TECHNISCHE UNIVERSITÄT DRESDEN

COMPLEXITY THEORY

Lecture 1: Introduction and Motivation

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Knowledge-Based Systems

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Course Tutors

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Lectures

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Exercises

Organisation

Exercise Sessions

- We will host the tutorials as "live sessions" on Tuesdays from 14:50 to 16:20.
- Exercise sheets preparing for the tutorials will be uploaded at least one week before the tutorial takes place.

Lectures

Every week on Tuesday, we will publish either one video (if there is a tutorial happening on that week) or two videos (if there is none) with the weekly lectures.

Web Page

`https://iccl.inf.tu-dresden.de/web/Complexity_Theory_(WS2020)`

Lecture Notes

Slides of current and past lectures will be online.

Goals and Prerequisites

Goals

- Introduce basic notions of **computational complexity theory**
- Introduce **commonly known complexity classes** (P, NP, PSpace, ...) and discuss relationships between them
- Develop **tools to classify problems** into their corresponding complexity classes
- Introduce some **advanced topics of complexity theory** (e.g., circuits and probabilistic computation)

(Non-)Prerequisites

- No particular prior courses needed
- Prior acquaintance with Turing Machines and basic topics in formal languages and complexity is helpful
- General mathematical and theoretical computer science skills necessary

Reading List

- **Michael Sipser: Introduction to the Theory of Computation, International Edition; 3rd Edition; Cengage Learning 2013**
- Sanjeev Arora and Boaz Barak: **Computational Complexity: A Modern Approach**; Cambridge University Press 2009
- Michael R. Garey and David S. Johnson: **Computers and Intractability**; Bell Telephone Laboratories, Inc. 1979
- Erich Grädel: **Complexity Theory**; Lecture Notes, Winter Term 2009/10
- John E. Hopcroft and Jeffrey D. Ullman: **Introduction to Automata Theory, Languages, and Computation**; Addison Wesley Publishing Company 1979
- Christos H. Papadimitriou: **Computational Complexity**; 1995 Addison-Wesley Publishing Company, Inc

Computational Problems are Everywhere

Example 1.1:

- What are the factors of 54,623?
- What is the shortest route by car from Berlin to Hamburg?
- My program now runs for two weeks. Will it ever stop?
- Is this C++ program syntactically correct?

Clear

Computational Problems are ubiquitous in our everyday life!
And, depending on what we want to do, those problems might be either **easily solvable** or **hardly solvable**.

Approach to problems:

> [T]he way is to avoid what is strong, and strike at what is weak.

(Sun Tzu: The Art of War, Chapter 6: Weak Points and Strong)

Examples

Example 1.2 (Shortest Path Problem): Given a weighted graph and two vertices s, t, find the shortest path between s and t.

Easily solvable using, e.g., Dijkstra's Algorithm.

Example 1.3 (Longest Path Problem): Given a weighted graph and two vertices s, t, find the **longest** path between s and t.

No efficient algorithm known, and believed to not exist (this problem is **NP-hard**)

Observation

Difficulty of a problem is hard to assess

Measuring the Difficulty of Problems

Question

How can we measure the complexity of a problem?

Approach

Estimate the resource requirements of the “best” algorithm that solves this problem.

Typical Resources:

- Running Time
- Memory Used

Note

To assess the complexity of a problem, we need to consider **all possible algorithms** that solve this problem.

Problems

What actually is ... a Problem?

(Decision) Problems are **word problems** of particular languages.

Example 1.4: "Problem: Is a given graph connected?" will be modelled as the word problem of the language

$$\text{GCONN} := \{ \langle G \rangle \mid G \text{ is a connected graph} \}.$$

Then for a graph G we have

$$G \text{ is connected} \iff \langle G \rangle \in \text{GCONN}.$$

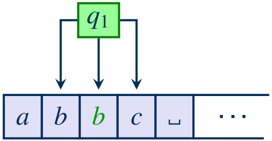
Note

The notation $\langle G \rangle$ denotes a suitable encoding of the graph G over some fixed alphabet (e.g., $\{ 0, 1 \}$).

What actually is … an Algorithm?

Different approaches to formalise the notion of an “algorithm”

- Turing Machines
- Lambda Calculus
- μ-Recursion
- …



Avoid What is Strong

Suppose we are given a language $\mathcal{L}$ and a word w.

Question

Does there need to exist **any** algorithm that decides whether $w \in \mathcal{L}$?

Answer

No. Some problems are **undecidable**.

Example 1.5:

- The Halting Problem of Turing machines
- The Entscheidungsproblem (Is a first-order logical statement true?)
- Finding the lowest air fare between two cities (→ Reference)
- Deciding syntactic validity of C++ programs (→ Reference)

Avoid: We will focus on decidable problems in this course.

Difficulty

Measuring running time and memory requirements depends highly on the **machine**, and not so much on the **problem**.

Resort

Measure time and space only **asymptotically** using **Big-$\mathcal{O}$**-Notation:

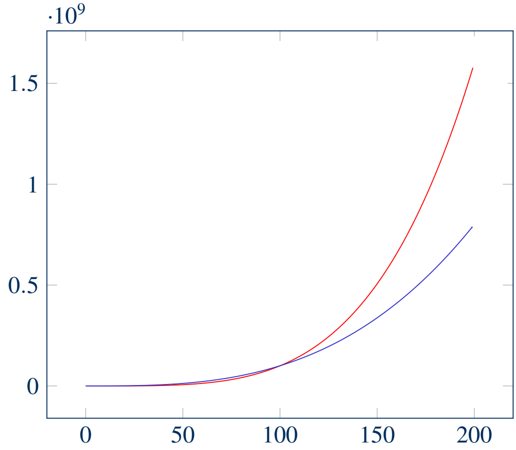
$$f(n) = \mathcal{O}(g(n)) \iff f(n) \text{ “asymptotically bounded by” } g(n)$$

More formally:

$$f(n) = \mathcal{O}(g(n)) \iff \exists c > 0\, \exists n_0 \in \mathbb{N}\, \forall n > n_0 : f(n) \leq c \cdot g(n).$$

Big-$\mathcal{O}$-Notation: Example

$100n^3 + 1729n = \mathcal{O}(n^4)$:



Complexity of Problems

Approach

The **time (space) complexity** of a problem is the asymptotic running time of a fastest (least memory consumptive) algorithm that solves the problem.

Problem

Still too difficult . . .

Example 1.6 (Travelling Salesman Problem): Given a weighted graph, find the shortest simple path visiting every node.

- Best known algorithm runs in time $O(n^2 2^n)$ (Bellman-Held-Karp algorithm)
- Best known lower bound is $O(n \log n)$
- Exact complexity of TSP unknown

Even more abstraction

Approach

Divide decision problems into the “quality” of their fastest algorithms:

- P is the class of problems **solvable in polynomial time**
- PSpace is the class of problems **solvable in polynomial space**
- ExpTime is the class of problems **solvable in exponential time**
- L is the class of problems **solvable in logarithmic space** (apart from the input)
- NP is the class of problems **verifiable in polynomial time**
- NL is the class of problems **verifiable in logarithmic space**

And many more!

$\oplus$P, #P, AC, AC^0, ACC0, AM, AP, APSpace, BPL, BPP, BQP, coNP, E, Exp, FP, IP, MA, MIP, NC, NExpTime, P/poly, PH, PP, RL, RP, Σ_i^{p}, TISP$(T(n), S(n))$, ZPP, ...

Strike at What is Weak

Approach (cf. Cobham–Edmonds Thesis)

The problems in P are “tractable” or “efficiently solvable”
(and those outside are not)

Example 1.7: The following problems are in P:

- Shortest Path Problem
- Satisfiability of Horn-Formulas
- Linear Programming
- Primality

Note

The Cobham-Edmonds-Thesis is only a **rule of thumb**: there are (practically) tractable problems outside of P, and (practically) intractable problems in P.

Friend or Foe?

Caveat

It is not known how big P is.
In particular, it is unknown whether $P \neq NP$ or not.

Approach

Try to find out which problems in a class are at least as hard as others.
Complete problems are then the hardest problems of a class.

Example 1.8: Satisfiability of propositional formulas is **NP-complete**: if we can efficiently decide whether a propositional formula is satisfiable, we can solve **any** problem in NP efficiently.

But: we still do not know whether we can or cannot solve satisfiability efficiently. We only know it will be difficult to find out . . .

Learning Goals

- Get an overview over the foundations of Complexity Theory
- Gain insights into advanced techniques and results in Complexity Theory
- Understand what it means to “compute” something, and what the strengths and limits of different computing approaches are
- Get a feeling of how hard certain problems are, and where this hardness comes from
- Appreciate how very little we actually know about the computational complexity of many problems

Lecture Outline (1)

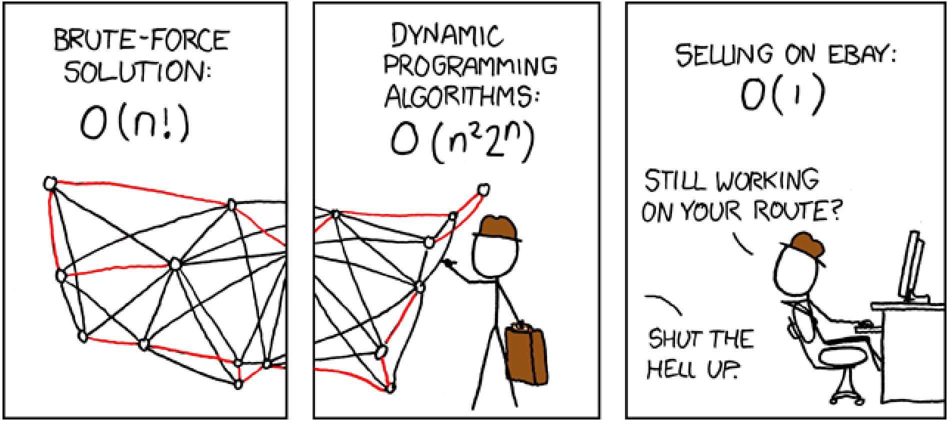
- **Turing Machines** (Revision)
  Definition of Turing Machines; Variants; Computational Equivalence; Decidability and Recognizability; Enumeration; Oracles
- **Time Complexity**
  Measuring Time Complexity; Many-One Reductions; Cook-Levin Theorem; Time Complexity Classes (P, NP, ExpTime); NP-completeness; pseudo-NP-complete problems
- **Space Complexity**
  Space Complexity Classes (PSpace, L, NL); Savitch's Theorem; PSpace-completeness; NL-completeness; NL = coNL

Lecture Outline (2)

- **Diagonalisation**
  Hierarchy Theorems (det. Time, non-det. Time, Space); Gap Theorem; Ladner's Theorem; Relativisation; Baker-Gill-Solovay Theorem
- **Alternation**
  Alternating Turing Machines; APTime = PSpace; APSpace = ExpTime; Polynomial Hierarchy
- **Circuit Complexity**
  Boolean Circuits; Alternative Proof of Cook-Levin Theorem; Parallel Computation (NC); P-completeness; P/poly; (Karp-Lipton Theorem, Meyer's Theorem)
- **Probabilistic Computation**
  Randomised Complexity Classes (RP, PP, BPP, ZPP); Sipser-Gács-Lautemann Theorem

Avoid what is Strong, and Strike at what is Weak

Sometimes the best way to solve a problem is to avoid it . . .



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