TECHNISCHE UNIVERSITÄT DRESDEN

Computational Logic ∴ Group

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Description Logics – Syntax and Semantics II

Lecture 5, 18th Nov 2024 // Foundations of Knowledge Representation, WS 2024/25

$\mathcal{ALC}$ Concepts

$\mathcal{ALC}$ is the basic description logic

$\mathcal{ALC}$ concepts C are inductively defined from atomic concepts A and roles R:

$$C ::= \top \mid \bot \mid A \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \forall R.C \mid \exists R.C$$

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The semantics is given through DL interpretations $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ with

$$\begin{aligned}
\top^{\mathcal{I}} &= \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} &= \emptyset \\
(\neg C)^{\mathcal{I}} &= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\exists R.C)^{\mathcal{I}} &= \{u \in \Delta^{\mathcal{I}} \mid \exists w \in \Delta^{\mathcal{I}} \text{ s.t. } \langle u, w \rangle \in R^{\mathcal{I}} \text{ and } w \in C^{\mathcal{I}}\} \\
(\forall R.C)^{\mathcal{I}} &= \{u \in \Delta^{\mathcal{I}} \mid \forall w \in \Delta^{\mathcal{I}},\ \langle u, w \rangle \in R^{\mathcal{I}} \text{ implies } w \in C^{\mathcal{I}}\}
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$\mathcal{ALC}$ Fragments

What happens to $\mathcal{ALC}$ if we disallow negation? That is, if we define "$\mathcal{ALC}^{+}$" via

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Nothing:
Instead of $\neg C$, we can use A_C for a new concept name A_C and add the GCIs

$$\top \sqsubseteq C \sqcup A_C$$
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What happens if we disallow negation, disjunction, and value restriction?
A lot – complexity (of concept satisfiability) drops from PSpace to PTime.

It is an important objective of DL (indeed KR) research to identify fragments that are "computationally well-behaved".

Basic Reasoning Problems and Services

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- Error-free:

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- Sufficiently detailed:

  Contain all relevant knowledge for our application

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\mathit{JuvArthritis} &\sqsubseteq \mathit{Arthritis} \sqcap \mathit{JuvDisease} \\
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- Juvenile arthritis is a kind of juvenile disease
  - Juvenile disease affects only children or teens, which are not adults
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- Juvenile arthritis is a kind of arthitis
  - Each arthritis affects some adult
    - Each juvenile arthritis affects some adult

Concept Satisfiability

What is the impact of the error?

- All models $\mathcal{I}$ of $\mathcal{T}$ must be such that $\mathit{JuvArthritis}^{\mathcal{I}} = \emptyset$
  - A juvenile arthritis cannot exist!
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Such errors can be detected by solving the following problem:

> Concept satisfiability w.r.t. a TBox:
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In a FOL setting, C is satisfiable w.r.t. $\mathcal{T}$ if and only if

$$\pi(\mathcal{T}) \wedge \exists x.(\pi_x(C)) \quad \text{is satisfiable}$$

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Parts of our arthritis TBox, however, do conform to our intuitions:

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We have discovered new interesting information:

All models $\mathcal{I}$ of $\mathcal{T}$ must be such that $\mathit{JuvArthritis}^{\mathcal{I}} \subseteq \mathit{JointDisease}^{\mathcal{I}}$

Juvenile arthritis is a sub-type of joint disease

All instances of juvenile arthitis are also joint diseases

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In a FOL setting, C is subsumed by D w.r.t. $\mathcal{T}$ if and only if

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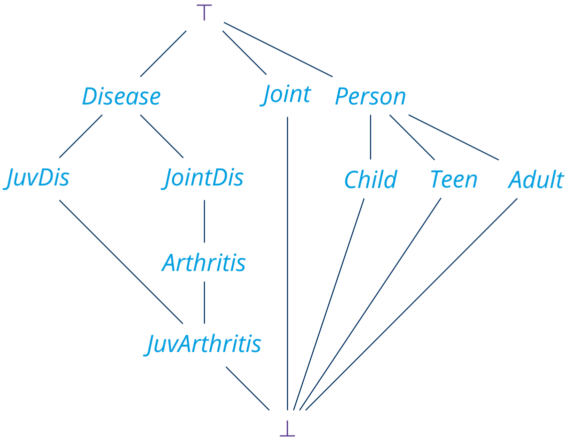
TBox Classification

The problem of finding all subsumptions between atomic concepts in $\mathcal{T}$.

Allows us to organise atomic concepts in a subsumption hierarchy:



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- Can we infer additional information about individuals?
  - Is D an instance of any class other than *Disease*?
  - Do we know if *MaryJones* is an *Adult* or a *Child*?

Summary of Basic Reasoning Problems

Definition

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$ knowledge base, C, D possibly compound $\mathcal{ALC}$ concepts, and b an individual name. We say that

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3. C and D are equivalent with respect to $\mathcal{T}$, written $\mathcal{T} \models C \equiv D$, if $C^{\mathcal{I}} = D^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{T}$;

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Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$ knowledge base, C, D possibly compound $\mathcal{ALC}$ concepts, and b an individual name. We say that

1. C is satisfiable with respect to $\mathcal{T}$ if there exists a model $\mathcal{I}$ of $\mathcal{T}$ and some $d \in \Delta^{\mathcal{I}}$ with $d \in C^{\mathcal{I}}$;
2. C is subsumed by D with respect to $\mathcal{T}$, written $\mathcal{T} \models C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{T}$;
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We write $C \sqsubseteq_{\mathcal{T}} D$ for $\mathcal{T} \models C \sqsubseteq D$ and $C \equiv_{\mathcal{T}} D$ for $\mathcal{T} \models C \equiv D$.

Important Properties of Subsumption

Lemma

Let C, D and E be concepts, b an individual name, and $(\mathcal{T}, \mathcal{A})$, $(\mathcal{T}', \mathcal{A}')$ knowledge bases with $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{A} \subseteq \mathcal{A}'$.

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Proofs follow easily from definition of semantics.

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Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$ knowledge base, C, D possibly compound $\mathcal{ALC}$ concepts and b an individual name.

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Consequently, all the previously mentioned reasoning problems can be reduced to KB (un)satisfiability.

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Additional Reasoning Services

We can define additional reasoning services in terms of basic ones:

- Classification of a TBox: given a TBox $\mathcal{T}$, compute the *subsumption hierarchy* of all concept names occurring in $\mathcal{T}$.
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- Realisation of an individual name: given an individual name b and a knowledge base $\mathcal{K}$, return all those concept names A such that b is an instance of A with respect to $\mathcal{K}$. That is, for each concept name A occurring in $\mathcal{K}$, check if $\mathcal{T} \models b : A$.

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We will use $\mathcal{L}$ as a placeholder for the name of a DL and write $\mathcal{LI}$ for $\mathcal{L}$ extended with inverse roles.

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Typically, DLs supporting inverse roles also allow for inverse roles to be used in axioms such as the following:

$$\mathit{AffectedBy} \equiv \mathit{Affects}^-$$

which establishes the intuitive semantic relationship.

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Note that an NR is equivalent to a QNR where the restriction concept is $\top$; e.g., $(\leqslant 2\, \mathit{sister})$ is equivalent to $(\leqslant 2\, \mathit{sister}.\top)$.

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For an interpretation $\mathcal{I}$, its mapping $\cdot^{\mathcal{I}}$ is extended as follows, where $\#M$ is used to denote the cardinality of a set M:

$$
\begin{aligned}
(\leqslant n\, r)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \#\{e \mid (d,e) \in r^{\mathcal{I}}\} \leq n\}, \\
(\geqslant n\, r)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \#\{e \mid (d,e) \in r^{\mathcal{I}}\} \geq n\}, \\
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Extensions: Number Restrictions

Definition

For n a non-negative integer, r an $\mathcal{L}$ role and C a (possibly compound) $\mathcal{L}$ concept description, a number restriction is a concept description of the form $(\leqslant n\, r)$ or $(\geqslant n\, r)$, and a qualified number restriction is a concept description of the form $(\leqslant n\, r.C)$ or $(\geqslant n\, r.C)$, where C is the qualifying concept.

For an interpretation $\mathcal{I}$, its mapping $\cdot^{\mathcal{I}}$ is extended as follows, where $\#M$ is used to denote the cardinality of a set M:

$$
\begin{aligned}
(\leqslant n\, r)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \#\{e \mid (d,e) \in r^{\mathcal{I}}\} \leq n\},\\
(\geqslant n\, r)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \#\{e \mid (d,e) \in r^{\mathcal{I}}\} \geq n\},\\
(\leqslant n\, r.C)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \#\{e \mid (d,e) \in r^{\mathcal{I}} \text{ and } e \in C^{\mathcal{I}}\} \leq n\},\\
(\geqslant n\, r.C)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \#\{e \mid (d,e) \in r^{\mathcal{I}} \text{ and } e \in C^{\mathcal{I}}\} \geq n\}.
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Concept descriptions $(=n\, r)$ and $(=n\, r.C)$ may be used as abbreviations for $(\leqslant n\, r) \sqcap (\geqslant n\, r)$ and $(\leqslant n\, r.C) \sqcap (\geqslant n\, r.C)$, respectively.

Extensions: Nominals

So far our use of individuals has been restricted to ABox axioms.

We may also want to use individuals in concept descriptions; e.g., to describe those individuals who are affected by some *Disease* that also affects the individual *JohnSmith*.

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$$\exists \mathit{Affects}^{-}.(\mathit{Disease} \sqcap \exists \mathit{Affects}.\,\mathit{JohnSmith})$$

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The fact that a DL provides nominals is normally indicated by the letter $\mathcal{O}$ in its name ($\mathcal{N}$ is already used for unqualified number restrictions).

[†] In fact this would be a syntax error if we use *JohnSmith* elsewhere as an individual (the set **C** of concept names and **I** of individual names must be disjoint).

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- With nominals, the separation between ABox and TBox is not meaningful:

$$\begin{aligned} C(a) &\equiv \{a\} \sqsubseteq C \\ R(a,b) &\equiv \{a\} \sqsubseteq \exists R.\{b\} \end{aligned}$$

Extensions: Role Hierarchies

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For an interpretation $\mathcal{I}$ to be a *model of* a role inclusion axiom $r \sqsubseteq s$, it has to satisfy

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Extensions: Transitive Roles

We can use the role *parent* to form descriptions such as:

$\exists \mathit{parent}.\mathit{Irish}$	having an *Irish* parent
$\exists \mathit{parent}.(\exists \mathit{parent}.\mathit{Irish})$	having an *Irish* grandparent
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But what if we want to mention *Irish* ancestors without specifying a generation?

We can do that by using a combination of role hierarchy and transitive roles:

$parent \sqsubseteq ancestor$	parent is a sub-role of ancestor
$\mathrm{Trans}(ancestor)$	ancestor is a transitive role
$\exists ancestor.Irish$	having an *Irish* ancestor

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- The use of $\mathcal{S}$ to replace $\mathcal{ALC}$ in DLs with transitive roles is inspired by similarities with the modal logic **S4** (and a desire for shorter names).
- However, in some cases (the subscript) $\cdot_{R^+}$ is used to indicate transitive roles; e.g., $\mathcal{SHIQ}$ could be written $\mathcal{ALCHIQ}_{R^+}$.

Extensions: Transitive Roles

It is important to understand the difference between transitive roles and the transitive closure of roles.

- Transitive closure is a role constructor: given a role r, transitive closure can be used to construct a role r^{+}, with the semantics being that $(r^{+})^{\mathcal{I}} = (r^{\mathcal{I}})^{+}$.

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- However, we cannot enforce that s is the smallest such transitive role: s is just some transitive role that includes r.
- In contrast, the transitive closure r^+ of r is, by definition, the smallest transitive role that includes r; thus we have:

$$\{\mathrm{Trans}(s), r \sqsubseteq s\} \models r \sqsubseteq r^+ \sqsubseteq s.$$

Relationships to FOL Revisited

As we have seen, $\mathcal{ALC}$ is in the 2-variable fragment of FOL (FO^2):

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\begin{aligned}
\pi_x(A) &= A(x) & \pi_y(A) &= A(y) \\
\pi_x(\neg C) &= \neg\pi_x(C) & \pi_y(\neg C) &= \neg\pi_y(C) \\
\pi_x(C \sqcap D) &= \pi_x(C) \wedge \pi_x(D) & \pi_y(C \sqcap D) &= \pi_y(C) \wedge \pi_y(D) \\
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Moreover, the translation uses quantification only in a restricted way, and therefore yields formulas in the guarded fragment for which satisfiability is known to be decidable in deterministic exponential time.

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- This gives us a nondeterministic exponential time upper bound for $\mathcal{SHOIQ}$ satisfiability.

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\end{aligned}$$

Complexity

Undecidable	FOL-3 sat
Non-Elementary decidable	…
…	…
NExpTime	FOL-2, $\mathcal{SHOIQ}$ sat
ExpTime	Datalog, $\mathcal{SHOI}$ sat
PSpace	$\mathcal{ALC}$ sat
NP	PL sat
P	Horn PL sat

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