# 4. Petri Nets: Boundedness and Undecidability of Equivalence Problems

May 24-25, 2022

# Warm-up: Something Useful

For $k \in \mathbb{N}$, $(\mathbb{N}^k, \leq)$ is a **well partial order** (antisymmetric wqo).

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For every infinite sequence $(a_i)_{i \in \mathbb{N}}$ ($a_j \in \mathbb{N}^k$ for each $j \in \mathbb{N}$), there is an infinite increasing subsequence, that is $a_{n_0} \leq a_{n_1} \leq a_{n_2} \leq \ldots$ with $n_0 < n_1 < n_2 < \ldots$.

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**Proof.**

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $\ldots$ |

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|---|---|---|---|---|---|---|---|---|---|---|
| $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $\cdots$ |

$$A = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \ldots\}$$

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|---|---|---|---|---|---|---|---|---|---|---|
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$$A = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \ldots\}$$

$\min A = a$ and there is an $n_0 \geq 0$, such that $a_{n_0} = a$

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**Proof.**

$\ldots$ $a_{n_0-2}$ | $a_{n_0-1}$ | $a_{n_0}$ | $a_{n_0+1}$ | $a_{n_0+2}$ | $a_{n_0+3}$ | $a_{n_0+4}$ | $a_{n_0+5}$ | $a_{n_0+6}$ | $a_{n_0+7}$ $\ldots$

$$A = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \ldots\}$$

$\min A = a$ and there is an $n_0 \geq 0$, such that $a_{n_0} = a$

$a_{n_0}$ and $n_0$

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| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\ldots$ | $a_{n_0-2}$ | $a_{n_0-1}$ | $a_{n_0}$ | $a_{n_0+1}$ | $a_{n_0+2}$ | $a_{n_0+3}$ | $a_{n_0+4}$ | $a_{n_0+5}$ | $a_{n_0+6}$ | $a_{n_0+7}$ | $\ldots$ |

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$\square$

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**Proof.**

$\ldots$ | $a_{n_0+1}$ | $a_{n_0+2}$ | $a_{n_0+3}$ | $a_{n_0+4}$ | $a_{n_0+5}$ | $a_{n_0+6}$ | $a_{n_0+7}$ | $\ldots$

$B = \{a_{n_0+1}, a_{n_0+2}, a_{n_0+3}, a_{n_0+4}, a_{n_0+5}, a_{n_0+6}, \ldots\}$

$\min A = a$ and there is an $n_0 \geq 0$, such that $a_{n_0} = a$

$a_{n_0}$ and $n_0$

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$\ldots$

| | | | | | | |
|---|---|---|---|---|---|---|
| $a_{n_0+1}$ | $a_{n_0+2}$ | $a_{n_0+3}$ | $a_{n_0+4}$ | $a_{n_0+5}$ | $a_{n_0+6}$ | $a_{n_0+7}$ |

$\ldots$

$$B = \{a_{n_0+1}, a_{n_0+2}, a_{n_0+3}, a_{n_0+4}, a_{n_0+5}, a_{n_0+6}, \ldots\}$$

$\min B = b$ and there is an $n_1 > n_0$, such that $a_{n_1} = b$

$a_{n_0}$ and $n_0$

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| … | $a_{n_1-2}$ | $a_{n_1-1}$ | $a_{n_1}$ | $a_{n_1+1}$ | $a_{n_1+2}$ | $a_{n_1+3}$ | $a_{n_1+4}$ | $a_{n_1+5}$ | $a_{n_1+6}$ | $a_{n_1+7}$ | … |
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$$B = \{a_{n_0+1}, a_{n_0+2}, a_{n_0+3}, a_{n_0+4}, a_{n_0+5}, a_{n_0+6}, \ldots\}$$

$\min B = b$ and there is an $n_1 \geq 0$, such that $a_{n_1} = b$

$a_{n_0} \leq a_{n_1}$ and $n_0 < n_1$

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$$C = \{a_{n_1+1}, a_{n_1+2}, a_{n_1+3}, a_{n_1+4}, a_{n_1+5}, a_{n_1+6}, \ldots\}$$

$\min C = c$ and there is an $n_2 \geq n_1$, such that $a_{n_2} = c$

$a_{n_0} \leq a_{n_1} \leq a_{n_2} \leq \ldots$ and $n_0 < n_1 < n_2 < \ldots$

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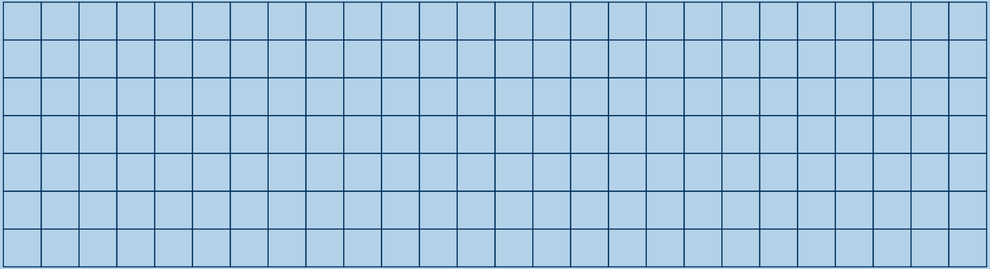

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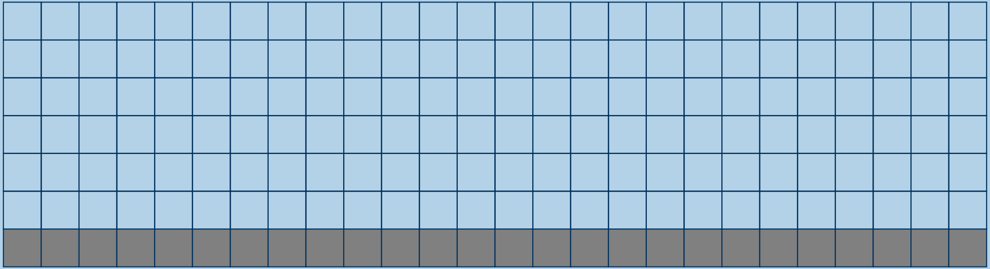

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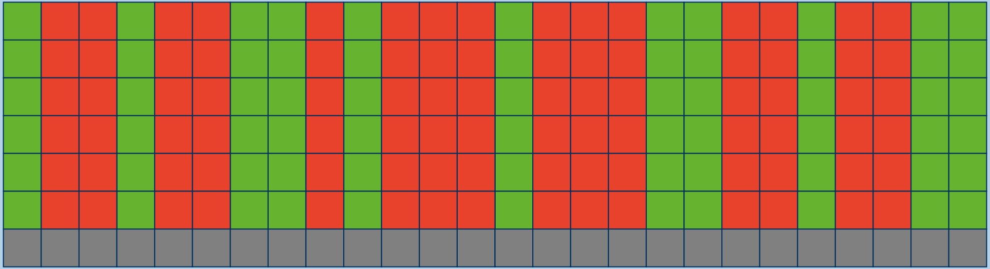

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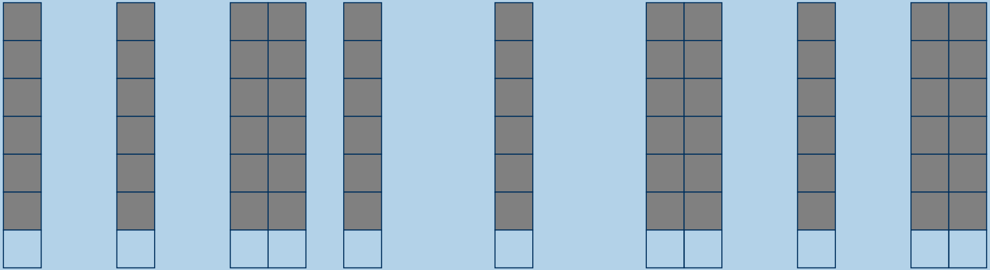

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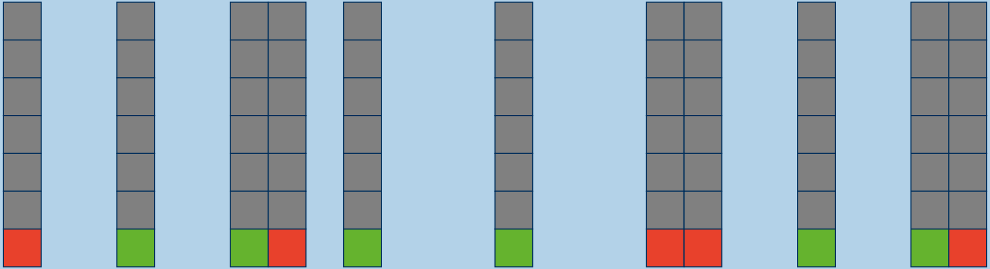

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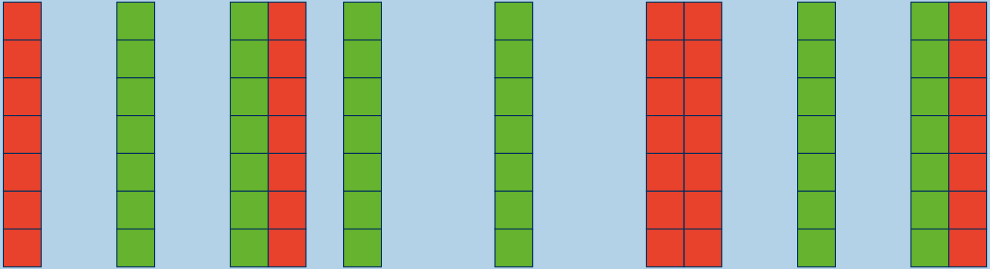

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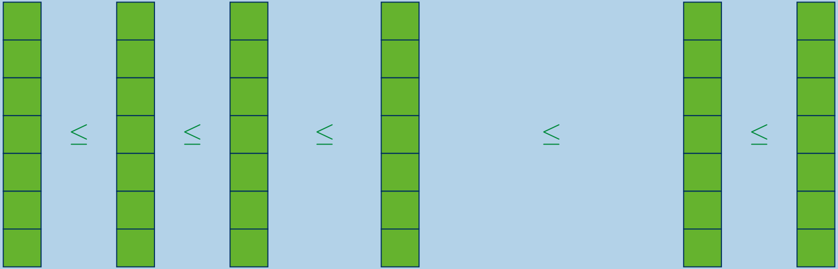

## Disclaimer

I will break with any conventions you may have heard of . . .

(e. g., P/T nets or S/T nets, elementary net systems, net systems, Petri nets, . . . will all be called **Petri nets**)

# Net Structure

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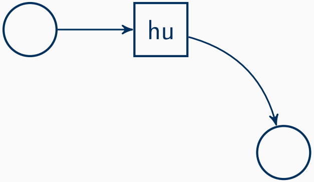
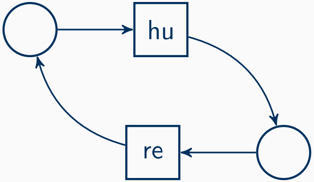


## Net Structure



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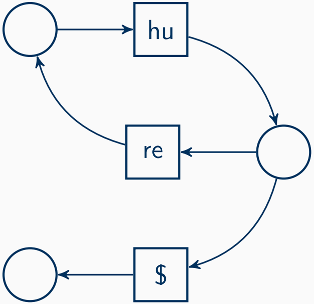

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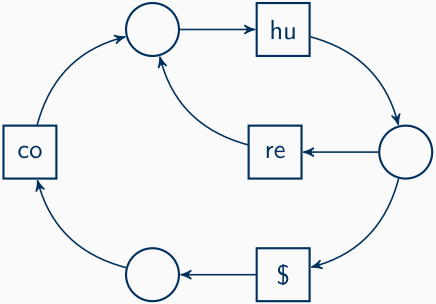

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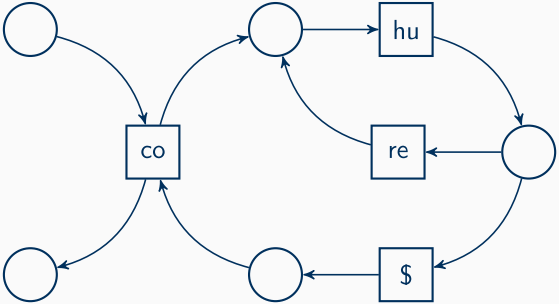

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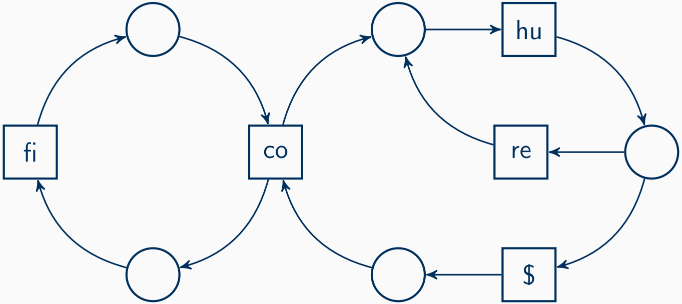


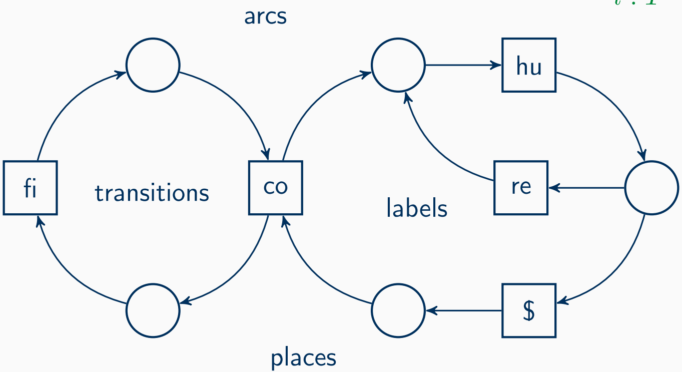
# Net Structure

$(P, T, F, l)$

$P, T$ disjoint and finite sets

$F \subseteq (P \times T) \cup (T \times P)$

$l : T \to \Sigma$ ($\Sigma$ is an alphabet)



# Markings and the Token Game

$N = (P, T, F, l, m_0)$
$P, T$ disjoint and finite sets
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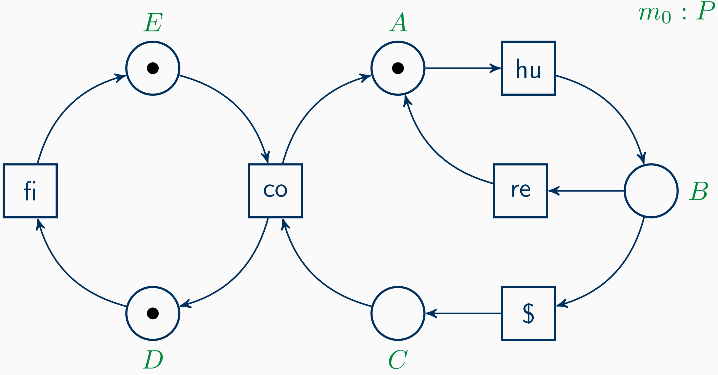
$l : T \rightarrow \Sigma$ ($\Sigma$ is an alphabet)
$m_0 : P \rightarrow \mathbb{N}$ (multiset)



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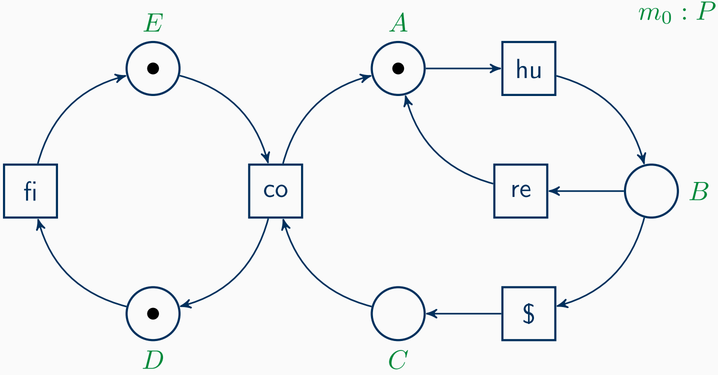
$m :$

$$\begin{array}{lcl} A & \mapsto & 1 \\ B & \mapsto & 0 \\ C & \mapsto & 0 \\ D & \mapsto & 1 \\ E & \mapsto & 1 \end{array}$$



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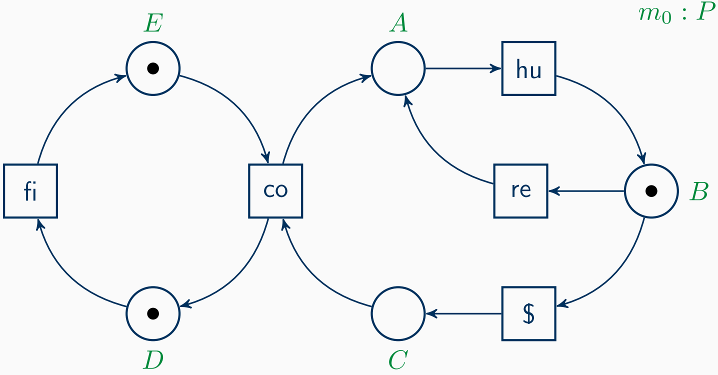

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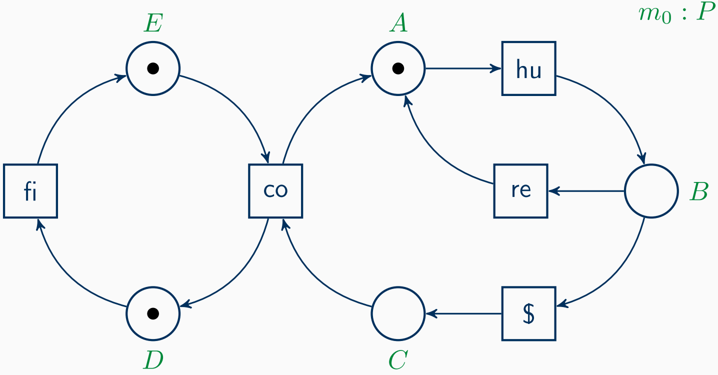

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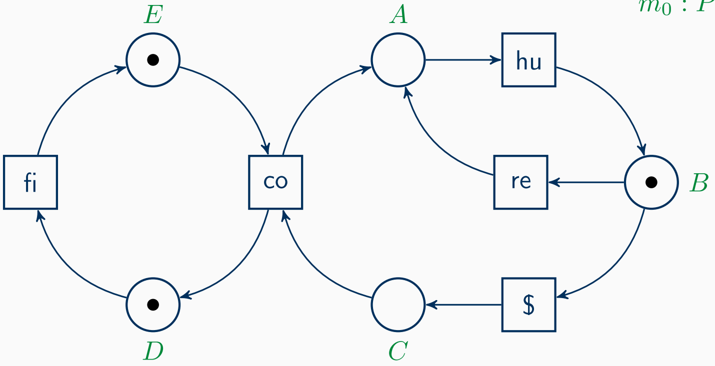

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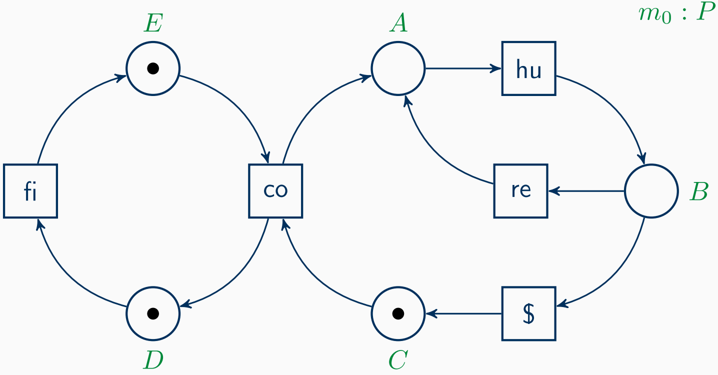

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$N = (P, T, F, l, m_0)$

$P, T$ disjoint and finite sets

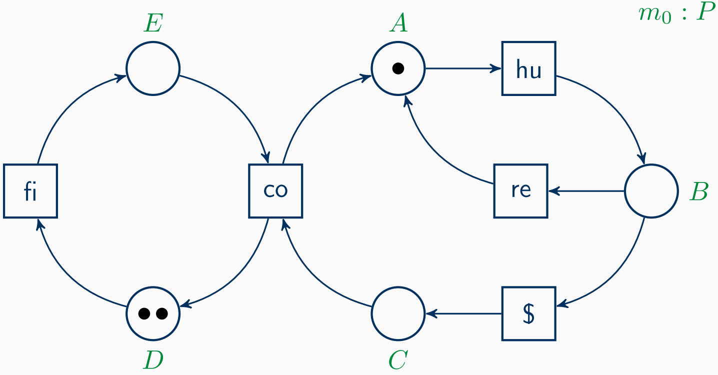
$F \subseteq (P \times T) \cup (T \times P)$

$l : T \to \Sigma$ ($\Sigma$ is an alphabet)

$m_0 : P \to \mathbb{N}$ (multiset)

$m :$

$$\begin{array}{ccc} A & \mapsto & 1 \\ B & \mapsto & 0 \\ C & \mapsto & 0 \\ D & \mapsto & 2 \\ E & \mapsto & 0 \end{array}$$



# Definitions and Observations

**Definition 4.1 (Net Structure)**

Let $\Sigma$ be an alphabet. A **($\Sigma$-labeled) net structure** is a quadruple $(P, T, F, l)$ with disjoint finite sets $P$ of places and $T$ of transitions, $F \subseteq (P \times T) \cup (T \times P)$, and $l : T \to \Sigma$.

For nodes $v \in P \cup T$, ${}^\bullet v := \{u \mid (u, v) \in F\}$ and $v^\bullet := \{w \mid (v, w) \in F\}$.

**Definition 4.2 (Marking, Firing Rule)**

For (labeled) net structure $N = (P, T, F, l)$, we call a multiset $m$ over $P$ a **marking** of $N$. A transition $t \in T$ is **enabled under marking** $m$ if ${}^\bullet t \leq m$. An enabled transition $t$ under marking $m$ may fire, producing the successor marking $m'$ such that for all $p \in P$,

$$m(p) := \begin{cases} m(p) - 1 & \text{if } p \in {}^\bullet t \setminus t^\bullet \\ m(p) + 1 & \text{if } p \in t^\bullet \setminus {}^\bullet t \\ m(p) & \text{otherwise.} \end{cases}$$

We also write $m \xrightarrow{t} m'$ or even $m \xrightarrow{l(t)} m'$.

# Definitions and Observations

**Definition 4.3 (Petri net, reachability graph)**

A ($\Sigma$-labeled) Petri net is a quintuple $N = (P, T, F, l, m_0)$ where $(P, T, F, l)$ is a labeled net structure and $m_0$ is a marking for it (**initial marking**).
The set of **reachable markings of** $N$ $[N\rangle$ is defined inductively by (1) $m_0 \in [N\rangle$ and (2) $m \in [N\rangle$ and $m \xrightarrow{t} m'$ implies $m' \in [N\rangle$.
The **reachability graph of** $N$ $\mathcal{R}(N)$ is induced by the set of reachable markings $[N\rangle$ as the set of nodes and $(\xrightarrow{t})_{t \in T}$ forming the edge relation.

We sometimes needs $[N, m\rangle$ for arbitrary markings $m$ of $N$ to be the set of reachable markings of $N$ where $m_0$ is replaced by $m$. **Special case:** $[N, m_0\rangle = [N\rangle$.

# The Boundedness Problem

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Given a Petri net $N = (P, T, F, l, m_0)$, is $[N\rangle$ finite?

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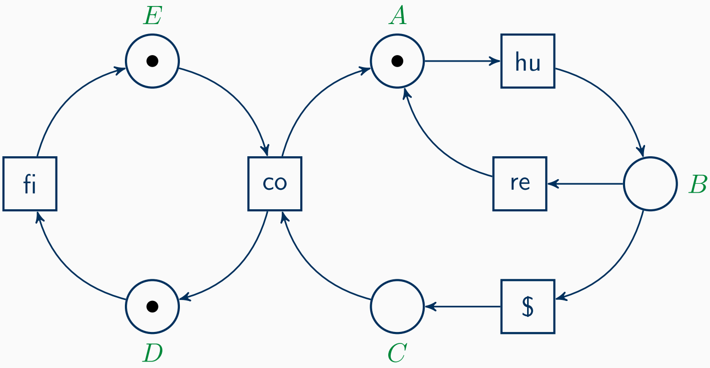
**Definition 4.4 (Bounded Petri net)**

Let $k \in \mathbb{N}$. A Petri net $N = (P, T, F, l, m_0)$ is $k$-**bounded** if for all $m \in [N\rangle$ and all places $p \in P$, $m(p) \leq k$. $N$ is **bounded** if there is a $k$, such that $N$ is $k$-bounded. If no such $k$ exists, $N$ is **unbounded**.

# Bounded and Unbounded Nets



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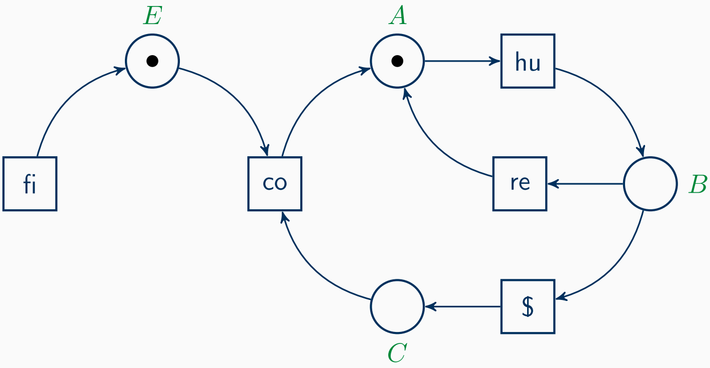

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**Lemma 4.5**

*The following statements are equivalent for Petri nets $N = (P, T, F, l, m_0)$:*

1. $[N\rangle$ *is infinite.*
2. $N$ *is unbounded.*
3. *There are markings $m_1, m_2$ of $N$, such that*
   *(a) $m_1 \in [N\rangle$, (b) $m_2 \in [N, m_1\rangle$, (c) $m_1 \leq m_2$, and (d) $m_1(p) < m_2(p)$ for some $p \in P$.*

# From 3 to 2

For Petri net $N = (P, T, F, l, m_0)$, let $m_1, m_2$ be markings, such that

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$m_2 = m_1 + s$ for some non-empty marking $s$! In particular, $s(p) > 0$

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**Lemma 4.6 (Monotonicity)**

*For Petri net $N = (P, T, F, l, m_0)$, $t \in T$, and markings $m, m', s$ of $N$, $m \xrightarrow{t} m'$ implies $m + s \xrightarrow{t} m' + s$.*

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For every $k \in \mathbb{N}$, repeat transition sequence $\sigma$ $k + 1$ times, reaching a marking $m^k$ with $m^k(p) > k$.

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For every $k \in \mathbb{N}$, repeat transition sequence $\sigma$ $k+1$ times, reaching a marking $m^k$ with $m^k(p) > k$.

Thus, $N$ is unbounded.

# From 1 to 3

Let $N = (P, T, F, l, m_0)$ be a Petri net, such that $[N\rangle$ is infinite.

1. As $[N\rangle$ is infinite, $\mathcal{R}(G)$ is infinite.

# From 1 to 3

Let $N = (P, T, F, l, m_0)$ be a Petri net, such that $[N\rangle$ is infinite.

1. As $[N\rangle$ is infinite, $\mathcal{R}(G)$ is infinite.
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5. Due to Dickson's Lemma, there is an infinite chain $n_0 < n_1 < n_2 < \ldots$ of indices, such that $m_{n_0} \leq m_{n_1} \leq m_{n_2} \leq \ldots$.

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7. By construction (a) $m_1 \in [N\rangle$, (b) $m_2 \in [N, m_1\rangle$, and (c) $m_1 \leq m_2$.
8. As $m_1$ and $m_2$ stem from a simple path, there is at least one place $p \in P$ with $m_2(p) > m_1(p)$.

# Theorem: Boundedness is Decidable

Start constructing $\mathcal{R}(N)$ by BFS:

- either the construction terminates (**bounded**), or
- a marking $m_2$ is constructed with a respective marking $m_1 \leq m_2$ earlier on a path from $m_0$, such that $m_1(p) < m_2(p)$ for some $p \in P$ (**unbounded**).

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Many more decidable problems:

- Reachability
- Coverability
- Deadlock-freedom
- Liveness
- Language inclusion/equivalence (?)
- Bisimilarity (?)

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- Coverability
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- Liveness
- Language inclusion/equivalence (?)
- Bisimilarity (?)

Yes to both (?), but not for labeled Petri nets!

# The Equivalence Problem(s)

The **(prefix) language** $\mathcal{L}(N)$ **of a labeled Petri net** $N = (P, T, F, l, m_0)$ is the set of all words $w \in \Sigma^*$, such that $w = \varepsilon$ or $m_0 \xrightarrow{t_1}\xrightarrow{t_2} \cdots \xrightarrow{t_{|w|}}$ such that $l^*(t_1 t_2 \ldots t_{|w|}) = w$.

Two Petri nets $N_1, N_2$ are **language equivalent** if $\mathcal{L}(N_1) = \mathcal{L}(N_2)$.

**Theorem 4.1:** Language equivalence is undecidable for labeled Petri nets.

We reduce from the halting problem of Minsky machines with two counters.

Petri nets are not Turing-complete!
$\rightsquigarrow$ weak simulation of Turing machines/Minsky machines

# Minsky Machines

A **Minsky machine** is a pair $\langle P, \{c_1, c_2, \ldots, c_k\}\rangle$, where $c_1, \ldots, c_k$ are counters and $P$ is a finite sequence of commands $l_1 l_2 \ldots l_n$, such that $l_n =$ **HALT** and $l_i$ $(i = 1, \ldots, n-1)$ is

1. $i$: `c_j:=c_j+1; goto` $k$, or
2. $i$: `if c_j=0 then goto` $k_1$ `else c_j:=c_j-1; goto` $k_2$

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**Example 4.7**

We consider two counter $c_1$ and $c_2$.

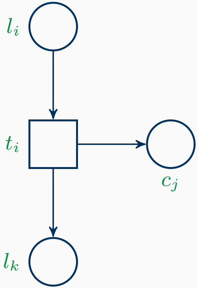
```
1: if c_2=0 then goto 3 else c_2:=c_2-1; goto 2
2: c_1:=c_1+1; goto 1
3: HALT
```

If $c_1$ and $c_2$ are initialized with $m$ and $n$, then the program halts with value $m + n$ in $c_1$.

# Constructing a Petri Net

For Minsky machine $\mathcal{M} = \langle l_1 l_2 \ldots l_m, \{c_1, \ldots, c_n\} \rangle$,
$N(\mathcal{M}) = (\{l_1, \ldots, l_m, c_1, \ldots, c_n\}, T, F, l, m_0)$ where for each $i \in \{1, \ldots, m-1\}$:

$l_i = i$: $c_j$ := $c_j$+1; goto $l_k$:

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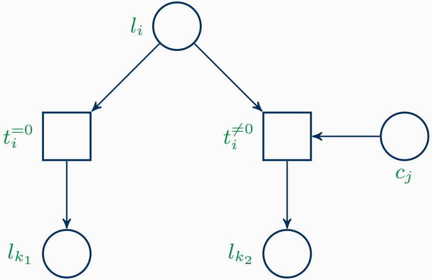

# Constructing a Petri Net

For Minsky machine $\mathcal{M} = \langle l_1 l_2 \dots l_m, \{c_1, \dots, c_n\}\rangle$,
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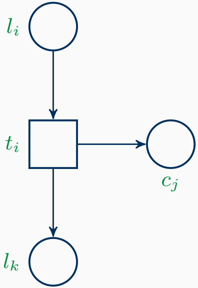


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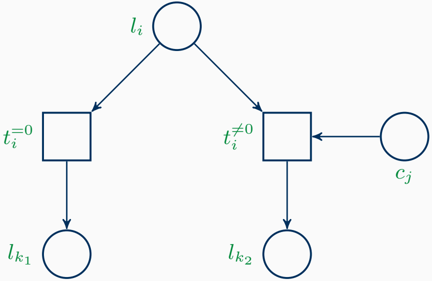


The labeling can be arbitrary but injective.

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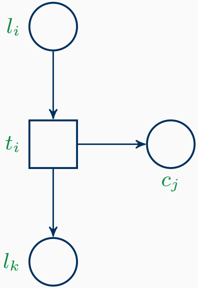


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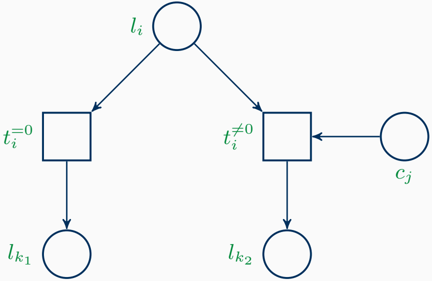
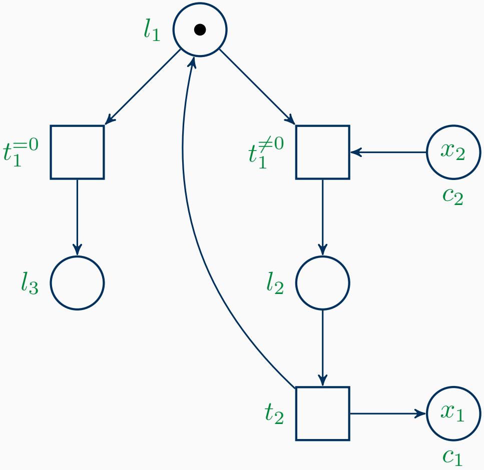


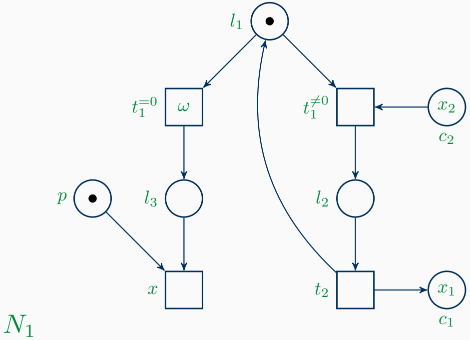

The labeling can be arbitrary but injective.

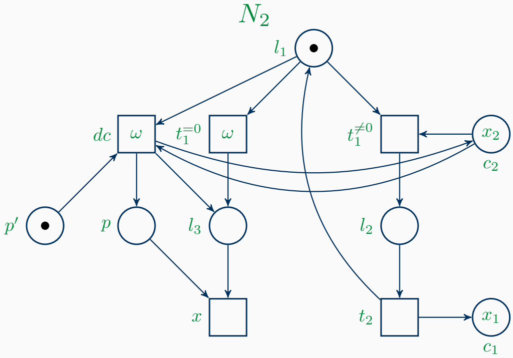
For input $x_1, \ldots, x_n \in \mathbb{N}$, define $m_0 = \{c_1 \mapsto x_1, \ldots, c_n \mapsto x_n, l_1 \mapsto 1\}$.

# Petri Net Construction by Example

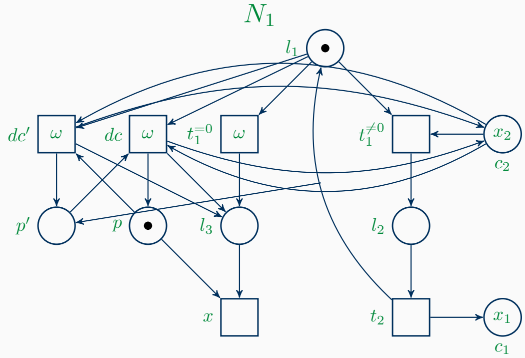


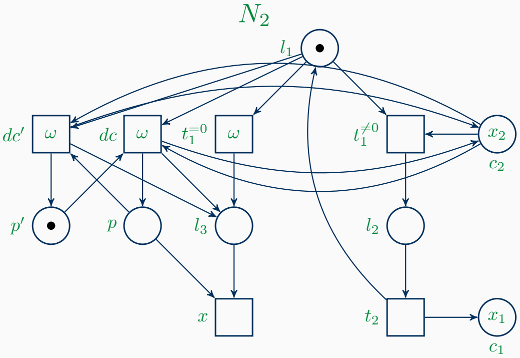
# Undecidability of Language Equivalence: The Reduction



# Undecidability of Bisimilarity: The Reduction



# The Coverability Graph

**Definition 4.8 ($\omega$-marking)**

For a net $(P, T, F)$, $m : P \to \mathbb{N} \cup \{\omega\}$ is called an $\omega$-marking.

Note, $\omega > n$ and $\omega + / - n = \omega$ for all $n \in \mathbb{N}$.

For directed graph $G = (V, E)$ and $v \in V$, defined $v \Downarrow$ to be the smallest set, such that (1) $v \in v \Downarrow$ and (2) if $w \in v \Downarrow$ and $u \to w$, then $u \in v \Downarrow$.

**Definition 4.9**

Let $N = (P, T, F, m_0, l)$ be a (labeled) Petri net. The coverability graph (of $N$) is the graph $\mathcal{C}(N) = (V, E)$, such that

1. $m_0 \in V$;
2. if $m \in V$ and $m \xrightarrow{t} m'$, then $\omega(m') \in V$ and $(m, \omega(m')) \in E$ such that for all $p \in P$,

$$\omega(m')(p) = \begin{cases} \omega & \text{if } m'' \in m \Downarrow \text{ with } m''(p) < m'(p) \\ m'(p) & \text{otherwise.} \end{cases}$$

# Properties of the Coverability Graph

**Theorem 4.2:** The coverability graph $\mathcal{C}(N)$ of a Petri net $N$ is finite.

$\rightsquigarrow$ follows the same argument as for the decidability proof of the boundedness problem.

# Properties of the Coverability Graph

**Theorem 4.3:** The coverability problem — given a Petri net $N$ and a marking $m$, is there a reachable marking $m'$, such that $m \leq m'$? — is decidable.

1. Construct $C(N)$
2. Check if there is an $\omega$-marking $m^\omega$ with $m \leq m^\omega$
3. Consider the path $m_0 \xrightarrow{t_1} \dots \xrightarrow{t_n} m^\omega$ and the marking $m'$ reached after firing the sequence $t_1 \dots t_n$
4. If $m \leq m'$, witness found.
5. If $m \not\leq m'$, then there is at least one $\omega$ in $m^\omega$ and there are markings on the path from $m_0$ to $m^\omega$ that led to the addition of $\omega$
6. Repeat the respective firing sequences until a covering marking is reached.
7. Hence, it is sufficient to check only $m^\omega$.
8. If $m$ is not coverable, then there is no marking $m'$ in the coverability graph with $m \leq m'$.

# Equivalence of Unlabeled Nets

**Theorem 4.4:** Bisimilarity and language equivalence of Petri nets is decidable for unlabeled Petri nets.

- Given are $N_1 = (P_1, T_1, F_1, m_0^1, l_1)$ and $N_2 = (P_2, T_2, F_2, m_0^2, l_2)$ ($P_1 \cap P_2 = \emptyset = T_1 \cap T_2$).
- Construct $N_1 + N_2 = (P_1 \cup P_2, T_1 \cup T_2, F_1 \cup F_2, m_0^1 + m_0^2)$.
- Each transition $t \in T_1 \cup T_2$ is duplicated to $t'$ with the same in-/outputs and label as $t$.
- Add a fresh place $p$ and add $\{p\} \times (T_1 \cup T_2)$ and $\{t' \mid t' \text{ is a duplicate}\} \times \{p\}$ to the arc relation.
- For each label $a \in \Sigma$, add places $p_1^a, p_2^a$ and for $t \in T_i$ with $l_i(t) = a$, add arcs $(t, p_j^a), (p_j^a, u')$ for transition duplicate $u'$ with $l_j(u) = a$.
- If the nets are language equivalent, then every transition firing of $t \in T_i$ can be reproduced in $N_j$ by $u'$, such that $l_i(t) = l_j(u)$.
- If the nets are not language equivalent, then there is a shortest word $w$ of $L(N_i) \setminus L(N_j)$. After firing the last transition of $w$ in $N_i$, no duplicate can be fired in $N_j$.
- Unlabledness is important to not leave $N_j$ the chance to use more clever $a$-labeled transitions.