A Generalized Next-Closure Algorithm – Enumerating Semilattice Elements from a Generating Set

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Motivation

Reminder

The Next-Closure algorithm successively computes all closed sets of a closure operator c on a finite set M.

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Definition (Closure Operators on Sets)

Let M be a set. Then $c\colon \mathfrak{P}(M) \longrightarrow \mathfrak{P}(M)$ is called a *closure operator* on M if and only if

- c is *extensive*, i. e. $A \subseteq c(A)$ for all $A \subseteq M$
- c is *monotone*, i. e. $A \subseteq B \subseteq M$ implies $c(A) \subseteq c(B)$
- c is *idempotent*, i. e. $c(c(A)) = A$ for all $A \subseteq M$.

Goal

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Generalize Next-Closure to arbitrary closure operators *once and for all*.

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Furthermore, $A \prec B$ if and only if $A \prec_i B$ for some $i \in M$.

Definition

Let $i \in M$, $A \subseteq M$ such that $c(A) = A$. Then define

$$A \oplus i := c(\{\, j \in A \mid j < i \,\} \cup \{\, i \,\}).$$

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Theorem (Next-Closure)

Let $A \neq M$ be such that $c(A) = A$. Define

$$A^{+} := \min_{<}\{\, B \subseteq M \mid c(B) = B, A < B \,\}.$$

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Then

$$A^+ = A \oplus i$$

where i is maximal with $A <_i A \oplus i$.

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Observation

We can solely work in the *semilattice* $(\mathrm{im}(c), \vee)$ of all closed sets of c!

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Generalize Next-Closure to work on *arbitrary, abstractly given semilattices*.

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Let $\{ x_1, \ldots, x_n \} \subseteq L$ be a generating set of $(L, \leqslant_L)$, i. e. for each $y \in L$ it is true that

$$y = \bigvee_{x_i \leqslant_L y} x_i.$$

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Recall

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where

$$\Delta_{a,b} := \{ i \mid (x_i \leqslant_L a \text{ and } x_i \nleqslant_L b) \text{ or } (x_i \nleqslant_L a \text{ and } x_i \leqslant_L b) \}.$$

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Consider the case of computing intents of a formal context $\mathbb{K} = (G, M, I)$.

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  Yields an algorithm with a lot of similarities to *Close-by-One*!

Thank You for Your Attention!

Questions Welcome!