TECHNISCHE UNIVERSITÄT DRESDEN

Computational Logic Group

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Description Logics – Syntax and Semantics II

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$\mathcal{ALC}$ Concepts

$\mathcal{ALC}$ is the basic description logic

$\mathcal{ALC}$ concepts C are inductively defined from atomic concepts A and roles R:

$$C ::= \top \mid \bot \mid A \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \forall R.C \mid \exists R.C$$

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Semantics given through DL interpretations $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ with

$$\begin{aligned}
\top^{\mathcal{I}} &= \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} &= \emptyset \\
(\neg C)^{\mathcal{I}} &= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\exists R.C)^{\mathcal{I}} &= \{u \in \Delta^{\mathcal{I}} \mid \exists w \in \Delta^{\mathcal{I}} \text{ s.t. } \langle u, w \rangle \in R^{\mathcal{I}} \text{ and } w \in C^{\mathcal{I}}\} \\
(\forall R.C)^{\mathcal{I}} &= \{u \in \Delta^{\mathcal{I}} \mid \forall w \in \Delta^{\mathcal{I}}, \langle u, w \rangle \in R^{\mathcal{I}} \text{ implies } w \in C^{\mathcal{I}}\}
\end{aligned}$$

$\mathcal{ALC}$ Fragments

What happens to $\mathcal{ALC}$ if we disallow negation? That is, if we define "$\mathcal{ALC}^+$" via

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Nothing:
Instead of $\neg C$, we can use A_C for a new concept name A_C and add the GCIs

$$\top \sqsubseteq C \sqcup A_C$$
$$C \sqcap A_C \sqsubseteq \bot$$

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What happens if we disallow negation, disjunction, and value restriction?
A lot – complexity (of concept satisfiability) drops from PSpace to PTime.

It is an important objective of DL (indeed KR) research to identify fragments that are "computationally well-behaved".

Basic Reasoning Problems and Services

What kinds of reasoning problems and services might be interesting?

Scenario: Ontology design

- We are building a conceptual model (a TBox) for our domain
- At this design stage we haven't yet included the data (no ABox)

Our TBox should be

- Error-free:

  No unintended logical consequences

- Sufficiently detailed:

  Contain all relevant knowledge for our application

Ontology Design

$$
\begin{aligned}
JuvArthritis &\sqsubseteq Arthritis \sqcap JuvDisease \\
Arthritis &\sqsubseteq \exists Damages.Joint \sqcap \forall Damages.Joint \sqcap \exists Affects.Adult \\
JuvDisease &\sqsubseteq Disease \sqcap \forall Affects.(Child \sqcup Teen) \\
Disease \sqcap \exists Damages.Joint &\sqsubseteq JointDisease \\
Child \sqcup Teen &\sqsubseteq \neg Adult
\end{aligned}
$$

This TBox contains modeling errors:

- Juvenile arthritis is a kind of juvenile disease
  - Juvenile disease affects only children or teens, which are not adults
    - A juvenile arthritis cannot affect any adult
- Juvenile arthritis is a kind of arthitis
  - Each arthritis affects some adult
    - Each juvenile arthritis affects some adult

Concept Satisfiability

What is the impact of the error?

- All models $\mathcal{I}$ of $\mathcal{T}$ must be such that $\mathit{JuvArthritis}^{\mathcal{I}} = \emptyset$
  - A juvenile arthritis cannot exist!
    - We cannot add data concerning juvenile arthritis

Such errors can be detected by solving the following problem:

> Concept satisfiability w.r.t. a TBox:
> An instance is a pair $\langle C, \mathcal{T} \rangle$ with C a concept and $\mathcal{T}$ a TBox.
> The answer is true iff a model $\mathcal{I} \models \mathcal{T}$ exists such that $C^{\mathcal{I}} \neq \emptyset$.

In a FOL setting, C is satisfiable w.r.t. $\mathcal{T}$ if and only if

$$\pi(\mathcal{T}) \land \exists x.(\pi_x(C)) \quad \text{is satisfiable}$$

Concept Subsumption

Parts of our arthritis TBox, however, do conform to our intuitions:

$$\begin{aligned}
\mathit{JuvArthritis} &\sqsubseteq \mathit{Arthritis} \sqcap \mathit{JuvDisease} \\
\mathit{Arthritis} &\sqsubseteq \exists \mathit{Damages}.\mathit{Joint} \sqcap \forall \mathit{Damages}.\mathit{Joint} \sqcap \exists \mathit{Affects}.\mathit{Adult} \\
\mathit{JuvDisease} &\sqsubseteq \mathit{Disease} \sqcap \forall \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}) \\
\mathit{Disease} \sqcap \exists \mathit{Damages}.\mathit{Joint} &\sqsubseteq \mathit{JointDisease} \\
\mathit{Child} \sqcup \mathit{Teen} &\sqsubseteq \neg \mathit{Adult}
\end{aligned}$$

Juvenile arthritis is a kind of juvenile disease

Juvenile disease is a kind of disease

Juvenile arthritis is a kind of disease

Juvenile arthritis is a kind of arthitis

Each arthritis damages some joint

Each juvenile arthritis damages some joint

Juvenile arthritis is a joint disease.

Concept Subsumption

We have discovered new interesting information

- All models $\mathcal{I}$ of $\mathcal{T}$ must be such that $\mathit{JuvArthritis}^{\mathcal{I}} \subseteq \mathit{JointDisease}^{\mathcal{I}}$
  - Juvenile arthritis is a sub-type of joint disease
    - All instances of juvenile arthitis are also joint diseases

Such implicit information is detectable by solving the following problem:

> Concept subsumption w.r.t. a TBox:
> An instance is a triple $\langle C, D, \mathcal{T} \rangle$ with C, D concepts, $\mathcal{T}$ a TBox.
> The answer is true iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for each $\mathcal{I} \models \mathcal{T}$ (written $\mathcal{T} \models C \sqsubseteq D$).

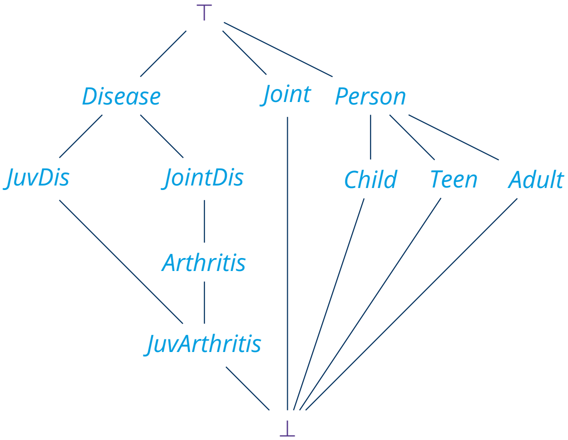
In a FOL setting, C is subsumed by D w.r.t. $\mathcal{T}$ if and only if

$$\pi(\mathcal{T}) \models \forall x.(\pi_x(C) \rightarrow \pi_x(D))$$

TBox Classification

Problem of finding all subsumptions between atomic concepts in $\mathcal{T}$.

Allows us to organise atomic concepts in a subsumption hierarchy:



Knowledge Base Reasoning

TBox:

$$JuvArthritis \sqsubseteq Arthritis \sqcap JuvDisease$$
$$JuvDisease \sqsubseteq Disease$$
$$Arthritis \sqsubseteq \exists Damages.Joint \sqcap \forall Damages.Joint$$
$$JuvDisease \sqsubseteq \forall Affects.(Child \sqcup Teen)$$
$$Child \sqcup Teen \sqsubseteq \neg Adult$$
$$Disease \sqcap \exists Damages.Joint \sqsubseteq JointDisease$$

ABox:

$$JuvArthritis(JRA)$$
$$Affects(JRA, MaryJones)$$
$$Disease(D)$$
$$Joint(J)$$
$$Damages(D, J)$$
$$\neg Teen(MaryJones)$$

May want to answer questions about individuals and/or KB as a whole:

- Is KB (TBox + ABox) consistent, i.e., does there exist a model?
  - What if we add $\neg JointDisease(JRA)$?
- Can we infer additional information about individuals?
  - Is D an instance of any class other than $Disease$?
  - Do we know if $MaryJones$ is an $Adult$ or a $Child$?

Summary of Basic Reasoning Problems

Definition

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$ knowledge base, C, D possibly compound $\mathcal{ALC}$ concepts, and b an individual name. We say that

1. C is satisfiable with respect to $\mathcal{T}$ if there exists a model $\mathcal{I}$ of $\mathcal{T}$ and some $d \in \Delta^{\mathcal{I}}$ with $d \in C^{\mathcal{I}}$;
2. C is subsumed by D with respect to $\mathcal{T}$, written $\mathcal{T} \models C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{T}$;
3. C and D are equivalent with respect to $\mathcal{T}$, written $\mathcal{T} \models C \equiv D$, if $C^{\mathcal{I}} = D^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{T}$;
4. $\mathcal{K}$ is consistent if there exists a model of $\mathcal{K}$;
5. b is an instance of C with respect to $\mathcal{K}$, written $\mathcal{K} \models b : C$, if $b^{\mathcal{I}} \in C^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{K}$.

We write $C \sqsubseteq_{\mathcal{T}} D$ for $\mathcal{T} \models C \sqsubseteq D$ and $C \equiv_{\mathcal{T}} D$ for $\mathcal{T} \models C \equiv D$.

Important Properties of Subsumption

Lemma

Let C, D and E be concepts, b an individual name, and $(\mathcal{T}, \mathcal{A})$, $(\mathcal{T}', \mathcal{A}')$ knowledge bases with $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{A} \subseteq \mathcal{A}'$.

1. $C \sqsubseteq_{\mathcal{T}} C$.
2. If $C \sqsubseteq_{\mathcal{T}} D$ and $D \sqsubseteq_{\mathcal{T}} E$, then $C \sqsubseteq_{\mathcal{T}} E$.
3. If b is an instance of C with respect to $(\mathcal{T}, \mathcal{A})$ and $C \sqsubseteq_{\mathcal{T}} D$, then b is an instance of D with respect to $(\mathcal{T}, \mathcal{A})$.
4. If $\mathcal{T} \models C \sqsubseteq D$ then $\mathcal{T}' \models C \sqsubseteq D$.
5. If $\mathcal{T} \models C \equiv D$ then $\mathcal{T}' \models C \equiv D$.
6. If $(\mathcal{T}, \mathcal{A}) \models b : E$ then $(\mathcal{T}', \mathcal{A}') \models b : E$.

Proofs follow easily from semantics

Reasoning Problem Reductions

Theorem

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$ knowledge base, C, D possibly compound $\mathcal{ALC}$ concepts and b an individual name.

1. $C \equiv_{\mathcal{T}} D$ if and only if $C \sqsubseteq_{\mathcal{T}} D$ and $D \sqsubseteq_{\mathcal{T}} C$.
2. $C \sqsubseteq_{\mathcal{T}} D$ if and only if $C \sqcap \neg D$ is not satisfiable with respect to $\mathcal{T}$.
3. C is satisfiable with respect to $\mathcal{T}$ if and only if $C \not\sqsubseteq_{\mathcal{T}} \bot$.
4. C is satisfiable with respect to $\mathcal{T}$ if and only if $(\mathcal{T}, \{b : C\})$ is consistent.
5. $(\mathcal{T}, \mathcal{A}) \models b : C$ if and only if $(\mathcal{T}, \mathcal{A} \cup \{b : \neg C\})$ is *not* consistent.

Consequently, all the previously mentioned reasoning problems can be reduced to KB (in)consistency.

Basic Reasoning Services

Correspond one-to-one with basic reasoning problems:

1. Given a TBox $\mathcal{T}$ and a concept C, check whether C is *satisfiable* with respect to $\mathcal{T}$.
2. Given a TBox $\mathcal{T}$ and two concepts C and D, check whether C is *subsumed by* D with respect to $\mathcal{T}$.
3. Given a TBox $\mathcal{T}$ and two concepts C and D, check whether C and D are *equivalent* with respect to $\mathcal{T}$.
4. Given a knowledge base $(\mathcal{T}, \mathcal{A})$, check whether $(\mathcal{T}, \mathcal{A})$ is *consistent*.
5. Given a knowledge base $(\mathcal{T}, \mathcal{A})$, an individual name a, and a concept C, check whether a is an *instance of* C w.r.t. $(\mathcal{T}, \mathcal{A})$.

All can be realised via KB consistency checks, e.g.:

$$(\mathcal{T}, \mathcal{A}) \models C \sqsubseteq D \quad \text{iff} \qquad\qquad \text{is not consistent}$$

for a an individual name not occurring in $\mathcal{A}$.

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3. Given a TBox $\mathcal{T}$ and two concepts C and D, check whether C and D are *equivalent* with respect to $\mathcal{T}$.
4. Given a knowledge base $(\mathcal{T}, \mathcal{A})$, check whether $(\mathcal{T}, \mathcal{A})$ is *consistent*.
5. Given a knowledge base $(\mathcal{T}, \mathcal{A})$, an individual name a, and a concept C, check whether a is an *instance of* C w.r.t. $(\mathcal{T}, \mathcal{A})$.

All can be realised via KB consistency checks, e.g.:

$$(\mathcal{T}, \mathcal{A}) \models C \sqsubseteq D \quad \text{iff} \quad (\mathcal{T}, \mathcal{A} \cup \{a : (C \sqcap \neg D)\}) \text{ is not consistent}$$

for a an individual name not occurring in $\mathcal{A}$.

Additional Reasoning Services

We can define additional reasoning services in terms of basic ones:

- Classification of a TBox: given a TBox $\mathcal{T}$, compute the *subsumption hierarchy* of all concept names occurring in $\mathcal{T}$.
  That is, for each pair A, B of concept names occurring in $\mathcal{T}$, check if $\mathcal{T} \models A \sqsubseteq B$ and if $\mathcal{T} \models B \sqsubseteq A$.
- Checking the satisfiability of concepts in $\mathcal{T}$: given a TBox $\mathcal{T}$, for each concept name A in $\mathcal{T}$, test if $\mathcal{T} \not\models A \sqsubseteq \bot$.
- Instance retrieval: given a concept C and a knowledge base $\mathcal{K}$, return all those individual names b such that b is an instance of C with respect to $\mathcal{K}$.
  That is, for each individual name b occurring in $\mathcal{K}$, check if $\mathcal{T} \models b : C$.
- Realisation of an individual name: given an individual name b and a knowledge base $\mathcal{K}$, return all those concept names A such that b is an instance of A with respect to $\mathcal{K}$. That is, for each concept name A occurring in $\mathcal{K}$, check if $\mathcal{T} \models b : A$.

Extensions: Inverse Roles

We might imagine that adding:

$$\mathit{Adult}(\mathit{JohnSmith}) \qquad \mathit{AffectedBy}(\mathit{JohnSmith}, \mathit{JRA})$$

would lead to an inconsistency.

However, this is not the case, because there is no semantic relationship between *Affects* and *AffectedBy*.

In order to relate roles such as Affects and AffectedBy in the desired way, DLs can be extended with inverse roles.

The fact that a DL provides inverse roles is normally indicated by the letter $\mathcal{I}$ in its name, e.g., $\mathcal{ALCI}$.

We will use $\mathcal{L}$ as a placeholder for the name of a DL and write $\mathcal{LI}$ for $\mathcal{L}$ extended with inverse roles.

Extensions: Inverse Roles

Definition

Let $\mathbf{R}$ be the set of role names. For $R \in \mathbf{R}$, R^{-} is an inverse role. The set of $\mathcal{I}$ roles is $\mathbf{R} \cup \{R^{-} \mid R \in \mathbf{R}\}$.

Let $\mathcal{L}$ be a description logic. The set of $\mathcal{LI}$ concepts is the smallest set of concepts that contains all $\mathcal{L}$ concepts and where $\mathcal{I}$ roles can occur in all places of role names.

An interpretation $\mathcal{I}$ maps inverse roles to binary relations as follows:

$$(r^{-})^{\mathcal{I}} = \{(y, x) \mid (x, y) \in r^{\mathcal{I}}\}$$

Typically, DLs supporting inverse roles also allow for inverse roles to be used in axioms such as the following:

$$\mathit{AffectedBy} \equiv \mathit{Affects}^{-}$$

which establishes the intuitive semantic relationship.

Extensions: Number Restrictions

We might want to state that *MildArthritis* *Affects* at most 2 *Joint*s, or that *SevereArthritis* *Affects* at least 5 *Joint*s.

In order to support this, DLs can be extended with (qualified) number restrictions, usually indicated by $\mathcal{N}$ for NRs and $\mathcal{Q}$ for QNRs.

NRs are concept descriptions whose instances are related to at least/most n other individuals via a given role; e.g., $(\leqslant 2\, \mathit{sister})$ describes individuals having at most 2 sisters.

QNRs additionally allow for restricting the type of the target individuals; e.g., $(\geqslant 2\, \mathit{sister}.\mathit{Graduate})$ describes individuals having at least 2 sisters who are graduates.

Note that an NR is equivalent to a QNR where the restriction concept is $\top$; e.g., $(\leqslant 2\, \mathit{sister})$ is equivalent to $(\leqslant 2\, \mathit{sister}.\top)$.

Extensions: Number Restrictions

Definition

For n a non-negative number, r an $\mathcal{L}$ role and C a (possibly compound) $\mathcal{L}$ concept description, a number restriction is a concept description of the form $(\leqslant n\, r)$ or $(\geqslant n\, r)$, and a qualified number restriction is a concept description of the form $(\leqslant n\, r.C)$ or $(\geqslant n\, r.C)$, where C is the qualifying concept.

For an interpretation $\mathcal{I}$, its mapping $\cdot^{\mathcal{I}}$ is extended as follows, where $\#M$ is used to denote the cardinality of a set M:

$$
\begin{aligned}
(\leqslant n\, r)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \#\{e \mid (d,e) \in r^{\mathcal{I}}\} \leq n\},\\
(\geqslant n\, r)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \#\{e \mid (d,e) \in r^{\mathcal{I}}\} \geq n\},\\
(\leqslant n\, r.C)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \#\{e \mid (d,e) \in r^{\mathcal{I}} \text{ and } e \in C^{\mathcal{I}}\} \leq n\},\\
(\geqslant n\, r.C)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \#\{e \mid (d,e) \in r^{\mathcal{I}} \text{ and } e \in C^{\mathcal{I}}\} \geq n\}.
\end{aligned}
$$

Concept descriptions $(=nr)$ and $(=n\, r.C)$ may be used as abbreviations for $(\leqslant n\, r) \sqcap (\geqslant n\, r)$ and $(\leqslant n\, r.C) \sqcap (\geqslant n\, r.C)$, respectively.

Extensions: Nominals

So far our use of individuals has been restricted to ABox axioms.

We may also want to use individuals in concept descriptions; e.g., to describe those individuals who are affected by some *Disease* that also affects the individual *JohnSmith*.

Intuitively, we might try the description

$$\exists \mathit{Affects}^{-}.(\mathit{Disease} \sqcap \exists \mathit{Affects}.\mathit{JohnSmith})$$

but this will not work, because in this context *JohnSmith* must be a concept.†

Nominals allow for the construction of a concept from an individual name; e.g.: $\{\mathit{JohnSmith}\}$ is the concept whose only instance is *JohnSmith*.

The fact that a DL provides nominals is normally indicated by the letter $\mathcal{O}$ in its name ($\mathcal{N}$ is already used for unqualified number restrictions).

† In fact this would be a syntax error if we use *JohnSmith* elsewhere as an individual (the set **C** of concept names and **I** of individual names must be disjoint).

Extensions: Nominals

Definition

Let $\mathbf{I}$ be the set of individual names. For $b \in \mathbf{I}$, $\{b\}$ is called a nominal.

Let $\mathcal{L}$ be a description logic. The description logic $\mathcal{LO}$ is obtained from $\mathcal{L}$ by allowing nominals as additional concepts.

For an interpretation $\mathcal{I}$, its mapping $\cdot^{\mathcal{I}}$ is extended as follows:

$$(\{a\})^{\mathcal{I}} = \{a^{\mathcal{I}}\}$$

- We can now form the desired concept description:

$$\exists \mathit{Affects}^{-}.(\mathit{Disease} \sqcap \exists \mathit{Affects}.\{\mathit{JohnSmith}\})$$

- With nominals, the separation between ABox and TBox is not meaningful:

$$\begin{aligned} C(a) &\equiv \{a\} \sqsubseteq C \\ R(a,b) &\equiv \{a\} \sqsubseteq \exists R.\{b\} \end{aligned}$$

Extensions: Role Hierarchies

We may want our KB to provide some structure for roles as well as concepts; e.g.: we may want to state that roles *brother* and *sister* are subsumed by the role *sibling*.

The fact that a DL provides such role inclusion axioms (RIAs) is normally indicated by the letter $\mathcal{H}$ in its name (there is a $\mathcal{H}$ierarchy of roles).

Definition

A *role inclusion axiom* (RIA) is an axiom of the form $r \sqsubseteq s$ for r, s $\mathcal{L}$ roles.

The DL $\mathcal{LH}$ is obtained from $\mathcal{L}$ by allowing, additionally, role inclusion axioms in TBoxes.

For an interpretation $\mathcal{I}$ to be a *model of* a role inclusion axiom $r \sqsubseteq s$, it has to satisfy

$$r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$$

Extensions: Transitive Roles

We can use the role *parent* to form descriptions such as:

$\exists parent.Irish$	having an *Irish* parent
$\exists parent.(\exists parent.Irish)$	having an *Irish* grandparent
$\exists parent.(\exists parent.(\exists parent.Irish))$	having an *Irish* greatgrandparent

But what if we want to mention *Irish* ancestors without specifying a generation?

We can do that by using a combination of role hierarchy and transitive roles:

$parent \sqsubseteq ancestor$	parent is a sub-role of ancestor
$\mathrm{Trans}(ancestor)$	ancestor is a transitive role
$\exists ancestor.Irish$	having an *Irish* ancestor

Extensions: Transitive Roles

Definition

A role transitivity axiom is an axiom of the form Trans(r) for r an $\mathcal{L}$ role.

The name of the DL that is the extension of $\mathcal{L}$ by allowing, additionally, transitivity axioms in TBoxes, is usually given by replacing $\mathcal{ALC}$ in $\mathcal{L}$'s name with $\mathcal{S}$.

For an interpretation $\mathcal{I}$ to be a model of a role transitivity axiom Trans(r), the relation $r^{\mathcal{I}}$ must be transitive.

- The use of $\mathcal{S}$ to replace $\mathcal{ALC}$ in DLs with transitive roles is inspired by similarities with the modal logic **S4** (and a desire for shorter names).
- However, in some cases (the subscript) $\cdot_{R^+}$ is used to indicate transitive roles; e.g., $\mathcal{SHIQ}$ could be written $\mathcal{ALCHIQ}_{R^+}$.

Extensions: Transitive Roles

It is important to understand the difference between transitive roles and the transitive closure of roles.

- Transitive closure is a role constructor: given a role r, transitive closure can be used to construct a role r^+, with the semantics being that $(r^+)^\mathcal{I} = (r^\mathcal{I})^+$.
- In a logic that includes both transitive roles and role inclusion axioms, e.g., $\mathcal{SH}$, adding axioms Trans(s) and $r \sqsubseteq s$ to a TBox $\mathcal{T}$ ensures that in every model $\mathcal{I}$ of $\mathcal{T}$, $s^\mathcal{I}$ is transitive, and $r^\mathcal{I} \subseteq s^\mathcal{I}$.
- However, we cannot enforce that s is the smallest such transitive role: s is just some transitive role that includes r.
- In contrast, the transitive closure r^+ of r is, by definition, the smallest transitive role that includes r; thus we have:

$$\{\mathrm{Trans}(s), r \sqsubseteq s\} \models r \sqsubseteq r^+ \sqsubseteq s.$$

Relationships to FOL Revisited

As we have seen, $\mathcal{ALC}$ is in the 2-variable fragment of FOL (FO^2):

$$\pi_x(A) = A(x) \qquad \pi_y(A) = A(y)$$
$$\pi_x(\neg C) = \neg \pi_x(C) \qquad \pi_y(\neg C) = \neg \pi_y(C)$$
$$\pi_x(C \sqcap D) = \pi_x(C) \land \pi_x(D) \qquad \pi_y(C \sqcap D) = \pi_y(C) \land \pi_y(D)$$
$$\pi_x(C \sqcup D) = \pi_x(C) \lor \pi_x(D) \qquad \pi_y(C \sqcup D) = \pi_y(C) \lor \pi_y(D)$$
$$\pi_x(\exists R.C) = \exists y.(R(x,y) \land \pi_y(C)) \qquad \pi_y(\exists R.C) = \exists x.(R(y,x) \land \pi_x(C))$$
$$\pi_x(\forall R.C) = \forall y.(R(x,y) \to \pi_y(C)) \qquad \pi_y(\forall R.C) = \forall x.(R(y,x) \to \pi_x(C))$$

$$\pi(C \sqsubseteq D) = \forall x.(\pi_x(C) \to \pi_x(D)) \qquad \pi(R(a,b)) = R(a,b) \qquad \pi(C(a)) = \pi_{x/a}(C)$$

FO^2 satisfiability is known to be decidable in NExpTime.
Moreover, the translation uses quantification only in a restricted way, and therefore yields formulas in the guarded fragment for which satisfiability is known to be decidable in deterministic exponential time.

Relationships to FOL Revisited

- Inverse roles can be captured easily in both the guarded and the two-variable fragments by simply swapping the variable places; e.g., $\pi_x(\exists r^-.C) = \exists y.(r(y,x) \wedge \pi_y(C))$.
- Number restrictions can be captured using (in)equality or so-called *counting quantifiers*; e.g., $\pi_x(\leqslant 2\, r.C) = \exists^{\leq 2} y.(r(x,y) \wedge \pi_y(C))$.
- It is known that the two-variable fragment with counting quantifiers (C^2) is still decidable in nondeterministic exponential time.
- Nominals can be captured using equality; e.g., $\pi_x(\{a\}) = (x = a)$.
- RIAs can also be captured in FO^2; e.g., $\pi(r \sqsubseteq s) = \forall x, y.(r(x,y) \rightarrow s(x,y))$.
- Transitive roles require three variables, and FO^3 is known to be undecidable; however, a satisfiability preserving transformation into FO^2 is still possible.
- This gives us a nondeterministic exponential time upper bound for $\mathcal{SHOIQ}$ satisfiability.

Relationships to Modal Logic

It is not hard to see that $\mathcal{ALC}$ concepts can be viewed as syntactic variants of formulae of multi-modal $\mathbf{K}_{(\mathbf{m})}$:

- Kripke structures can easily be viewed as DL interpretations, and vice versa;
- we can then view concept names as propositional variables, and role names as modal operators;
- we can realise this correspondence through the mapping π as follows:

$$
\begin{aligned}
\pi(A) &= A && \text{for concept names } A \\
\pi(C \sqcap D) &= \pi(C) \land \pi(D) \\
\pi(C \sqcup D) &= \pi(C) \lor \pi(D) \\
\pi(\neg C) &= \neg \pi(C) \\
\pi(\forall r.C) &= [r]\pi(C) \\
\pi(\exists r.C) &= \langle r \rangle \pi(C)
\end{aligned}
$$

Complexity

Undecidable	FOL-3 sat
|	|
Non-Elementary decidable	…
|	|
…	…
|	|
NExpTime	FOL-2, $\mathcal{SHOIQ}$ sat
|	|
ExpTime	Datalog, $\mathcal{SHOI}$ sat
|	|
PSpace	$\mathcal{ALC}$ sat
|	|
NP	PL sat
|	|
P	Horn PL sat

Conclusion

- For description logic knowledge bases, there are various relevant reasoning problems.
- All can be reduced to knowledge base (in)consistency.
- The basic description logic $\mathcal{ALC}$ can be extended in various ways:
  - Inverse Roles $\mathcal{I}$
  - (Qualified) Number Restrictions $(\mathcal{Q})\ \mathcal{N}$
  - Nominals $\mathcal{O}$
  - Role Hierarchies $\mathcal{H}$
  - Transitive Roles $\mathcal{ALC} \rightsquigarrow \mathcal{S}, \cdot_{R^+}$
- Description Logics have close connections with propositional modal logic ...
- ...and with the two-variable fragments of first-order logic (with counting quantifiers)