TECHNISCHE UNIVERSITÄT DRESDEN

International Center for Computational Logic

COMPLEXITY THEORY

Lecture 11: Games/Logarithmic Space

Markus Krötzsch

Knowledge-Based Systems

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Review

Review: PSpace-complete problems

We have encountered some PSpace-complete problems so far:

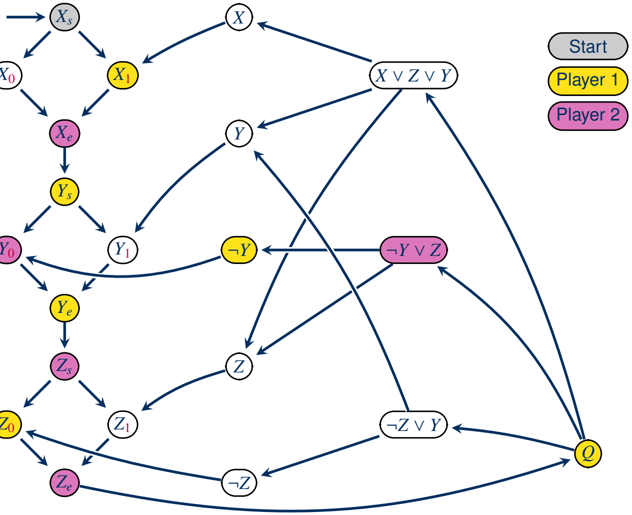
- The word problem for polynomially space bounded (N)TMs
- **True QBF**
- **FOL Model Checking** (and SQL query answering)

Several typical PSpace problems are related to the existence of winning strategies in 2-player games:

- **Formula Game**
- **Geography**

Review: Geography is PSpace-hard

We consider the formula $\exists X.\forall Y.\exists Z.(X \vee Z \vee Y) \wedge (\neg Y \vee Z) \wedge (\neg Z \vee Y)$



More Games

The characteristic of PSpace is quantifier alternation

This is closely related to taking turns in 2-player games.

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- Issue 1: many games are finite – that is: computationally trivial
  $\leadsto$ generalise games to arbitrarily large boards
  - generalised Tic-Tac-Toe is PSpace-complete
  - generalised Reversi is PSpace-complete
  - it is not always clear how to generalise a game (Generalised Backgammon?)

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- Issue 2: (generalised) games where moves can be reversed may require very long matches
  $\leadsto$ such games often are even harder
  - generalised Go with Japanese ko rule is ExpTime-complete
  - generalised Draughts (Checkers) is ExpTime-complete
  - generalised Chess (without 50-move no-capture draw rule) is ExpTime-complete

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Surprisingly, some of these games, e.g. Chess, are known to become even harder – namely ExpSpace-complete – if the exact same board position is not allowed to re-occur in a match. For Go, this case is open (link).

Logarithmic Space

Logarithmic Space

Polynomial space

As we have seen, polynomial space is already quite powerful.

We therefore consider more restricted space complexity classes.

Linear space

Even linear space is enough to solve **Sat**.

Sub-linear space

To get sub-linear space complexity, we consider Turing-machines with separate input tape and only count working space.

Recall:

$$\mathsf{L} = \mathsf{LogSpace} = \mathsf{DSpace}(\log n)$$
$$\mathsf{NL} = \mathsf{NLogSpace} = \mathsf{NSpace}(\log n)$$

Problems in L and NL

What sort of problems are in L and NL?

In logarithmic space we can store

- a fixed number of counters (up to length of input)
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- a fixed number of pointers to positions in the input string

Hence,

- L contains all problems requiring only a constant number of counters/pointers for solving.
- NL contains all problems requiring only a constant number of counters/pointers for verifying solutions.

Examples: Problems in L

Example 11.1: The language $\{0^n 1^n \mid n \geq 0\}$ is in L.

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Algorithm:

- Check that no 1 is ever followed by a 0
  Requires no working space (only movements of the read head)
- Count the number of 0's and 1's
- Compare the two counters

Examples: Problems in L

PALINDROMES

Input: Word w on some input alphabet Σ

Problem: Does w read the same forward and backward?

Example 11.2: **PALINDROMES** $\in$ L.

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Example 11.2: PALINDROMES $\in$ L.

Algorithm:

- Use two pointers, one to the beginning and one to the end of the input.
- At each step, compare the two symbols pointed to.
- Move the pointers one step inwards.

Example: A Problem in NL

REACHABILITY *a.k.a.* STCON *a.k.a.* PATH

Input:	Directed graph G, vertices $s, t \in V(G)$
Problem:	Does G contain a path from s to t?

Example 11.3: **REACHABILITY** $\in$ NL.

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Example 11.3: REACHABILITY $\in$ NL.

Algorithm:

- Use a pointer to the current vertex, starting in s
- Iteratively move pointer from current vertex to some neighbour vertex nondeterministically
- Accept when finding t; reject when searching for too long

An Algorithm for Reachability

More formally:

```
CanReach(G,s,t) :
   c := |V(G)| // counter
   p := s      // pointer
   while c > 0 :
      if p = t :
         return TRUE
      else :
         nondeterministically select G-successor p′ of p
         p := p′
         c := c − 1
   // eventually, if no success:
   return FALSE
```

Defining Reductions in Logarithmic Space

To compare the difficulty of problems in P or NL, polynomial-time reductions are useless. Recall the respective result from Lecture 5:

Theorem 5.22: If **B** is any language in P, $\mathbf{B} \neq \emptyset$, and $\mathbf{B} \neq \Sigma^*$, then $\mathbf{A} \leq_p \mathbf{B}$ for any $\mathbf{A} \in$ P.

This also applies to languages in NL ($\subseteq$ P).

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This also applies to languages in NL ($\subseteq$ P).

Definition 11.4: A log-space transducer $\mathcal{M}$ is a logarithmic space bounded Turing machine with a read-only input tape and a write-only, write-once output tape, and that halts on all inputs.

A log-space transducer $\mathcal{M}$ computes a function $f : \Sigma^* \to \Sigma^*$, where $f(w)$ is the content of the output tape of $\mathcal{M}$ running on input w when $\mathcal{M}$ halts.

In this case, f is called a log-space computable function.

Log-Space Reductions and NL-Completeness

Definition 11.5: A log-space reduction from $\mathbf{L} \subseteq \Sigma^*$ to $\mathbf{L}' \subseteq \Sigma^*$ is a log-space computable function $f : \Sigma^* \to \Sigma^*$ such that for all $w \in \Sigma^*$:

$$w \in \mathbf{L} \iff f(w) \in \mathbf{L}'$$

We write $\mathbf{L} \leq_L \mathbf{L}'$ in this case.

Definition 11.6: A problem $\mathbf{L} \in \mathrm{NL}$ is complete for NL if every other language in NL is log-space reducible to $\mathbf{L}$.

Detour: P-completeness

Log-space reductions are also used to define P-complete problems:

Definition 11.7: A problem $\mathbf{L} \in \mathsf{P}$ is complete for P if every other language in P is log-space reducible to $\mathbf{L}$.

We will see some examples in later lectures ...

Remark: Log-space Reductions for Larger Classes?

Could we use log-space reductions instead of polynomial reductions for defining hardness for other classes, e.g., for NP?

- Some authors do this (prominently Papadimitriou)
- All concrete polynomial reductions we have seen can be computed in logarithmic space

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Obvious question: Are the classes "NP-complete problems under polynomial time reductions" and "NP-complete problems under log-space reductions" different?

Today's answer: Nobody knows (YCTBF)

(at least we have not seen any example of such differences, so it might not matter much in practice)

An NL-Complete Problem

Theorem 11.8: **REACHABILITY** is NL-complete.

Proof idea: We already showed membership. What remains is hardness.

Let $\mathcal{M}$ be a non-deterministic log-space TM deciding **L**.

On input w:

(1) modify Turing machine to have a unique accepting configuration (easy)

(2) construct the configuration graph (graph whose nodes are configurations of $\mathcal{M}$ and edges represent possible computational steps of $\mathcal{M}$ on w)

(3) find a path from the start configuration to the accepting configuration

NL-Completeness

Proof sketch: We construct $\langle G, s, t\rangle$ from $\mathcal{M}$ and w using a log-space transducer:

(1) A configuration $(q, w_2, (p_1, p_2))$ of $\mathcal{M}$ can be described in $c \log n$ space for some constant c and $n = |w|$.

(2) List the nodes of G by going through all strings of length $c \log n$ and outputting those that correspond to legal configurations.

(3) List the edges of G by going through all pairs of strings (C_1, C_2) of length $c \log n$ and outputting those pairs where $C_1 \vdash_{\mathcal{M}} C_2$.

(4) s is the starting configuration of G.

(5) Assume w.l.o.g. that $\mathcal{M}$ has a single accepting configuration t.

$w \in$ **L** iff $\langle G, s, t\rangle \in$ **Reachability**

(see also Sipser, Theorem 8.25) □

coNL

As for time, we consider complement classes for space.

Recall Definition 9.6:
For a complexity class C, we define coC := $\{\mathbf{L} \mid \overline{\mathbf{L}} \in \mathsf{C}\}$.

Complement classes for space:

- coNL := $\{\mathbf{L} \mid \overline{\mathbf{L}} \in \mathsf{NL}\}$
- coNPSpace := $\{\mathbf{L} \mid \overline{\mathbf{L}} \in \mathsf{NPSpace}\}$

From Savitch's theorem:

PSpace = NPSpace and hence coNPSpace = PSpace,
but merely NL ⊆ DSpace $(\log^2 n)$ and hence coNL ⊆ DSpace $(\log^2 n)$

The NL vs. coNL Problem

Another famous problem in complexity theory: is NL = coNL?

- First stated in 1964 [Kuroda]
- Related question: are complements of context-sensitive languages also context-sensitive?
  (such languages are recognized by linear-space bounded TMs)
- Open for decades, although most experts believe NL $\neq$ coNL

The Immerman-Szelepcsényi Theorem

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Theorem 11.9 (Immerman 1987/Szelepcsényi 1987): NL = coNL.

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Proof: Show that $\overline{\textsc{Reachability}}$ is in NL. (Why does this suffice?)

Remark: alternative explanations provided by

- Sipser (Theorem 8.27)
- Dick Lipton's blog entry We All Guessed Wrong (link)
- Wikipedia Immerman–Szelepcsényi theorem

Towards Nondeterminsitic Nonreachability

How could we check in logarithmic space that t is not reachable from s?

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Initial idea: iterate through all reachable nodes looking for t

```
NaiveNonReach(G, s, t) :
  for each vertex v of G :
    if CanReach(G, s, v) and v = t :
      return FALSE
  // eventually, if FALSE was not returned above:
  return TRUE
```

Does this work?

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```

Does this work?

No: the check CanReach(G, s, v) may fail even if v is reachable from s
Hence there are many (nondeterministic) runs where the algorithm accepts, although t is reachable from s.

Towards Nondeterminsitic Nonreachability

Things would be different if we knew
the number *count* of vertices reachable from s:

```
CountingNonReach(G, s, t, count) :
  reached := 0
  for each vertex v of G :
    if CanReach(G, s, v) :
      reached := reached + 1
      if v = t :
        return FALSE
  // eventually, if FALSE was not returned above:
  return (count = reached)
```

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   // eventually, if FALSE was not returned above:
   return (count = reached)
```

Problem: how can we know *count*?

Counting Reachable Vertices – Intuition

Idea:

- Count number of vertices reachable in at most $length$ steps
  - we call this number $count_{length}$
  - then the number we are looking for is $count = count_{|V(G)|-1}$

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- Use a limited-length reachability test:
  CanReach$(G, s, v, length)$: "t reachable from s in G in $\leq length$ steps"
  (we actually implemented CanReach(G, s, v) as CanReach$(G, s, v, |V(G)| - 1)$)

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- Compute the count iteratively, starting with $length = 0$ steps:
  - for $length > 0$, go through all vertices u of G and check if they are reachable
  - to do this, for each such u, go through all v reachable by a shorter path, and check if you can directly reach u from them
  - use the counting trick to make sure you don't miss any v
    (the required number $count_{length}$ was computed before)

Counting Reachable Vertices – Algorithm

The count for $length = 0$ is 1. For $length > 0$, we compute as follows:

```
01 CountReachable(G, s, length, count_{length-1}) :
02    count := 1 // we always count s
03    for each vertex u of G such that u ≠ s :
04       reached := 0
05       for each vertex v of G :
06          if CanReach(G, s, v, length − 1) :
07             reached := reached + 1
08             if G has an edge v → u :
09                count := count + 1
10                GOTO 03 // continue with next u
11       if reached < count_{length-1} :
12          REJECT // whole algorithm fails
13    return count
```

Completing the Proof of NL = coNL

Putting the ingredients together:

```
NonReachable(G, s, t) :
   count := 1 // number of nodes reachable in 0 steps
   for ℓ := 1 to |V(G)| − 1 :
      count_prev := count
      count := CountReachable(G, s, ℓ, count_prev)
   return CountingNonReach(G, s, t, count)
```

It is not hard to see that this procedure runs in logarithmic space, since we use a fixed number of counters and pointers. □

Summary and Outlook

Winning board games that don't allow moves to be undone is often PSpace-complete

L is the class of problems solvable using only a fixed number of linearly bound counters and pointers to the input

NL is the corresponding non-deterministic class, but we do not know if L = NL

Summary:

$$
\begin{array}{ccccccccccc}
\text{L} & \subseteq & \text{NL} & \subseteq & \text{PTime} & \subseteq & \text{NP} & \subseteq & \text{PSpace} & = & \text{NPSpace} \\
\shortparallel & & \shortparallel & & \shortparallel & & ? & & \shortparallel & & \shortparallel \\
\text{coL} & \subseteq & \text{coNL} & \subseteq & \text{coP} & \subseteq & \text{coNP} & \subseteq & \text{coPSpace} & = & \text{coNPSpace}
\end{array}
$$

What's next?

- So many $\subseteq$! Will we ever get a strict $\subset$?
- More generally: can more resources solve more problems?