Finite and Algorithmic Model Theory

Lecture 3 (Dresden 26.10.22, Long version)

Lecturer: Bartosz "Bart" Bednarczyk

Technische Universität Dresden & Uniwersytet Wrocławski

TECHNISCHE UNIVERSITÄT DRESDEN

Uniwersytet Wrocławski

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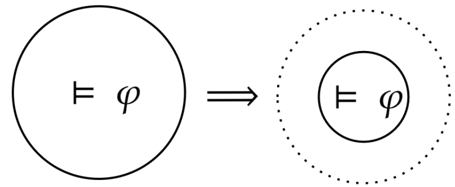
Today's agenda

Goal: Investigate important properties of FO and see whether they stay true in the finite.

1. Diagrams and embeddings.
2. Preservation Theorem of Łoś-Tarski.
3. Failure of Łoś-Tarski in the finite.
4. Discussion of related preservation theorems.
5. ~~Robinson's Joint-Consistency (without a proof).~~
6. ~~Craig Interpolation Property (CIP).~~
7. ~~Projective Beth's Definability Property (PBDP).~~

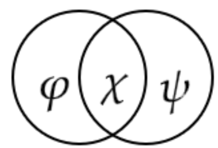


$\Longleftrightarrow \quad \varphi \equiv \forall^* \varphi'$

Based on Chapters 0.1, 0.2.1–0.2.3, 1.2 by [Otto]

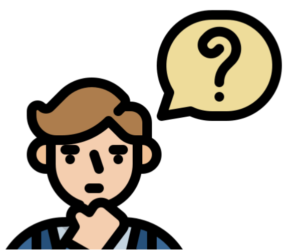
Chapters 1.9–1.11 by [Väänänen]

+ recent research papers.



$sig(\chi) \subseteq sig(\varphi) \cap sig(\psi)$

$\varphi \models \psi \implies \exists \chi \;\; \varphi \models \chi \models \psi$



Feel free to ask questions and interrupt me!

Don't be shy! If needed send me an email (bartosz.bednarczyk@cs.uni.wroc.pl) or approach me after the lecture!

Reminder: this is an advanced lecture. Target: people that had fun learning logic during BSc studies!

Algebraic Diagrams and Embeddings

Goal: Describe a τ-structure $\mathfrak{A}$ up to isomorphism with a (possibly infinite) FO theory $\mathcal{T}_{\mathfrak{A}}$

1. Start with $\mathcal{T}_{\mathfrak{A}} := \emptyset$.

2. With each domain element $a \in A$ we associate a constant symbol "a".

Let τ_A be the extended signature, and let $\mathfrak{A}_A := \mathfrak{A} +$ the interpretation of each a as the corresponding $a \in A$.

3. Append $\bigwedge_{\mathrm{a} \neq \mathrm{b} \in \tau_A \setminus \tau} \mathrm{a} \neq \mathrm{b}$ to $\mathcal{T}_{\mathfrak{A}}$.

4. For all $n \in \mathbb{N}$, all n-tuples of constant symb. $\bar{\mathrm{a}}$ from $\tau_A \setminus \tau$, and relational symb. $\mathrm{R} \in \tau$ of arity n:

- append $\mathrm{R}(\bar{\mathrm{a}})$ to $\mathcal{T}_{\mathfrak{A}}$ iff the corresponding n-tuple belongs to $\mathrm{R}^{\mathfrak{A}}$.
- proceed similarly with $\neg\mathrm{R}(\bar{\mathrm{a}})$ and n-tuples outside $\mathrm{R}^{\mathfrak{A}}$.

5. Close $\mathcal{T}_{\mathfrak{A}}$ under $\wedge$. We denote it $\mathrm{D}(\mathfrak{A})$ and call it the algebraic diagram of $\mathfrak{A}$.

Formally, $\mathrm{D}(\mathfrak{A}) := \left\{ \varphi \in \mathrm{FO}[\tau_A] \mid \mathfrak{A}_A \models \varphi,\ \varphi \text{ is quantifier free} \right\}$

fresh constants | make them different | iterate through τ | positive facts | negative facts

Preservation Theorems

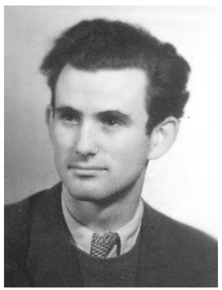
Common theme: Formulae having certain properties are precisely these of a certain fragment of FO

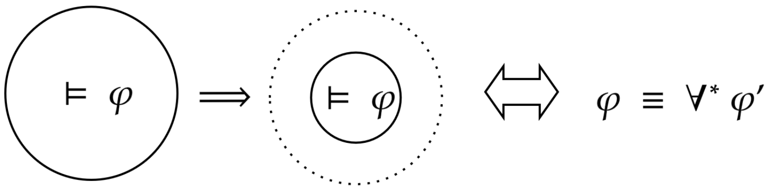
Theorem (Łoś-Tarski 1954)

An FO formula is preserved under substructures[a] iff it is equivalent to a universal[b] formula.

[a] i.e. $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \subseteq \mathfrak{A}$ then $\mathfrak{B} \models \varphi$

[b] (possibly negated) atomic symbols $+ \wedge, \vee$ and $\forall$





- Finitary analogous of Łoś-Tarski fails in the finite, c.f. [Tait 1933].
- Finitary generalisations of Łoś-Tarski by Abhisekh Sankaran [MFCS 2014].
- There are $\mathcal{L} \subseteq$ FO with Łoś-Tarski (also in the finite), e.g. the Guarded Neg. Frag. [B.B.tC. 2018]
- Open problem: Is there a non-trivial $\mathcal{L} \subseteq$ FO (without equality) without Łoś-Tarski? [B. 2022]

Proof of Łoś-Tarski Preservation Theorem: Part I

Theorem (Łoś-Tarski 1954)

An FO formula is preserved under substructures[a] iff it is equivalent to a universal[b] formula.

[a] i.e. $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \subseteq \mathfrak{A}$ then $\mathfrak{B} \models \varphi$

[b] (possibly negated) atomic symbols + $\wedge$, $\vee$ and $\forall$

Proof

Every universal formula is preserved under substructures, so let us focus on the other direction.

Assume that φ is preserved under substructures, and consider the set

$$\Psi := \left\{\psi \mid \varphi \models \psi, \psi \text{ is universal}\right\}.$$

Note that $\varphi \models \Psi$. It suffices to show that $\Psi \models \varphi$. Why?

By compactness there would be a finite subset $\Psi_0 \subseteq_{\text{fin}} \Psi$ such that $\Psi_0 \models \varphi$.

But then $\bigwedge_{\psi \in \Psi_0} \psi$ is the desired universal formula equivalent to φ.

collect universal consequences

compactness

universal formulae are closed under $\wedge$

Proof of Łoś-Tarski Preservation Theorem: Part II

Recall that: φ is preserved under substructures, $\Psi := \{\psi \mid \varphi \models \psi, \psi \text{ is universal}\}$ and our goal is: $\Psi \models \varphi$.

Let $\mathfrak{A} \models \Psi$. We want to show $\mathfrak{A} \models \varphi$. It suffices to find a model $\mathfrak{B}$ of φ containing $\mathfrak{A}$ as a substructure.

Indeed, as φ is preserved under substructures, from $\mathfrak{B} \models \varphi$ we conclude $\mathfrak{A} \models \varphi$.

How to find such $\mathfrak{B}$? Show that $\mathrm{D}(\mathfrak{A}) \cup \{\varphi\}$ is satisfiable!

Ad absurdum, assume that $\mathrm{D}(\mathfrak{A}) \cup \{\varphi\}$ has no model. So $\varphi \not\models \mathrm{D}(\mathfrak{A})$ holds, i.e. $\varphi \models \neg \bigwedge_{\psi(\bar{a}) \in \mathrm{D}(\mathfrak{A})} \psi(\bar{a})$.

By compactness there is a finite $D_0 \subseteq_{\text{fin}} \mathrm{D}(\mathfrak{A})$ such that $\varphi \models \neg \bigwedge_{\psi(\bar{a}) \in D_0} \psi(\bar{a})$.

But as diagrams are closed under conjunction, we get a single formula $\xi(\bar{a}) \in \mathrm{D}(\mathfrak{A})$ s.t. $\varphi \models \neg\xi(\bar{a})$.

Note that φ does not use extra constants from τ_A. Thus actually $\varphi \models \forall \bar{x}\ \neg\xi(\bar{x})$ holds.

As $\forall \bar{x}\ \neg\xi(\bar{x})$ is universal and follows from φ, we know that $\forall \bar{x}\ \neg\xi(\bar{x}) \in \Psi$.

From $\xi(\bar{a}) \in \mathrm{D}(\mathfrak{A})$ we infer $\mathfrak{A} \models \exists \bar{x} \xi(\bar{x})$. A contradiction with $\mathfrak{A} \models \Psi$. $\square$

Strengthen $\varphi \models \neg\xi(\bar{a})$ and use Ψ.

def of $\models$ | magic | assumption φ | diagrams | contradiction | def of $\models$ | compactness | $\mathrm{D}(\mathfrak{A})$ clos.u.$\wedge$ | Shape of ξ/φ

Failure of Łoś-Tarski in the finite. (Part I)

Theorem (Tait 1933)

There is an FO formula that is preserved under substructures of finite structures then $\mathfrak{B} \models \varphi$ but it is not equivalent (in the finite) to any universal formula.

Proof

Consider $\tau = \{\mathtt{min}^{(0)}, \mathtt{max}^{(0)}, <^{(2)}, \mathrm{Next}^{(2)}, \mathrm{P}^{(1)}\}$. Let φ_0 be a universal stating that

$\mathfrak{A} \models \varphi_0$ iff $<^{\mathfrak{A}}$ is a strict linear order with the minimal/maximal elements $\mathtt{min}^{\mathfrak{A}}$, $\mathtt{max}^{\mathfrak{A}}$, and $\mathrm{Next}^{\mathfrak{A}} \subseteq <^{\mathfrak{A}}$.

Moreover, take $\varphi_1 := \forall x \forall y\ \mathrm{Next}(x, y) \leftrightarrow (x < y \land \neg(\exists z\ x < z \land z < y))$.

Note: if $\mathfrak{A} \models \varphi_0 \land \varphi_1$, then $\mathrm{Next}^{\mathfrak{A}}$ is the induced successor of $<^{\mathfrak{A}}$. Finally, let $\varphi := \varphi_0 \land (\varphi_1 \rightarrow \exists x\ \mathrm{P}(x))$.

Observation (The set of finite models of φ is closed under substructures.)

Take a finite $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \subseteq \mathfrak{A}$. Observe that $\mathfrak{B} \models \varphi_0$ (because φ_0 is universal). If $\mathfrak{B} \not\models \varphi_1$ we are done.

If $\mathfrak{B} \models \varphi_1$ then $\mathfrak{A} = \mathfrak{B}$, concluding $\mathfrak{B} \models \varphi$. $\square$

universals are preserved under $\subseteq$

finiteness

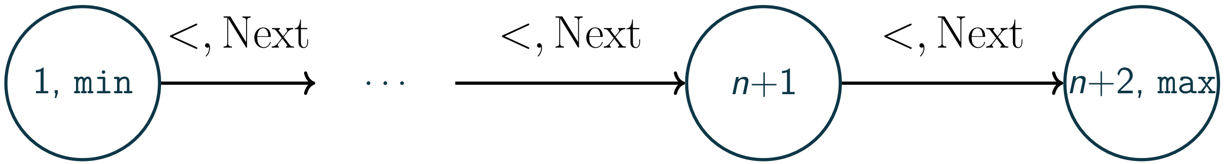
Failure of Łoś-Tarski in the finite. (Part II)

$\mathfrak{A} \models \varphi_0$ iff $<^{\mathfrak{A}}$ is a strict linear order with the minimal/maximal elements $\mathtt{min}^{\mathfrak{A}}$, $\mathtt{max}^{\mathfrak{A}}$, and $\mathrm{Next}^{\mathfrak{A}} \subseteq <^{\mathfrak{A}}$.

$\varphi_1 := \forall x \forall y\ \mathrm{Next}(x, y) \leftrightarrow (x < y \land \neg(\exists z\ x < z \land z < y))$ and $\varphi := \varphi_0 \land (\varphi_1 \to \exists x\ \mathrm{P}(x))$.

Lemma (φ is not equivalent (in the finite) to any universal formula.)

Ad absurdum, there exists quantifier-free $\chi(\bar{x})$ with n variables so that $\varphi \equiv_{\mathsf{fin}} \forall \bar{x}\ \chi(\bar{x})$. Take $\mathfrak{A}$ as below.



$:= \mathfrak{A}$

By construction $\mathfrak{A} \models \varphi_0 \land \varphi_1$. Moreover, observe that $(\mathfrak{A}, \mathrm{P}^{\mathfrak{A}}) \models \varphi$ iff $\mathrm{P}^{\mathfrak{A}} \neq \emptyset$.

Then $(\mathfrak{A}, \emptyset) \not\models \varphi$ implies $(\mathfrak{A}, \emptyset) \not\models \forall \bar{x}\ \chi(\bar{x})$. Thus $(\mathfrak{A}, \emptyset) \models \neg\chi(\bar{a})$ for suitable $\bar{a}$.

Take b to be different from $\bar{a}$, $\mathtt{max}^{\mathfrak{A}}$ and $\mathtt{min}^{\mathfrak{A}}$ (we have enough elements!). Then $(\mathfrak{A}, \{b\}) \models \varphi$.

But $(\mathfrak{A}, \{b\}) \models \neg\chi(\bar{a})$ ($\mathfrak{A} \restriction \bar{a}$ was not touched!). But it means $(\mathfrak{A}, \{b\}) \not\models \forall \bar{x}\ \chi(\bar{x}) \equiv \varphi$. A contradiction!

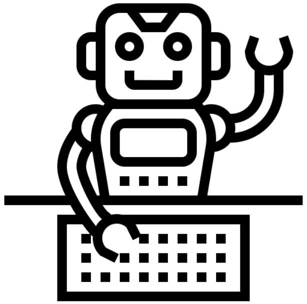
contradiction | def of P | when $\mathrm{P}^{\mathfrak{A}} = \emptyset$ | witness | select suitable b and make it satisfy P | def of φ

Can we make Łoś-Tarski theorem computable?

Input: First-Order φ closed under substructures (in the general setting).

Output: the equivalent universal formula.

Is this problem solvable?: **YES!** Ask Gödel for help!



Unfortunately, the finitary analog is unsolvable. [Chen and Flum 2022]

Other preservation theorems?

Theorem (Lyndon–Tarski 1956, Rossmann 2005)

An FO formula is preserved under homomorphic images[a] iff it is equivalent to a positive existential[b] formula.

[a] i.e. $\mathfrak{A} \models \varphi$ and there is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ then $\mathfrak{B} \models \varphi$

[b] atomic symbols + $\wedge$, $\vee$ and $\exists$

- A notable example of classical MT theorems working in the FMT. [Rossmann's paper]

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