Technische Universität Dresden

International Center for Computational Logic

COMPLEXITY THEORY

Lecture 4: Undecidability and Recursion

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Knowledge-Based Systems

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Undecidability so far

We have seen several undecidable problems for TMs:

- The Halting Problem: recognise TM-word pairs where the TM halts
- The Non-Halting Problem: recognise TM-word pairs where the TM does not halt
- The ε-Halting Problem: recognise TMs that halt on the empty input

Many further TM-related problems are undecidable . . .
. . . but we can use a shortcut to proving many of them:

Theorem 4.1 (Rice's Theorem, informal): Any interesting property related to the language recognised by a given TM is undecidable.

Rice's Theorem

We can make this formal as follows:

Definition 4.2: Let $\mathcal{P}$ be a set of languages. A language $\mathbf{L}$ has the property $\mathcal{P}$ if $\mathbf{L} \in \mathcal{P}$. Property $\mathcal{P}$ is a non-trivial property of recognisable languages if there are TM-recognisable languages that have it and others that do not have it.

Theorem 4.1 (Rice's Theorem): If $\mathcal{P}$ is a non-trivial property of recognisable languages, then the following problem is undecidable:

$$\mathcal{P}\text{-ness} = \{\langle \mathcal{M} \rangle \mid \mathbf{L}(\mathcal{M}) \in \mathcal{P}\}$$

Proof of Rice's Theorem

Theorem 4.1 (Rice's Theorem): If $\mathcal{P}$ is a non-trivial property of recognisable languages, then the following problem is undecidable:

$$\mathcal{P}\text{-ness} = \{\langle\mathcal{M}\rangle \mid \mathbf{L}(\mathcal{M}) \in \mathcal{P}\}$$

Proof: We reduce ε-Halting to $\mathcal{P}$-ness.

- Assume w.l.o.g. that $\emptyset \notin \mathcal{P}$ (otherwise do the proof for $\overline{\mathcal{P}}$)
- Let $\mathcal{M}_{\mathbf{L}}$ be some TM that recognises a language $\mathbf{L} \in \mathcal{P}$
- Given any TM $\mathcal{M}$, compute a TM $\mathcal{M}^*$ that behaves as follows:

  On input $w \in \Sigma^*$:
  (1) Simulate $\mathcal{M}$ on input ε
  (2) If $\mathcal{M}$ halts, simulate $\mathcal{M}_{\mathbf{L}}$ on w

- Then $\mathbf{L}(\mathcal{M}^*) = \mathbf{L} \in \mathcal{P}$ if $\mathcal{M}$ halts on ε, and
  $\mathbf{L}(\mathcal{M}^*) = \emptyset \notin \mathcal{P}$ if $\mathcal{M}$ does not halt on ε

For the required Turing reduction, we construct a TM that:
(Step 1) checks if the input is a TM encoding $\langle\mathcal{M}\rangle$ and rejects otherwise,
(Step 2) returns the result of the check $\langle\mathcal{M}^*\rangle \in \mathcal{P}$-ness. This would decide ε-Halting. □

Using Rice's Theorem

Here are some simple results that Rice gives us:

Corollary 4.3: Given an arbitrary TM $\mathcal{M}$, it is undecidable whether the language recognised by $\mathcal{M}$ has any of the following properties:

- emptiness
- finiteness
- decidability
- regularity
- context-freedom
- contains any given word w (word problem for TMs)

Attention: There are of course many non-trivial properties of TMs that can be decided, and which do not relate to their language:

Example 4.4: It is decidable if a TM has at least three states.

Semi-decidability and Co-semi-decidability

We can distinguish the following two cases:

(1) $\mathbf{L}$ is Turing-recognisable: $\mathbf{L}$ is semi-decidable

(2) $\overline{\mathbf{L}}$ is Turing-recognisable: $\mathbf{L}$ is co-semi-decidable

We have seen examples for both:

Theorem 4.5: The Halting Problem is semi-decidable.

Proof: Use the universal TM to simulate an input TM, and accept if it halts. □

Corollary 4.6: The Non-Halting Problem is co-semi-decidable.

Semi-decidable + Co-semi-decidable = Decidable

An easy but important observation:

Theorem 4.7: If $\mathbf{L}$ is semi-decidable and co-semi-decidable, then $\mathbf{L}$ is decidable.

Proof: On input w, simulate, in parallel, a recogniser for $\mathbf{L}$ and a recogniser for $\overline{\mathbf{L}}$. At least one of them eventually must halt, so we can decide if $w \in \mathbf{L}$. □

We thus obtain an example of a problem that is not Turing-recognisable.

Corollary 4.8: The Non-Halting Problem is not Turing-recognisable.

Turing reductions and semi-decidability

Observation:

- If **Q** is decidable and **P** $\leq_T$ **Q**, then **P** is decidable (Theorem 3.17)
- But: if **Q** is semi-decidable and **P** $\leq_T$ **Q**, then **P** may or may not be semi-decidable

Reason: An oracle for Halting is as good as an oracle for Non-Halting, since we are free to complement the answer in an oracle machine.

This is a general insight: complementing oracles has no effect

To preserve (co-)semi-decidability, one needs a more restricted form of reduction:

Definition 4.9: A language **P** is many-one reducible to a language **Q**, written **P** $\leq_m$ **Q** if there exists a total computable function $f : \Sigma^* \to \Sigma^*$ such that, for all $w \in \Sigma^*$:

$$w \in \mathbf{P} \qquad \text{if and only if} \qquad f(w) \in \mathbf{Q}.$$

This is sometimes called a mapping-reduction or an m-reduction.

Properties of Many-One-Reductions

Many-one reductions are special kinds of Turing reductions:

Theorem 4.10: If **P** $\leq_m$ **Q** then **P** $\leq_T$ **Q**.

Proof: We obtain an OTM with oracle **Q** that recognises **P** as follows:

- On input w, compute $f(w)$
- Call the oracle and return its result (yes = accept; no = reject) □

An easy consequence of Theorem 3.17 therefore is:

Corollary 4.11: If **P** $\leq_m$ **Q** and **Q** is decidable, then **P** is decidable.

However, now we also have the following:

Theorem 4.12: If **P** $\leq_m$ **Q** and **Q** is semi-decidable, then **P** is semi-decidable.

Proof: Given a TM that recognises **Q**, we obtain a TM that recognises **P** as follows:

- On input w, compute $f(w)$
- Simulate the TM for **Q** and return the result (if any) □

Example: Many-one Reduction

Some of our previous Turing-reductions can easily be described as many-one, e.g., Halting can be many-one reduced to ε-Halting. Here is another example:

Definition 4.13: Two TMs $\mathcal{M}$ and $\mathcal{N}$ are equivalent if $\mathbf{L}(\mathcal{M}) = \mathbf{L}(\mathcal{N})$.

Theorem 4.14: Equivalence of Turing machines is undecidable.

(Note that we could also get this from Rice's Theorem, but we want to try out our new machinery.)

Proof: We define f such that $w \in \varepsilon$-Halting iff $f(w) \in$ Equivalence.

Let $\mathcal{M}_a$ be a TM that accepts all inputs.

For a TM $\mathcal{M}$, we define the following TM $\mathcal{M}^*$:

- Simulate $\mathcal{M}$ on the empty input.
- If $\mathcal{M}$ halts, accept.

Then $\mathcal{M}^*$ is equivalent to $\mathcal{M}_a$ iff $\mathcal{M}$ halts on the empty input. We define f:

$$f(w) = \begin{cases} \langle \mathcal{M}^*, \mathcal{M}_a \rangle & \text{if } w = \langle \mathcal{M} \rangle \\ \varepsilon & \text{(an invalid input) if } w \text{ is no encoded TM} \end{cases} \qquad \square$$

Equivalence is Hard

We can show a somewhat stronger result:

Theorem 4.15: Equivalence of Turing machines is neither semi-decidable nor co-semi-decidable.

Proof: We have already shown ε-Halting $\leq_m$ Equivalence. Since we know that ε-Halting is not co-semi-decidable (similar to Halting), we conclude that Equivalence is neither.

However, we can also show that $\overline{\varepsilon\text{-Halting}} \leq_m$ Equivalence.

- Note that the TM $\mathcal{M}^*$ defined on the previous slide either accepts all inputs (if $\mathcal{M}$ halts on ε) or none (if it doesn't)
- Equivalence to $\mathcal{M}_a$ corresponds to ε-Halting
- On the other hand, equivalence to a TM $\mathcal{M}_\emptyset$, which rejects all inputs, corresponds to ε-non-Halting

We can therefore use the reduction f:

$$f(w) = \begin{cases} \langle \mathcal{M}^*, \mathcal{M}_\emptyset \rangle & \text{if } w = \langle \mathcal{M} \rangle \\ \langle \mathcal{M}_\emptyset, \mathcal{M}_\emptyset \rangle & \text{(equivalent TMs) if } w \text{ is no encoded TM (since then } w \in \overline{\varepsilon\text{-Halting}}) \end{cases}$$

□

Recursion

A Paradox

A Paradox in the Study of Life:

(1) Living things are machines.

(2) Living things can reproduce.

(3) Machines cannot reproduce.

Rationale:

(1) Viewpoint of modern biology.

(2) Evident.

(3) If a machine A produces a machine B, then A must be more complex than B. For example, a car-producing factory is **more complex** than the cars it produces, as it contains the design of the cars and, **in addition**, the design of all manufacturing robots, among others. Since no machine is more complex than itself, a machine cannot reproduce itself.

Resolving the Paradox

A Paradox in the Study of Life:

(1) Living things are machines.

(2) Living things can reproduce.

(3) Machines cannot reproduce.

Question: How to resolve this paradox?

Answer: Assertion (3) is wrong.

In particular, the underlying argument of “more information” and “greater complexity” needed by the producing machine is flawed: there are TMs that reproduce themselves

Quines

Reproduction of TMs is closely related to the task of creating a program that prints its own source code:

Definition 4.16: A quine is a program that, when started without any input, will print out its own source code, and then stop.

Can Quines be created? How?

Example 4.17 (A quine in English): Print this sentence.

However, we cannot turn this into a program, since “this sentence” does often not correspond to available programming constructs.

Example 4.18 (Another quine in English): Print the following sentence twice, the second time in quotes. "Print the following sentence twice, the second time in quotes."

Some Real Quines

Example 4.19 (A classic C quine): `main()char *c="main()char *c=%c%s%c;printf(c,34,c,34);";printf(c,34,c,34);`

Example 4.20 (The shortest C quine, by Szymon Rusinkiewicz):

Example 4.21 (A Python quine by Frank Stajano):

```
l='l=%s;print l%%`l`';print l%`l`
```

Note: A variation are ouroboros quines that print out another program that prints out the original again. More steps are possible. See, e.g., `https://github.com/mame/quine-relay` for one with 100 steps.

Other variations exist (see Wikipedia).

Towards a TM Quine

We define a TM SELF that ignores its input and prints out a description of itself.
(A TM quine, where "source code" is interpreted as "encoding of the TM")

The following small result is helpful:

Lemma 4.22: There is a computable function $q : \Sigma^* \to \Sigma^*$ such that, for each $w \in \Sigma^*$, the word $q(w)$ is (the encoding of) a TM that prints w and halts.

Proof: For any word w, let $\mathcal{P}_w$ be a TM that replaces the tape contents with the word w (clearly, this can easily be found for any w).

Now q is simply computed by a TM that, given w as input, constructs $\mathcal{P}_w$ and then computes and outputs $\langle \mathcal{P}_w \rangle$. □

Intuition: If we were using another programming language, the TM $\mathcal{P}_w$ might be, e.g., `print(`w`)`, and the function we seek would simply turn input string w into output string "`print(`w`)`".

Defining the TM SELF

Like other quines, SELF consists of two parts:

A Compute the "source code" $\langle B\rangle$ of a suitable program B

B Use $\langle B\rangle$ to print out:
(1) source code $\langle A\rangle$ that computes $\langle B\rangle$ and (2) the source code $\langle B\rangle$ itself

We know how to implement part A: use the TM $\mathcal{P}_{\langle B\rangle}$

(however, to actually do this, we need to know B first)

B in turn can work as follows:

Given some input string $\langle M\rangle$:

- compute $q(\langle M\rangle)$
- concatenate the TMs given by $q(\langle M\rangle)$ and $\langle M\rangle$
  (take a disjoint union of states where any halting state of $q(\langle M\rangle)$ gets a transition to the starting state of $\langle M\rangle$)
- output the encoding of the resulting machine

Then part B does not depend on A, so we can really define A as $\mathcal{P}_{\langle B\rangle}$

Summary: SELF TM

So how did we construct our TM quine now?

Step 1: We define some TM B that behaves as follows:

Given some input string $\langle M\rangle$:

- compute $q(\langle M\rangle)$
- concatenate the TMs given by $q(\langle M\rangle)$ and $\langle M\rangle$
  (take a disjoint union of states where any halting state of $q(\langle M\rangle)$ gets a transition to the starting state of $\langle M\rangle$)
- output the encoding of the resulting machine

Step 2: We define SELF to be the TM constructed by B on input $\langle B\rangle$

Exercise: Use this recipe to create a quine in your favourite programming language (or just use Python). What is the equivalent of "TM concatenation" here? Also note that the function q is often more complicated than one might think, due to character escaping.

The Recursion Theorem

Going further, we can allow any TM to access its own description during the computation:

Theorem 4.23 (Recursion Theorem): Let $t : \Sigma^* \times \Sigma^* \to \Sigma^*$ be a function computed by some TM $\mathcal{T}$ (assuming a suitable encoding of pairs of words over Σ^*). Then there is a TM $\mathcal{R}$ that computes a function $r : \Sigma^* \to \Sigma^*$ such that

$$r(w) = t(\langle \mathcal{R} \rangle, w)$$

for every $w \in \Sigma^*$.

Intuition: To make a TM that can use its own description, we first devise a TM $\mathcal{T}$ (to compute t) that receives the description of a machine as extra input. The theorem yields a TM $\mathcal{R}$ that operates like $\mathcal{T}$ does but with $\mathcal{R}$'s description filled in automatically.

The Recursion Theorem: Proof

Theorem 4.23 (Recursion Theorem): Let $t : \Sigma^* \times \Sigma^* \to \Sigma^*$ be a function computed by some TM $\mathcal{T}$ (assuming a suitable encoding of pairs of words over Σ^*). Then there is a TM $\mathcal{R}$ that computes a function $r : \Sigma^* \to \Sigma^*$ such that $r(w) = t(\langle \mathcal{R} \rangle, w)$ for every $w \in \Sigma^*$.

Proof: The proof is similar to the construction of SELF, using a TM with three parts A, B and T:

- A: print $\langle BT \rangle$ (like $\mathcal{P}_{\langle BT \rangle}$ but without deleting the input)

  we use BT to denote the concatenation of the TM parts B and T in one TM

- B: on an input of form $w\langle M \rangle$, replace $\langle M \rangle$ by an encoding of the concatenation of $q'(\langle M \rangle)$ and $\langle M \rangle$

  where $q'(v)$ is like q but returns a TM that adds v at the end of the tape

- T: run $\mathcal{T}$ on an input of form $w\langle N \rangle$

We assume here that our TM encoding can be written next to the input w without risk of confusion. Then $\mathcal{R}$ is the TM obtained as the concatenation of A, B, and T.

We can get an encoding of this TM by running B on input $\varepsilon\langle BT \rangle$. □

Using the Recursion Theorem

By the Recursion Theorem, we can now use instructions like “obtain own description $\langle \mathcal{M} \rangle$” in our informal descriptions of TMs.

Example 4.24: We can describe a TM quine in the style of our previous SELF as follows:

On any input:

- Obtain own description $\langle \mathcal{M} \rangle$
- Print $\langle \mathcal{M} \rangle$

We can construct such a TM by applying the Recursion Theorem to the TM $\mathcal{T}$ described as follows:

$$\text{On input } \langle w, \mathcal{M} \rangle \text{, print } \langle \mathcal{M} \rangle$$

The Recursion Theorem turns this into a TM $\mathcal{R}$ that is a quine.

Halting is Undecidable: Proof by Introspection

We can also use the Recursion Theorem for alternative proofs:

Theorem 3.11 The Halting Problem $\mathbf{P}_{\text{Halt}}$ is undecidable.

Proof: By contradiction: Suppose there is a decider $\mathcal{H}$ for the Halting Problem

We construct a TM $\mathcal{M}$ that, on input w, acts as follows:

(1) Obtain own description $\langle\mathcal{M}\rangle$

(2) Simulate $\mathcal{H}$ on input $\langle\mathcal{M}\rangle\#\#\langle w\rangle$, that is, check if $\mathcal{M}$ halts on w

(3) If yes, enter an infinite loop;
if no, halt and accept

Then $\mathcal{M}$ halts on w if and only if it doesn't – contradiction. □

Minimal TMs

Definition 4.25: A TM $\mathcal{M}$ is called minimal if there is no TM equivalent to $\mathcal{M}$ that has a shorter description. The problem of deciding if a TM is minimal is:

$$\textbf{MIN}_{\text{TM}} = \{\langle\mathcal{M}\rangle \mid \mathcal{M} \text{ is a minimal } TM\}$$

Theorem 4.26: $\textbf{MIN}_{\text{TM}}$ is not Turing-recognisable.

Proof: Assume there is some TM $\mathcal{E}$ enumerating MIN_{TM}.

We define a TM C that processes an input w as follows:

(1) Obtain own description $\langle C\rangle$

(2) Simulate $\mathcal{E}$ until some TM $\mathcal{D}$ is printed such that $\langle\mathcal{D}\rangle$ is longer than $\langle C\rangle$

(3) Simulate $\mathcal{D}$ on w

Then C is equivalent to $\mathcal{D}$, but it has a shorter description, contradicting the assumption that $\mathcal{D}$ is minimal. □

Summary and Outlook

Most properties related to the computation of TMs are undecidable

Many-one reductions establish a closer relationship between two problems than Turing reductions

There are non-semi-decidable problems

Turing machines can work with their own description

What's next?

- Defining complexity classes
- Time complexity
- Non-deterministic time