EXERCISE 8

Science of Computational Logic

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Problem 8.1

Consider the default knowledge base $\langle \mathcal{F}_D, \mathcal{F}_W \rangle$ with

$$\mathcal{F}_D = \left\{ \frac{bird(X) : fly(X)}{fly(X)} \; , \; \frac{fly(X) : happy(X)}{happy(X)} \; , \; \frac{fly(X) : hungry(X)}{hungry(X)} \right\}$$

$$\mathcal{F}_W = \{ \; bird(tweety) \; , \;\; hungry(X) \rightarrow \neg happy(X) \; \}$$

1. Find two different extensions of $\langle \mathcal{F}_D, \mathcal{F}_W \rangle$
   and verify them by means of Theorem 11.7.
2. Find formulas G and G' such that $\langle \mathcal{F}_D, \mathcal{F}_W \rangle \models_c G$ and $\langle \mathcal{F}_D, \mathcal{F}_W \rangle \models_s G'$.

Solution

1. I) $\mathcal{F} = \mathcal{C}(\{\text{bird(tweety)}, \forall \text{hungry}(X) \rightarrow \neg\text{happy}(X),$
$\text{fly(tweety)},$
$\text{happy(tweety)},$
$\neg\text{hungry(tweety)}\})$

(verifying this by Theorem 11.7)

$$\begin{aligned} \mathcal{F}_0 &= \mathcal{F}_W \\ \mathcal{F}_1 &= \mathcal{C}(\mathcal{F}_W) \cup \{\text{fly(tweety)}\} && (\mathcal{C}(\mathcal{F}_W) = \mathcal{F}_W) \\ \mathcal{F}_2 &= \mathcal{C}(\mathcal{F}_1) \cup \{\text{happy(tweety)}\} && (\mathcal{C}(\mathcal{F}_1) = \mathcal{F}_1) \\ \mathcal{F} &= \mathcal{C}(\mathcal{F}_2) \end{aligned}$$

($\neg\text{hungry(tweety)}$ is in $\mathcal{C}(\mathcal{F}_2)$, because
$\{\text{happy(tweety)}, \text{hungry}(X) \rightarrow \neg\text{happy}(X)\} \models \neg\text{hungry(tweety)}$)

II) $\mathcal{F}' = \mathcal{C}(\{\text{bird(tweety)}, \forall \text{hungry}(X) \rightarrow \neg\text{happy}(X)$
$\text{fly(tweety)},$
$\text{hungry(tweety)},$
$\neg\text{happy(tweety)}\})$

(verifying this by Theorem 11.7)

$$\begin{aligned} \mathcal{F}_0 &= \mathcal{F}_W \\ \mathcal{F}_1 &= \mathcal{C}(\mathcal{F}_W) \cup \{\text{fly(tweety)}\} \\ \mathcal{F}_2 &= \mathcal{C}(\mathcal{F}_1) \cup \{\text{hungry(tweety)}\} \\ \mathcal{F}' &= \mathcal{C}(\mathcal{F}_2) \end{aligned}$$

2. $G = \text{hungry(tweety)}$
   $\langle F_D, F_W \rangle \models_C G$ because $G \in \mathcal{F}'$

   $G' = \text{bird(tweety)}$
   $\langle F_D, F_W \rangle \models_S G'$ because $G' \in \mathcal{F} \wedge G' \in \mathcal{F}'$

Problem 8.2

Prove theorem 11.7 of the lectures:

> Let (K_D, K_W) be a closed default knowledge base and K be a set of sentences.
> Define $K_0 = K_W$
> and for $i \geq 1$:
> $K_{i+1} = C(K_i) \cup \{H \mid G : G_1, ..., G_n/H \in K_D, G \in K_i$ and for all $1 \leq j \leq n : \neg G_j \notin K\}$.
> Then, K is an extension of (K_D, K_W) if $K = \bigcup_{i=0}^{\infty} K_i$.

Solution

1. $K_W \subseteq K$, because $K_W = K_0 \subseteq \bigcup_{i=0}^{\infty} K_i = K$
2. To show: $C(K) = K$: $K \subseteq C(K)$: trivial; Remains to show: $C(K) \subseteq K$:

   Assume: There is a $F \in C(K) \setminus K$.

   $F \in C(K)$ implies $K \models F$.

   Consequently, $K \cup \{\neg F\}$ is unsatisfiable.

   Therfore, there exists $\{F_1, \ldots, F_n\} \subseteq K \cup \{\neg F\}$ which is unsatisfiable (compactness theorem).
   Moreover, we may assume that $\{F_1, \ldots, F_n\}$ is an unsatisfiable set.

   (a) $\neg F \notin \{F_1, \ldots, F_n\}$:
   Then $\{F_1, \ldots, F_n\} \subseteq K$.
   There exists $j \in \mathbb{N}$ with $\{F_1, \ldots, F_n\} \subseteq K_j$
   (j is the maximum of the indices of the K_i which contain the F_i
   and the sequence of K_i is growing w.r.t. $\subseteq$.)
   Then K_j is unsatisfiable.
   Then $C(K_j)$ is the entire language,
   and then with $C(K_j) \subseteq K_{j+1} \subseteq \bigcup_{i=0}^{\infty} K_i = K$,
   it follows that K is the entire language.
   But then $C(K) \subseteq K$. **Contradiction**

   (b) $\neg F \in \{F_1, \ldots, F_n\}$: without loss of generality $F = F_1$
   Then $\{F_2, \ldots, F_n\} \models F$.
   As above, there exists $j \in \mathbb{N}$ with $\{F_2, \ldots, F_n\} \subseteq K_j$.
   Then $F \in C(K_j)$ and thus $F \in K$. **Contradiction**

3. Let $G : G_1, ..., G_n/H \in K_D$, $G \in K$ and for all $1 \leq j \leq n$ holds: $\neg G_j \notin K$.

   We have to show that $H \in K$ holds.

But $G \in K$ implies that there is a K_i with $G \in K_i$ (because $K = \bigcup_{i=0}^{\infty} K_i$), and from this follows $H \in K_{i+1}$, and thus $H \in K$.