinality constraints as heads

- A rule of the form

\[ l \{a_1, \ldots, a_m, \textit{not } a_{m+1}, \ldots, \textit{not } a_n\} \ u \leftarrow a_{n+1}, \ldots, a_o, \textit{not } a_{o+1}, \ldots, \textit{not } a_p \]

where \(0 \leq m \leq n \leq o \leq p\) and each \(a_i\) is an atom for \(1 \leq i \leq p\);
\(l\) and \(u\) are non-negative integers

stands for

\[ b \leftarrow a_{n+1}, \ldots, a_o, \textit{not } a_{o+1}, \ldots, \textit{not } a_p \]

\[ \{a_1, \ldots, a_m\} \leftarrow b \]

\[ c \leftarrow l \{a_1, \ldots, a_m, \textit{not } a_{m+1}, \ldots, \textit{not } a_n\} \ u \leftarrow b, \textit{not } c \]

where \(b\) and \(c\) are new symbols

- Example

\[ \text{Example } \{ \text{color(v42,red); color(v42,green); color(v42,blue) } \} 1.\]
Full-fledged cardinality rules

- A rule of the form

\[ l_0 S_0 u_0 \leftarrow l_1 S_1 u_1, \ldots, l_n S_n u_n \]

where for \( 0 \leq i \leq n \) each \( l_i S_i u_i \)
A rule of the form
\[ l_0 S_0 u_0 \leftarrow l_1 S_1 u_1, \ldots, l_n S_n u_n \]
where for \(0 \leq i \leq n\) each \(l_i S_i u_i\) stands for \(0 \leq i \leq n\)

\[ a \leftarrow b_1, \ldots, b_n, \text{not } c_1, \ldots, \text{not } c_n \]

\[ S_0^+ \leftarrow a \]
\[ \leftarrow a, \text{not } b_0 \]
\[ \leftarrow a, c_0 \]
\[ b_i \leftarrow l_i S_i \]
\[ c_i \leftarrow u_i + 1 S_i \]

where \(a, b_i, c_i\) are new symbols (and \(S_0^+ = \{p \in \mathcal{A} | p \in S_0\}\))
Full-fledged cardinality rules

- A rule of the form

\[ l_0 S_0 u_0 \leftarrow l_1 S_1 u_1, \ldots, l_n S_n u_n \]

where for \( 0 \leq i \leq n \) each \( l_i S_i u_i \) stands for \( 0 \leq i \leq n \)

\[ a \leftarrow b_1, \ldots, b_n, \text{not } c_1, \ldots, \text{not } c_n \]

\[ S_0^+ \leftarrow a \]
\[ \leftarrow a, \text{not } b_0 \]
\[ \leftarrow a, c_0 \]

\[ b_i \leftarrow l_i S_i \]
\[ c_i \leftarrow u_i+1 S_i \]

where \( a, b_i, c_i \) are new symbols (and \( S_0^+ = \{ p \in A \mid p \in S_0 \} \))
Full-fledged cardinality rules

- A rule of the form

\[ l_0 S_0 u_0 \leftarrow l_1 S_1 u_1, \ldots, l_n S_n u_n \]

where for \(0 \leq i \leq n\) each \(l_i S_i u_i\) stands for \(0 \leq i \leq n\)

\[ a \leftarrow b_1, \ldots, b_n, \text{not } c_1, \ldots, \text{not } c_n \]

\[ S_0^+ \leftarrow a \]
\[ \leftarrow a, \text{not } b_0 \]
\[ \leftarrow a, c_0 \]
\[ b_i \leftarrow l_i S_i \]
\[ c_i \leftarrow u_i+1 S_i \]

where \(a, b_i, c_i\) are new symbols (and \(S_0^+ = \{p \in \mathcal{A} \mid p \in S_0\}\))
Full-fledged cardinality rules

- A rule of the form

\[
l_0 S_0 u_0 \leftarrow l_1 S_1 u_1, \ldots, l_n S_n u_n
\]

where for \(0 \leq i \leq n\) each \(l_i S_i u_i\) stands for \(0 \leq i \leq n\)

\[
a \leftarrow b_1, \ldots, b_n, \text{not } c_1, \ldots, \text{not } c_n
\]

\[
S_0^+ \leftarrow a
\]

\[
\leftarrow a, \text{not } b_0
\]

\[
\leftarrow a, c_0
\]

\[
b_i \leftarrow l_i S_i
\]

\[
c_i \leftarrow u_i+1 S_i
\]

where \(a, b_i, c_i\) are new symbols (and \(S_0^+ = \{p \in A \mid p \in S_0\}\))
Full-fledged cardinality rules

- A rule of the form

\[ \begin{align*}
  l_0 & \ S_0 & u_0 & \leftarrow & l_1 & S_1 & u_1, \ldots, & l_n & S_n & u_n \\
\end{align*} \]

where for \( 0 \leq i \leq n \) each \( l_i S_i u_i \)

stands for \( 0 \leq i \leq n \)

\[ \begin{align*}
  a & \leftarrow & b_1, \ldots, b_n, & not & c_1, \ldots, & not & c_n \\
  S_0^+ & \leftarrow & a \\
  & \leftarrow & a, not \ b_0 & b_i & \leftarrow & l_i S_i \\
  & \leftarrow & a, c_0 & c_i & \leftarrow & u_i + 1 S_i
\end{align*} \]

where \( a, b_i, c_i \) are new symbols (and \( S_0^+ = \{ p \in A \mid p \in S_0 \} \))
3 Motivation

4 Core language
   - Integrity constraint
   - Choice rule
   - Cardinality rule
   - Weight rule

5 Extended language
   - Conditional literal
   - Optimization statement
Weight rule

- **Syntax** A weight rule is the form

  \[ a_0 \leftarrow l \{ \ w_1 : a_1, \ldots, w_m : a_m, w_{m+1} : \text{not } a_{m+1}, \ldots, w_n : \text{not } a_n \ \} \]

  where \( 0 \leq m \leq n \) and each \( a_i \) is an atom;
  \( l \) and \( w_i \) are integers for \( 1 \leq i \leq n \)

- A weighted literal \( w_i : \ell_i \) associates each literal \( \ell_i \) with a weight \( w_i \)
Weight rule

- **Syntax** A weight rule is the form

  \[ a_0 \leftarrow l \{ w_1 : a_1, \ldots, w_m : a_m, w_{m+1} : not a_{m+1}, \ldots, w_n : not a_n \} \]

  where \( 0 \leq m \leq n \) and each \( a_i \) is an atom;
  \( l \) and \( w_i \) are integers for \( 1 \leq i \leq n \)

- **A weighted literal** \( w_i : \ell_i \) associates each literal \( \ell_i \) with a weight \( w_i \)

- **Note** A cardinality rule is a weight rule where \( w_i = 1 \) for \( 0 \leq i \leq n \)
Weight constraints

- **Syntax** A *weight constraint* is of the form

\[
\begin{align*}
l \{ & w_1 : a_1, \ldots, w_m : a_m, w_{m+1} : not a_{m+1}, \ldots, w_n : not a_n \} \ u \\
\end{align*}
\]

where \(0 \leq m \leq n\) and each \(a_i\) is an atom; \(l, u\) and \(w_i\) are integers for \(1 \leq i \leq n\).
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  where \( 0 \leq m \leq n \) and each \( a_i \) is an atom; 
  \( l, u \) and \( w_i \) are integers for \( 1 \leq i \leq n \)

- **Meaning** A weight constraint is satisfied by a stable model \( X \), if

  \[ l \leq \left( \sum_{1 \leq i \leq m, a_i \in X} w_i + \sum_{m < i \leq n, a_i \not\in X} w_i \right) \leq u \]
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- **Note** (Cardinality and) weight constraints amount to constraints on (count and) sum aggregate functions
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• **Note** (Cardinality and) weight constraints amount to constraints on (count and) sum aggregate functions

• **Example**

\[ 10 \{ 4:course(db); 6:course(ai); 8:course(project); 3:course(xml) \} 20 \]
Outline

3 Motivation

4 Core language

5 Extended language
Outline

3 Motivation

4 Core language
   - Integrity constraint
   - Choice rule
   - Cardinality rule
   - Weight rule

5 Extended language
   - Conditional literal
   - Optimization statement
Conditional literals

- **Syntax** A conditional literal is of the form

  \[ \ell : \ell_1, \ldots, \ell_n \]

  where \( \ell \) and \( \ell_i \) are literals for \( 0 \leq i \leq n \)

- **Informal meaning** A conditional literal can be regarded as the list of elements in the set \( \{ \ell | \ell_1, \ldots, \ell_n \} \)
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Example given: 'p(1..3). q(2).

\[ \text{r}(X) : \text{p}(X), \text{not q}(X) \iff \text{r}(X) : \text{p}(X), \text{not q}(X); 1 \{ \text{r}(X) : \text{p}(X), \text{not q}(X) \} \]

is instantiated to

\[ \text{r}(1); \text{r}(3) \iff \text{r}(1), \text{r}(3), 1 \{ \text{r}(1), \text{r}(3) \} \]
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  p(1..3). \quad q(2). \n  \]

  \[
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  \]

  is instantiated to

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  r(1); r(3) :- r(1), r(3), 1 \{ r(1), r(3) \}.
  \]
Outline

3 Motivation

4 Core language
   - Integrity constraint
   - Choice rule
   - Cardinality rule
   - Weight rule

5 Extended language
   - Conditional literal
   - **Optimization statement**
Optimization statement

- **Idea** Express (multiple) cost functions subject to minimization and/or maximization
- **Syntax** A minimize statement is of the form

\[
\text{minimize } \{ w_1 @ p_1 : \ell_1, \ldots, w_n @ p_n : \ell_n \}.
\]

where each \( \ell_i \) is a literal; and \( w_i \) and \( p_i \) are integers for \( 1 \leq i \leq n \)
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Priority levels, $p_i$, allow for representing lexicographically ordered minimization objectives
Optimization statement

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  $\text{minimize } \{ w_1 \oplus p_1 : \ell_1, \ldots, w_n \oplus p_n : \ell_n \}.$

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  Priority levels, $p_i$, allow for representing lexicographically ordered minimization objectives

- **Meaning** A minimize statement is a directive that instructs the ASP solver to compute optimal stable models by minimizing a weighted sum of elements
Optimization statement

- A maximize statement of the form

\[
\text{maximize } \{ \ w_1 @ p_1 : \ell_1, \ldots, w_n @ p_n : \ell_n \ } \]

stands for \textit{minimize} \{ \ -w_1 @ p_1 : \ell_1, \ldots, -w_n @ p_n : \ell_n \ }

Example When configuring a computer, we may want to maximize hard disk capacity, while minimizing price.

\#maximize \{ 250@1:hd(1), 500@1:hd(2), 750@1:hd(3), 1000@1:hd(4) \}.
\#minimize \{ 30@2:hd(1), 40@2:hd(2), 60@2:hd(3), 80@2:hd(4) \}.

The priority levels indicate that (minimizing) price is more important than (maximizing) capacity.
Optimization statement

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Language Extensions: Overview

6 Two kinds of negation
7 Disjunctive logic programs
8 Aggregates
Outline

6 Two kinds of negation

7 Disjunctive logic programs

8 Aggregates
Motivation

- Classical versus default negation
  - Symbol \( \neg \) and *not*
Motivation

• Classical versus default negation
  – Symbol \(\neg\) and \textit{not}
  – Idea
    • \(\neg a \approx \neg a \in X\)
    • \textit{not} \(a \approx a \notin X\)
Motivation

- Classical versus default negation
  - Symbol $\neg$ and *not*
  - Idea
    - $\neg a \approx \neg a \in X$
    - *not a* $\approx a \notin X$
  - Example
    - *cross* $\leftarrow \neg$*train*
    - *cross* $\leftarrow$ *not train*
Classical negation

- We consider logic programs in negation normal form
  - That is, classical negation is applied to atoms only
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- Given an alphabet $\mathcal{A}$ of atoms, let $\overline{\mathcal{A}} = \{\neg a \mid a \in \mathcal{A}\}$ such that $\mathcal{A} \cap \overline{\mathcal{A}} = \emptyset$
Classical negation

- We consider logic programs in negation normal form
  - That is, classical negation is applied to atoms only
- Given an alphabet $\mathcal{A}$ of atoms, let $\overline{\mathcal{A}} = \{ \neg a \mid a \in \mathcal{A} \}$ such that $\mathcal{A} \cap \overline{\mathcal{A}} = \emptyset$
- Given a program $P$ over $\mathcal{A}$, classical negation is encoded by adding

$$P^\neg = \{ a \leftarrow b, \neg b \mid a \in (\mathcal{A} \cup \overline{\mathcal{A}}), b \in \mathcal{A} \}$$
Classical negation

- Given an alphabet $\mathcal{A}$ of atoms, let $\overline{\mathcal{A}} = \{\neg a \mid a \in \mathcal{A}\}$ such that $\mathcal{A} \cap \overline{\mathcal{A}} = \emptyset$
- Given a program $P$ over $\mathcal{A}$, classical negation is encoded by adding

\[ P^- = \{a \leftarrow b, \neg b \mid a \in (\mathcal{A} \cup \overline{\mathcal{A}}), b \in \mathcal{A}\} \]

- A set $X$ of atoms is a **stable model** of a program $P$ over $\mathcal{A} \cup \overline{\mathcal{A}}$, if $X$ is a stable model of $P \cup P^-$.
An example

- The program

\[ P = \{ a \leftarrow \text{not } b, \ b \leftarrow \text{not } a \} \cup \{ c \leftarrow b, \ b \leftarrow c \} \]
An example

- The program

\[ P = \{a \leftarrow \text{not } b, \ b \leftarrow \text{not } a\} \cup \{c \leftarrow b, \ \neg c \leftarrow b\} \]

induces

\[ P^- = \begin{cases} 
  a &\leftarrow a, \neg a \\
  \neg a &\leftarrow a, \neg a \\
  b &\leftarrow a, \neg a \\
  \neg b &\leftarrow a, \neg a \\
  c &\leftarrow a, \neg a \\
  \neg c &\leftarrow a, \neg a \\
  a &\leftarrow b, \neg b \\
  \neg a &\leftarrow b, \neg b \\
  b &\leftarrow b, \neg b \\
  \neg b &\leftarrow b, \neg b \\
  c &\leftarrow b, \neg b \\
  \neg c &\leftarrow b, \neg b \\
  a &\leftarrow c, \neg c \\
  \neg a &\leftarrow c, \neg c \\
  b &\leftarrow c, \neg c \\
  \neg b &\leftarrow c, \neg c \\
  c &\leftarrow c, \neg c \\
  \neg c &\leftarrow c, \neg c 
\end{cases} \]
An example

- The program

\[ P = \{ a \leftarrow \text{not } b, \ b \leftarrow \text{not } a \} \cup \{ c \leftarrow b, \ \neg c \leftarrow b \} \]

induces

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 \neg a \leftarrow a, \neg a \\
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 c \leftarrow a, \neg a \\
 \neg c \leftarrow a, \neg a 
\end{cases} \]

- The stable models of \( P \) are given by the ones of \( P \cup P^- \), viz \( \{ a \} \)
Properties

- The only inconsistent stable “model” is $X = \mathcal{A} \cup \overline{\mathcal{A}}$.
Properties

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- **Note** Strictly speaking, an inconsistent set like $\mathcal{A} \cup \overline{\mathcal{A}}$ is not a model
Properties

- The only inconsistent stable “model” is $X = \mathcal{A} \cup \overline{\mathcal{A}}$
- **Note** Strictly speaking, an inconsistent set like $\mathcal{A} \cup \overline{\mathcal{A}}$ is not a model.
- For a logic program $P$ over $\mathcal{A} \cup \overline{\mathcal{A}}$, exactly one of the following two cases applies:
  1. All stable models of $P$ are consistent or
  2. $X = \mathcal{A} \cup \overline{\mathcal{A}}$ is the only stable model of $P$
Train spotting

- $P_1 = \{\text{cross} \leftarrow \text{not train}\}$
- $P_2 = \{\text{cross} \leftarrow \lnot\text{train}\}$
- $P_3 = \{\text{cross} \leftarrow \lnot\text{train}, \lnot\text{train} \leftarrow\}$
- $P_4 = \{\text{cross} \leftarrow \lnot\text{train}, \lnot\text{train} \leftarrow, \lnot\text{cross} \leftarrow\}$
- $P_5 = \{\text{cross} \leftarrow \lnot\text{train}, \lnot\text{train} \leftarrow, \lnot\text{train} \leftarrow \text{not train}\}$
- $P_6 = \{\text{cross} \leftarrow \lnot\text{train}, \lnot\text{train} \leftarrow \text{not train}, \lnot\text{cross} \leftarrow\}$
Train spotting

- $P_1 = \{\text{cross } \leftarrow \text{ not train}\}$
  - stable model: $\{\text{cross}\}$
Train spotting

- $P_2 = \{\text{cross} \leftarrow \neg \text{train}\}$
Train spotting

- \( P_2 = \{ \text{cross } \leftarrow \neg \text{train} \} \)
  - stable model: \( \emptyset \)
Train spotting

- $P_3 = \{\text{cross} \leftarrow \neg \text{train}, \neg \text{train} \leftarrow\}$
Train spotting

- $P_3 = \{\text{cross} \leftarrow \neg\text{train}, \neg\text{train} \leftarrow\}$
  - stable model: $\{\text{cross}, \neg\text{train}\}$
Train spotting

• \( P_4 = \{\text{cross} \leftarrow \neg \text{train}, \neg \text{train} \leftarrow, \neg \text{cross} \leftarrow\} \)
Train spotting

- $P_4 = \{cross \leftarrow \neg train, \neg train \leftarrow, \neg cross \leftarrow\}$
  - stable model: $\{cross, \neg cross, train, \neg train\}$
Train spotting

- $P_5 = \{cross \leftarrow \neg \text{train}, \neg \text{train} \leftarrow \text{not train}\}$
• $P_5 = \{\text{cross} \leftarrow \neg\text{train}, \neg\text{train} \leftarrow \text{not train}\}$
  
  – stable model: $\{\text{cross}, \neg\text{train}\}$
Train spotting

- $P_1 = \{\text{cross} \leftarrow \neg \text{train}\}$
- $P_2 = \{\text{cross} \leftarrow \neg \text{train}\}$
- $P_3 = \{\text{cross} \leftarrow \neg \text{train}, \neg \text{train} \leftarrow \}$
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- $P_5 = \{\text{cross} \leftarrow \neg \text{train}, \neg \text{train} \leftarrow \neg \text{train}\}$
- $P_6 = \{\text{cross} \leftarrow \neg \text{train}, \neg \text{train} \leftarrow \neg \text{train}, \neg \text{cross} \leftarrow\}$

$P_6$ has no stable model.
Train spotting

- $P_6 = \{\text{cross } \leftarrow \neg \text{train}, \; \neg \text{train } \leftarrow \neg \text{not train}, \; \neg \text{cross } \leftarrow \}$
  - no stable model
Train spotting

• \( P_1 = \{\text{cross} \leftarrow \neg \text{train}\} \)
  – stable model: \{cross\}

• \( P_2 = \{\text{cross} \leftarrow \neg \text{train}\} \)
  – stable model: \emptyset

• \( P_3 = \{\text{cross} \leftarrow \neg \text{train}, \neg \text{train} \leftrightsquigarrow\} \)
  – stable model: \{cross, \neg \text{train}\}

• \( P_4 = \{\text{cross} \leftarrow \neg \text{train}, \neg \text{train} \leftrightsquigarrow, \neg \text{cross} \leftrightsquigarrow\} \)
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• \( P_5 = \{\text{cross} \leftarrow \neg \text{train}, \neg \text{train} \leftrightsquigarrow \neg \text{train}\} \)
  – stable model: \{cross, \neg \text{train}\}

• \( P_6 = \{\text{cross} \leftarrow \neg \text{train}, \neg \text{train} \leftrightsquigarrow \neg \text{train}, \neg \text{cross} \leftrightsquigarrow\} \)
  – no stable model
Default negation in rule heads

- We consider logic programs with default negation in rule heads

Given an alphabet $A$ of atoms, let $\tilde{A} = \{ \tilde{a} | a \in A \}$ such that $A \cap \tilde{A} = \emptyset$.

Given a program $P$ over $A$, consider the program $\tilde{P} = \{ r \in P | \text{head}(r) \neq \text{not } a \} \cup \{ \leftarrow \text{body}(r) \cup \{ \text{not } \tilde{a} \} | r \in P \text{ and } \text{head}(r) = \text{not } a \} \cup \{ \tilde{a} \leftarrow \text{not } a | r \in P \text{ and } \text{head}(r) = \text{not } a \}$.

A set $X$ of atoms is a stable model of a program $P$ (with default negation in rule heads) over $A$, if $X = Y \cap A$ for some stable model $Y$ of $\tilde{P}$ over $A \cup \tilde{A}$.
Default negation in rule heads

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$$
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Outline

6 Two kinds of negation
7 Disjunctive logic programs
8 Aggregates
Disjunctive logic programs

- A disjunctive rule, $r$, is of the form

  $a_1; \ldots; a_m \leftarrow a_{m+1}, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_o$

  where $0 \leq m \leq n \leq o$ and each $a_i$ is an atom for $0 \leq i \leq o$

- A disjunctive logic program is a finite set of disjunctive rules
Disjunctive logic programs

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a_1 ; \ldots ; a_m \leftarrow a_{m+1}, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_o
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• A disjunctive logic program is a finite set of disjunctive rules

• Notation

\[
\begin{align*}
\text{head}(r) & = \{a_1, \ldots, a_m\} \\
\text{body}(r) & = \{a_{m+1}, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_o\} \\
\text{body}(r)^+ & = \{a_{m+1}, \ldots, a_n\} \\
\text{body}(r)^- & = \{a_{n+1}, \ldots, a_o\} \\
atom(P) & = \bigcup_{r \in P} \left( \text{head}(r) \cup \text{body}(r)^+ \cup \text{body}(r)^- \right) \\
\text{body}(P) & = \{\text{body}(r) \mid r \in P\}
\end{align*}
\]
Disjunctive logic programs

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  \[
  a_1 ; \ldots ; a_m \leftarrow a_{m+1}, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_o
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  \text{head}(r) = \{a_1, \ldots, a_m\} \\
  \text{body}(r) = \{a_{m+1}, \ldots, a_n, \text{not } a_{n+1}, \ldots, \text{not } a_o\} \\
  \text{body}(r)^+ = \{a_{m+1}, \ldots, a_n\} \\
  \text{body}(r)^- = \{a_{n+1}, \ldots, a_o\} \\
  \text{atom}(P) = \bigcup_{r \in P} \left( \text{head}(r) \cup \text{body}(r)^+ \cup \text{body}(r)^- \right) \\
  \text{body}(P) = \{\text{body}(r) \mid r \in P\}
  \]

- A program is called positive if \( \text{body}(r)^- = \emptyset \) for all its rules
Stable models

- **Positive programs**
  - A set $X$ of atoms is *closed under* a positive program $P$ iff for any $r \in P$, $\text{head}(r) \cap X \neq \emptyset$ whenever $\text{body}(r)^+ \subseteq X$
  - $X$ corresponds to a model of $P$ (seen as a formula)
  - The set of all $\subseteq$-minimal sets of atoms being closed under a positive program $P$ is denoted by $\text{min}_{\subseteq}(P)$
    - $\text{min}_{\subseteq}(P)$ corresponds to the $\subseteq$-minimal models of $P$ (ditto)
Stable models

- **Positive programs**
  - A set $X$ of atoms is **closed under** a positive program $P$ iff for any $r \in P$, $\text{head}(r) \cap X \neq \emptyset$ whenever $\text{body}(r)^+ \subseteq X$
  
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- **Disjunctive programs**
  
  - The **reduct**, $P^X$, of a disjunctive program $P$ relative to a set $X$ of atoms is defined by

  \[
P^X = \{\text{head}(r) \leftarrow \text{body}(r)^+ \mid r \in P \text{ and } \text{body}(r)^- \cap X = \emptyset\}\]
Stable models

- **Positive programs**
  - A set $X$ of atoms is **closed under** a positive program $P$ iff for any $r \in P$, $\text{head}(r) \cap X \neq \emptyset$ whenever $\text{body}(r)^+ \subseteq X$
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- **Disjunctive programs**
  - The **reduct**, $P_X^X$, of a disjunctive program $P$ relative to a set $X$ of atoms is defined by
    $$P_X^X = \{ \text{head}(r) \leftarrow \text{body}(r)^+ \mid r \in P \text{ and } \text{body}(r)^- \cap X = \emptyset \}$$
  - A set $X$ of atoms is a **stable model** of a disjunctive program $P$, if $X \in \text{min}_{\subseteq}(P_X^X)$
A “positive” example

\[ P = \{ \begin{array}{c}
    a \\
    b \\
    ; c \leftrightarrow a
\end{array} \} \]
A “positive” example

\[
P = \left\{ \begin{array}{c}
a \\
b \\
c \leftarrow a \end{array} \right. \]

- The sets \{a, b\}, \{a, c\}, and \{a, b, c\} are closed under \( P \)
A “positive” example

\[ P = \left\{ \begin{array}{c} a \\ b ; c \leftrightarrow a \end{array} \right\} \]

- The sets \{a, b\}, \{a, c\}, and \{a, b, c\} are closed under \( P \).
- We have \( \min_{\subseteq} (P) = \{\{a, b\}, \{a, c\}\} \).
Graph coloring (reloaded)

node(1..6).

edge(1, (2;3;4)). edge(2, (4;5;6)). edge(3, (1;4;5)).
edge(4, (1;2)). edge(5, (3;4;6)). edge(6, (2;3;5)).

color(X,r) ; color(X,b) ; color(X,g) :- node(X).

:- edge(X,Y), color(X,C), color(Y,C).
Graph coloring (reloaded)

node(1..6).

dge(1, (2;3;4)).  edge(2, (4;5;6)).  edge(3, (1;4;5)).
edge(4, (1;2)).  edge(5, (3;4;6)).  edge(6, (2;3;5)).

col(r).  col(b).  col(g).

color(X,C) : col(C) :- node(X).

:- edge(X,Y), color(X,C), color(Y,C).
More Examples

- $P_1 = \{a \land b \land c \leftarrow\}$
More Examples

- $P_1 = \{a ; b ; c \leftarrow\}$
  - stable models $\{a\}$, $\{b\}$, and $\{c\}$
More Examples

- $P_2 = \{a ; b ; c \leftarrow , \leftarrow a\}$
More Examples

- $P_2 = \{a ; b ; c \leftarrow , \leftarrow a\}$
  - stable models $\{b\}$ and $\{c\}$
More Examples

- $P_3 = \{a; b; c \leftarrow, \leftarrow a, b \leftarrow c, c \leftarrow b\}$
More Examples

• $P_3 = \{a ; b ; c ← , ← a , b ← c , c ← b\}$
  – stable model $\{b , c\}$
• $P_4 = \{a ; b \leftarrow c, \ b \leftarrow \text{not } a, \text{not } c, \ a ; c \leftarrow \text{not } b\}$
More Examples

- $P_4 = \{ a ; b \leftarrow c , b \leftarrow \text{not } a, \text{not } c , a ; c \leftarrow \text{not } b \}$
  - stable models $\{a\}$ and $\{b\}$
More Examples

- $P_1 = \{a \mid b ; c \leftarrow \}$
  - stable models $\{a\}$, $\{b\}$, and $\{c\}$

- $P_2 = \{a \mid b ; c \leftarrow , \leftarrow a\}$
  - stable models $\{b\}$ and $\{c\}$

- $P_3 = \{a \mid b ; c \leftarrow , \leftarrow a , b \leftarrow c , c \leftarrow b\}$
  - stable model $\{b, c\}$

- $P_4 = \{a \mid b \leftarrow c , b \leftarrow not a, not c , a ; c \leftarrow not b\}$
  - stable models $\{a\}$ and $\{b\}$
Some properties

- A disjunctive logic program may have zero, one, or multiple stable models.
- If \( X \) is a stable model of a disjunctive logic program \( P \), then \( X \) is a model of \( P \) (seen as a formula).
- If \( X \) and \( Y \) are stable models of a disjunctive logic program \( P \), then \( X \not\subset Y \).
Some properties

- A disjunctive logic program may have zero, one, or multiple stable models.
- If $X$ is a stable model of a disjunctive logic program $P$, then $X$ is a model of $P$ (seen as a formula).
- If $X$ and $Y$ are stable models of a disjunctive logic program $P$, then $X \nsubseteq Y$.
- If $A \in X$ for some stable model $X$ of a disjunctive logic program $P$, then there is a rule $r \in P$ such that $\text{body}(r)^+ \subseteq X$, $\text{body}(r)^- \cap X = \emptyset$, and $\text{head}(r) \cap X = \{A\}$.
An example with variables

\[ P = \begin{cases} 
    a(1,2) & \leftarrow \ a(X, Y), \text{not } c(Y) \\
    b(X) ; c(Y) & \leftarrow \ a(X, Y), \text{not } c(Y) 
\end{cases} \]
An example with variables

\[ P = \{ \begin{array}{c} a(1, 2) \leftarrow a(X, Y), \text{not } c(Y) \\ b(X); c(Y) \leftarrow a(X, Y), \text{not } c(Y) \end{array} \} \]

\[ \text{ground}(P) = \{ \begin{array}{c} a(1, 2) \leftarrow a(1, 1), \text{not } c(1) \\ b(1); c(1) \leftarrow a(1, 1), \text{not } c(1) \\ b(1); c(2) \leftarrow a(1, 2), \text{not } c(2) \\ b(2); c(1) \leftarrow a(2, 1), \text{not } c(1) \\ b(2); c(2) \leftarrow a(2, 2), \text{not } c(2) \end{array} \} \]
An example with variables

\[ P = \left\{ \begin{array}{l}
  a(1, 2) \leftarrow \\
  b(X) \land c(Y) \leftarrow a(X, Y), \text{not } c(Y)
\end{array} \right\} \]

\[ \text{ground}(P) = \left\{ \begin{array}{l}
  a(1, 2) \leftarrow \\
  b(1) \land c(1) \leftarrow a(1, 1), \text{not } c(1) \\
  b(1) \land c(2) \leftarrow a(1, 2), \text{not } c(2) \\
  b(2) \land c(1) \leftarrow a(2, 1), \text{not } c(1) \\
  b(2) \land c(2) \leftarrow a(2, 2), \text{not } c(2)
\end{array} \right\} \]

For every stable model \( X \) of \( P \), we have

- \( a(1, 2) \in X \) and
- \( \{a(1, 1), a(2, 1), a(2, 2)\} \cap X = \emptyset \)
An example with variables

\[
\text{ground}(P) = \begin{cases} 
  a(1, 2) & \leftarrow a(1, 1), \text{not } c(1) \\
  b(1) ; c(1) & \leftarrow a(1, 2), \text{not } c(2) \\
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  b(2) ; c(1) & \leftarrow a(2, 2), \text{not } c(2) \\
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\end{cases}
\]

- Consider \( X = \{a(1, 2), b(1)\} \)
An example with variables

\[ \text{ground}(P)^X = \begin{cases} 
    a(1, 2) & \leftarrow a(1, 1) \\
    b(1) ; c(1) & \leftarrow a(1, 2) \\
    b(2) ; c(1) & \leftarrow a(1, 2) \\
    b(2) ; c(2) & \leftarrow a(2, 1) \\
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• Consider \( X = \{a(1, 2), b(1)\} \)
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  b(2) ; c(1) & \leftarrow a(2, 1) \\
  b(2) ; c(2) & \leftarrow a(2, 2)
\end{cases}
\]

- Consider \( X = \{a(1, 2), b(1)\} \)
- We get \( \text{min}_{\subseteq}(\text{ground}(P)^X) = \{ \{a(1, 2), b(1)\}, \{a(1, 2), c(2)\} \} \)
An example with variables

\[
\text{ground}(P)^X = \left\{ \begin{array}{c}
  a(1,2) \quad \leftarrow \\
  b(1) ; c(1) \quad \leftarrow \quad a(1,1) \\
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  b(2) ; c(2) \quad \leftarrow \quad a(2,2) \\
\end{array} \right\}
\]

- Consider \( X = \{a(1,2), b(1)\} \)
- We get \( \min_{\subseteq} (\text{ground}(P)^X) = \{ \{a(1,2), b(1)\}, \{a(1,2), c(2)\} \} \)
- \( X \) is a stable model of \( P \) because \( X \in \min_{\subseteq} (\text{ground}(P)^X) \)
An example with variables

\[\text{ground}(P) = \begin{cases}
  a(1, 2) & \leftarrow \\
  b(1) ; c(1) & \leftarrow a(1, 1), \text{not } c(1) \\
  b(1) ; c(2) & \leftarrow a(1, 2), \text{not } c(2) \\
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\end{cases} \]

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    b(2) ; c(1) & \leftarrow a(2, 1) 
\end{cases}
\]

- Consider \( X = \{a(1, 2), c(2)\} \)
- We get \( \min_{\subseteq} (\ground(P)^X) = \{\{a(1, 2)\}\} \)
- \( X \) is no stable model of \( P \) because \( X \notin \min_{\subseteq} (\ground(P)^X) \)
Default negation in rule heads

- Consider disjunctive rules of the form

\[
a_1 ; \ldots ; a_m ; not \ a_{m+1} ; \ldots ; not \ a_n \leftarrow a_{n+1}, \ldots, a_o, not \ a_{o+1}, \ldots, not \ a_p
\]

where \(0 \leq m \leq n \leq o \leq p\) and each \(a_i\) is an atom for \(0 \leq i \leq p\)
Default negation in rule heads

- Consider disjunctive rules of the form

  \[a_1; \ldots; a_m; not \ a_{m+1}; \ldots; not \ a_n \leftarrow a_{n+1}, \ldots, a_o, not \ a_{o+1}, \ldots, not \ a_p\]

  where \(0 \leq m \leq n \leq o \leq p\) and each \(a_i\) is an atom for \(0 \leq i \leq p\)

- Given a program \(P\) over \(A\), consider the program

\[
\tilde{P} = \{\text{head}(r)^+ \leftarrow \text{body}(r) \cup \{\text{not } \tilde{a} \mid a \in \text{head}(r)^-\} \mid r \in P\} \\
\cup \{\tilde{a} \leftarrow \text{not } a \mid r \in P \text{ and } a \in \text{head}(r)^-\}
\]

A set \(X\) of atoms is a stable model of a disjunctive program \(P\) (with default negation in rule heads) over \(A\), if \(X = Y \cap A\) for some stable model \(Y\) of \(\tilde{P}\) over \(A \cup \tilde{A}\).
Default negation in rule heads

- Consider disjunctive rules of the form

\[ a_1; \ldots; a_m; \text{not } a_{m+1}; \ldots; \text{not } a_n \leftarrow a_{n+1}, \ldots, a_o, \text{not } a_{o+1}, \ldots, \text{not } a_p \]

where \(0 \leq m \leq n \leq o \leq p\) and each \(a_i\) is an atom for \(0 \leq i \leq p\)

- Given a program \(P\) over \(\mathcal{A}\), consider the program

\[
\tilde{P} = \{ \text{head}(r)^+ \leftarrow \text{body}(r) \cup \{\text{not } \tilde{a} \mid a \in \text{head}(r)^- \} \mid r \in P \} \\
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\]

- A set \(X\) of atoms is a **stable model** of a disjunctive program \(P\) (with default negation in rule heads) over \(\mathcal{A}\), if \(X = Y \cap \mathcal{A}\) for some stable model \(Y\) of \(\tilde{P}\) over \(\mathcal{A} \cup \tilde{\mathcal{A}}\)
An example

- The program

\[ P = \{ a ; \text{not } a \leftrightarrow \} \]
An example

- The program

\[ P = \{ a ; \text{not } a \leftarrow \} \]

yields

\[ \tilde{P} = \{ a \leftarrow \text{not } \tilde{a} \} \cup \{ \tilde{a} \leftarrow \text{not } a \} \]
An example

- The program

\[ P = \{a ; not a \leftarrow\} \]

yields

\[ \tilde{P} = \{a \leftarrow not \tilde{a}\} \cup \{\tilde{a} \leftarrow not a\} \]

- \( \tilde{P} \) has two stable models, \( \{a\} \) and \( \{\tilde{a}\} \)
An example

- The program
  \[ P = \{ a ; \text{not } a \} \]
  yields
  \[ \tilde{P} = \{ a \leftarrow \text{not } \tilde{a} \} \cup \{ \tilde{a} \leftarrow \text{not } a \} \]
- \( \tilde{P} \) has two stable models, \( \{ a \} \) and \( \{ \tilde{a} \} \)
- This induces the stable models \( \{ a \} \) and \( \emptyset \) of \( P \)
Motivation

- Aggregates provide a general way to obtain a single value from a collection of input values
- Popular aggregate (functions)
  - average
  - count
  - maximum
  - minimum
  - sum
- Cardinality and weight constraints rely on count and sum aggregates
Syntax

- An **aggregate** has the form:

  \[ \alpha \{ w_1 : a_1, \ldots, w_m : a_m, w_{m+1} : not a_{m+1}, \ldots, w_n : not a_n \} \prec k \]

  where for \( 1 \leq i \leq n \)
  - \( \alpha \) stands for a function mapping multisets over \( \mathbb{Z} \) to \( \mathbb{Z} \cup \{ +\infty, -\infty \} \)
  - \( \prec \) stands for a relation between \( \mathbb{Z} \cup \{ +\infty, -\infty \} \) and \( \mathbb{Z} \)
  - \( k \in \mathbb{Z} \)
  - \( a_i \) are atoms and
  - \( w_i \) are integers

- **Example** \( \text{sum} \ \{ 30 : hd(a), \ldots, 50 : hd(m) \} \leq 300 \)
Semantics

- A (positive) aggregate $\alpha \{w_1 : a_1, \ldots, w_n : a_n\} \prec k$ can be represented by the formula:

$$\bigwedge_{I \subseteq \{1, \ldots, n\}, \alpha\{w_i | i \in I\} \not\prec k} \left( \bigwedge_{i \in I} a_i \rightarrow \bigvee_{i \in \overline{I}} a_i \right)$$

where $\overline{I} = \{1, \ldots, n\} \setminus I$ and $\not\prec$ is the complement of $\prec$

- Then, $\alpha \{w_1 : a_1, \ldots, w_n : a_n\} \prec k$ is true in $X$ iff the above formula is true in $X$
Example

- Consider \( \text{sum}\{1 : p, 1 : q\} \neq 1 \)
  
  That is, \( a_1 = p, a_2 = q \) and \( w_1 = 1, w_2 = 1 \)

- Calculemus!

<table>
<thead>
<tr>
<th>( I )</th>
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  That is, $a_1 = p$, $a_2 = q$ and $w_1 = 1$, $w_2 = 1$

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- We get $(p \rightarrow q) \land (q \rightarrow p)$
Example

- Consider \( \text{sum}\{1 : p, 1 : q\} \neq 1 \)
  That is, \( a_1 = p, a_2 = q \) and \( w_1 = 1, w_2 = 1 \)
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- We get \((p \to q) \land (q \to p)\)
- Analogously, we obtain \((p \lor q) \land \neg(p \land q)\) for \( \text{sum}\{1 : p, 1 : q\} = 1 \).
Monotonicity

- **Monotone aggregates**
  - For instance,
    - \( \text{body}(r)^+ \)
    - \( \text{sum}\{1 : p, 1 : q\} > 1 \)
  - We get a simpler characterization: \( \wedge_{I \subseteq \{1, \ldots, n\}, \alpha \{ w_i | i \in I \} \not\prec k} \bigvee_{i \in I} a_i \)

- **Anti-monotone aggregates**
  - For instance,
    - \( \text{body}(r)^- \)
    - \( \text{sum}\{1 : p, 1 : q\} < 1 \)
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- **Non-monotone aggregates**
  - For instance, \( \text{sum}\{1 : p, 1 : q\} \neq 1 \) is non-monotone.
Monotonicity

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  - For instance,
    - \( \text{body}(r)^+ \)
    - \( \text{sum}\{1 : p, 1 : q\} > 1 \) amounts to \( p \land q \)
  - We get a simpler characterization:
    \[ \bigwedge_{I \subseteq \{1, \ldots, n\}, \alpha\{w_i | i \in I\} \not≺ k} \bigvee_{i \in I} a_i \]

- **Anti-monotone aggregates**
  - For instance,
    - \( \text{body}(r)^- \)
    - \( \text{sum}\{1 : p, 1 : q\} < 1 \) amounts to \( \neg p \land \neg q \)
  - We get a simpler characterization:
    \[ \bigwedge_{I \subseteq \{1, \ldots, n\}, \alpha\{w_i | i \in I\} \not≺ k} \neg \bigwedge_{i \in I} a_i \]

- **Non-monotone aggregates**
  - For instance, \( \text{sum}\{1 : p, 1 : q\} \neq 1 \) is non-monotone.
Computational Aspects: Overview

9 Complexity
Complexity

Let $a$ be an atom and $X$ be a set of atoms
Let $a$ be an atom and $X$ be a set of atoms

- For a positive normal logic program $P$:
  - Deciding whether $X$ is the stable model of $P$ is P-complete
  - Deciding whether $a$ is in the stable model of $P$ is P-complete
Complexity

Let $a$ be an atom and $X$ be a set of atoms

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- For a normal logic program $P$:
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Complexity

Let \( a \) be an atom and \( X \) be a set of atoms

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- For a normal logic program \( P \):
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  - Deciding whether \( a \) is in a stable model of \( P \) is NP-complete

- For a normal logic program \( P \) with optimization statements:
  - Deciding whether \( X \) is an optimal stable model of \( P \) is co-NP-complete
  - Deciding whether \( a \) is in an optimal stable model of \( P \) is \( \Delta^p_2 \)-complete
Complexity

Let $a$ be an atom and $X$ be a set of atoms

- For a positive disjunctive logic program $P$:
  - Deciding whether $X$ is a stable model of $P$ is co-NP-complete
  - Deciding whether $a$ is in a stable model of $P$ is $NP^{NP}$-complete

- For a disjunctive logic program $P$:
  - Deciding whether $X$ is a stable model of $P$ is co-NP-complete
  - Deciding whether $a$ is in a stable model of $P$ is $NP^{NP}$-complete

- For a disjunctive logic program $P$ with optimization statements:
  - Deciding whether $X$ is an optimal stable model of $P$ is co-$NP^{NP}$-complete
  - Deciding whether $a$ is in an optimal stable model of $P$ is $\Delta_{3}^{P}$-complete
Let $a$ be an atom and $X$ be a set of atoms

- For a positive disjunctive logic program $P$:
  - Deciding whether $X$ is a stable model of $P$ is co-NP-complete
  - Deciding whether $a$ is in a stable model of $P$ is NP$^N_P$-complete

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  - Deciding whether $a$ is in a stable model of $P$ is NP$^N_P$-complete

- For a disjunctive logic program $P$ with optimization statements:
  - Deciding whether $X$ is an optimal stable model of $P$ is co-NP$^N_P$-complete
  - Deciding whether $a$ is in an optimal stable model of $P$ is $\Delta^P_3$-complete

- For a propositional theory $\Phi$:
  - Deciding whether $X$ is a stable model of $\Phi$ is co-NP-complete
  - Deciding whether $a$ is in a stable model of $\Phi$ is NP$^N_P$-complete
Martin Gebser, Benjamin Kaufmann Roland Kaminski, and Torsten Schaub.
*Answer Set Solving in Practice.*
doi=10.2200/S00457ED1V01Y201211AIM019.

- See also: [http://potassco.sourceforge.net](http://potassco.sourceforge.net)