

# DATABASE THEORY

## Lecture 6: Conjunctive Queries

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Knowledge-Based Systems

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More recent versions of this slide deck might be available.  
For the most current version of this course, see  
[https://iccl.inf.tu-dresden.de/web/Database\\_Theory/en](https://iccl.inf.tu-dresden.de/web/Database_Theory/en)

# Review: FO Query Complexity

The evaluation of FO queries is

- PSpace-complete for combined complexity
- PSpace-complete for query complexity
- $AC^0$ -complete for data complexity

~> PSpace is rather high

~> Are there relevant query languages that are simpler than that?

# Conjunctive Queries

Idea: restrict FO queries to conjunctive, positive features

**Definition 6.1:** A **conjunctive query** (CQ) is an expression of the form

$$\exists y_1, \dots, y_m. A_1 \wedge \dots \wedge A_\ell$$

where each  $A_i$  is an atom of the form  $R(t_1, \dots, t_k)$ . In other words, a conjunctive query is an FO query that only uses conjunctions of atoms and (outer) existential quantifiers.

Example: “Find all lines that depart from an accessible stop” (as seen in earlier lectures)

$$\exists y_{\text{SID}}, y_{\text{Stop}}, y_{\text{To}}. \text{Stops}(y_{\text{SID}}, y_{\text{Stop}}, \text{"true"}) \wedge \text{Connect}(y_{\text{SID}}, y_{\text{To}}, x_{\text{Line}})$$

# Conjunctive Queries in Relational Calculus

The expressive power of CQs can also be captured in the relational calculus

**Definition 6.2:** A **conjunctive query** (CQ) is a relational algebra expression that uses only the operations select  $\sigma_{n=m}$ , project  $\pi_{a_1, \dots, a_n}$ , join  $\bowtie$ , and renaming

$\delta_{a_1, \dots, a_n \rightarrow b_1, \dots, b_n}$ .

Renaming is only relevant in named perspective

↪ CQs are also known as **SELECT-PROJECT-JOIN** queries

# Extensions of Conjunctive Queries

Two features are often added:

- **Equality:** CQs with equality can use atoms of the form  $t_1 \approx t_2$   
(in relational calculus: table constants)
- **Unions:** unions of conjunctive queries are called UCQs  
(in this case the union is only allowed as outermost operator)

Both extensions truly increase expressive power  
(as shown in exercise)

**Features omitted on purpose:** negation and universal quantifiers  
→ the reason for this is query complexity (as we shall see)

# Boolean Conjunctive Queries

A **Boolean conjunctive query (BCQ)** asks for a mapping from query variables to domain elements such that all atoms are true

**Example:** “Is there an accessible stop where some line departs?”

$$\exists y_{\text{SID}}, y_{\text{Stop}}, y_{\text{To}}, y_{\text{Line}}. \text{Stops}(y_{\text{SID}}, y_{\text{Stop}}, \text{"true"}) \wedge \text{Connect}(y_{\text{SID}}, y_{\text{To}}, y_{\text{Line}})$$

Stops:

SID	Stop	Accessible
17	Hauptbahnhof	true
42	Helmholtzstr.	true
57	Stadtgutstr.	true
123	Gustav-Freytag-Str.	false
...	...	...

Connect:

From	To	Line
57	42	85
17	789	3
...	...	...

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# How Hard is it to Answer CQs?

If we know the variable mappings, it is easy to check:

- Checking if a single ground atom  $R(c_1, \dots, c_k)$  holds can be done in linear time
- Checking if a conjunction of ground atoms holds can be done in quadratic time

# How Hard is it to Answer CQs?

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~> A candidate BCQ match can be verified in P

(There are  $n^m$  candidates:  $n$  size of domain;  $m$  number of query variables)

**Theorem 6.3:** BCQ query answering is in NP for combined complexity (and also for query complexity).

~> Better than PSpace (presumably)

# Can we do any better?

Not really. To see this, let's look at some other problems.

Consider two relational structures  $\mathcal{I}$  and  $\mathcal{J}$   
(= database instances, interpretations, hypergraphs)

**Definition 6.4:** A **homomorphism**  $h$  from  $\mathcal{I}$  to  $\mathcal{J}$  is a function  $h : \Delta^{\mathcal{I}} \rightarrow \Delta^{\mathcal{J}}$  such that, for all relation names  $R$ :

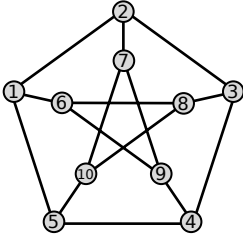
$$\text{if } \langle d_1, \dots, d_n \rangle \in R^{\mathcal{I}} \quad \text{then} \quad \langle h(d_1), \dots, h(d_n) \rangle \in R^{\mathcal{J}}.$$

The **homomorphism problem** is the question if there is a homomorphism from  $\mathcal{I}$  to  $\mathcal{J}$ .

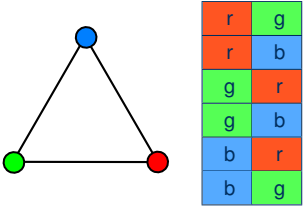
# Example: Three-colouring as Homomorphism

$\mathcal{I}$ :

1	2
1	5
1	6
2	3
2	7
3	4
3	8
...	...



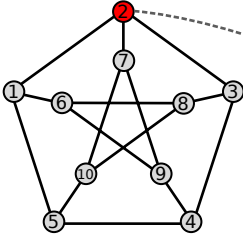
$\mathcal{J}$ :



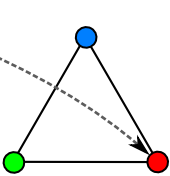
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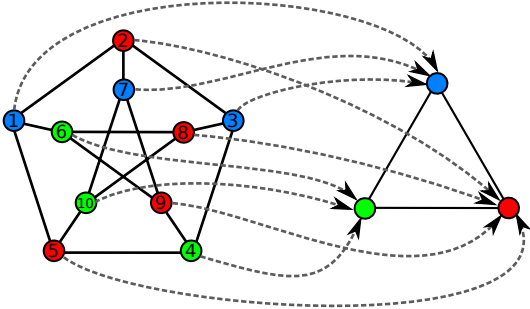
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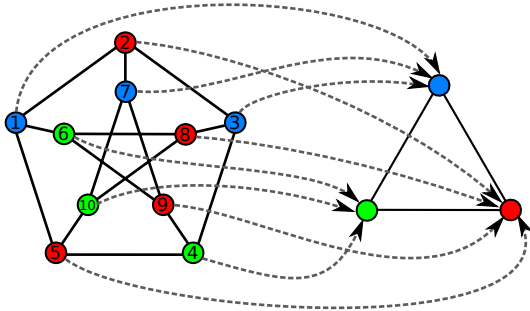
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$\mathcal{J}$ :



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g	r
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b	g

3-colouring is NP-hard

$\leadsto$  the homomorphism problem is NP-hard

# BCQ Answering as Homomorphism Problem

The homomorphism problem can be reduced to BCQ answering:

- A relational structure  $\mathcal{I}$  gives rise to a CQ  $Q_{\mathcal{I}}$ :  
replace domain elements by variables (one-to-one); add one query atom per relational tuple; existentially quantify all variables
- $\mathcal{I}$  has a homomorphism to  $\mathcal{J}$  if and only if  $\mathcal{J} \models Q_{\mathcal{I}}$



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BCQ answering can be reduced to the homomorphism problem:

- Clear for BCQs that don't contain constants
- Eliminate query constants  $a$ : create new relation  $R_a = \{\langle a \rangle\}$ ; replace  $a$  by a fresh variable  $x$  and add a query atom  $R_a(x)$

$\leadsto$  both problems are equivalent

# Complexity of Conjunctive Query Answering

We showed that BCQ answering is in NP and that the homomorphism problem is NP-hard, therefore:

**Theorem 6.5:** BCQ answering is

- NP-complete for combined complexity
- NP-complete for query complexity
- in  $AC^0$  for data complexity (inherited from FO queries)

# Constraint Satisfaction Problems

Another important problem equivalent to BCQ answering

**Definition 6.6:** A **constraint satisfaction problem** (CSP) over a domain  $\Delta$  is given by a set of variables  $\{x_1, \dots, x_n\}$  and a set of constraints  $\{C_1, \dots, C_m\}$ , where each constraint  $C_i$  has the form  $\langle X_i, R_i \rangle$  with

- $X_i$  a list of variables from  $\{x_1, \dots, x_n\}$ ,
- $R_i$  a  $|X_i|$ -ary relation over  $\Delta$ .

A **solution** to the CSP is an assignment of variables to values from  $\Delta$  such that all constraints are satisfied (=all tuples occur in the respective relations).

↪ alternative notation for BCQ answering/homomorphism problem

# CSP Example

A combinatorial crossword puzzle:

Domain:  $\Delta = \{A, \dots, Z\}$

Variables:  $x_1, \dots, x_{26}$

Constraints:

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$		$x_6$
$x_7$				$x_8$	$x_9$	$x_{10}$
$x_{11}$	$x_{12}$	$x_{13}$		$x_{14}$		$x_{15}$
$x_{16}$		$x_{17}$		$x_{18}$		$x_{19}$
$x_{20}$	$x_{21}$	$x_{22}$	$x_{23}$	$x_{24}$	$x_{25}$	$x_{26}$

1 vertically:

H	E	A	R	T
H	O	N	E	Y
I	R	O	N	Y
L	O	G	I	C

1 horizontally:

H	A	P	P	Y
I	N	F	E	R
L	A	B	O	R
L	A	T	E	R

5 vertically:

R	A	D	I	O
R	E	T	R	O
Y	A	C	H	T
Y	E	R	B	A

...

# Equivalent Problems

Summing up, the following problems are equivalent:

- Answering a conjunctive query over a database instance
- Finding a homomorphism from a relational structure to another
- Solving a constraint satisfaction problem

Each of these problems is NP-complete

# Tractable CQ Answering

# How to reduce complexities?

NP-complete query complexity is still intractable

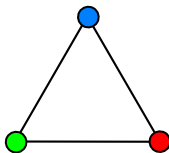
Can we do better?

## Idea:

- We have encoded 3-colourability to show NP-hardness
- Can we avoid hardness by restricting to certain cases?

## Excursion: CSP Complexity

The problem of 3-colourability was captured by the target structure of the homomorphism:

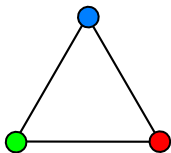


This part of the problem is called the **template** in CSP.



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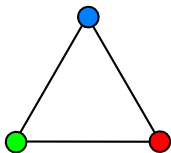
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Another template:



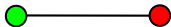
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Another template:



↪ **2-colourability**, a well-known problem in P

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Can we study CSP complexity based on a given template?

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In 1993, Feder and Vardi famously conjecture the following

**Feder–Vardi Conjecture (simplified):** For any fixed template, the CSP-problem is either NP-complete or contained in P.

**What's the big deal?**

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### What's the big deal?

- According to [Ladner's Theorem](#), if  $P \neq NP$ , then there are problems that are neither in P nor hard for NP— so-called [NP-intermediate](#) problems<sup>1</sup>
- According to Feder/Vardi, such problems do not exist in CSPs (which might contribute to explaining why they seem to be relatively rare)

<sup>1</sup> For more details, see our lecture [Complexity Theory](#)

# The End of the Feder–Vardi Conjecture

In 2017, the Feder–Vardi Conjecture has been proven independently by two authors:

- Andrei A. Bulatov: **A Dichotomy Theorem for Nonuniform CSPs** (FOCS 2017)
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Can we exploit this for better BCQ answering complexities?



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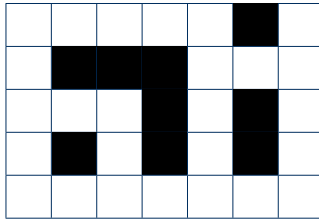
Not really:

- The template corresponds to the database in CQ answering
- We do not want to answer many queries over fixed databases
- Assuming that only families of “easy” databases are considered is not realistic

# Towards Better Complexities

## Idea 2:

- Searching a match may require backtracking, eventually exploring all options
- Can we constrain the query to avoid this?



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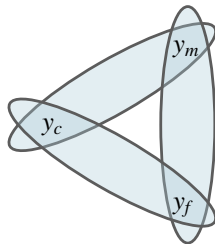
Intuition: life would be easier if we would not have to go back so much ...

↪ the problem is with the cycles

## Example: Cyclic CQs

“Is there a child whose parents are married with each other?”

$$\exists y_c, y_m, y_f. \text{mother}(y_c, y_m) \wedge \text{father}(y_c, y_f) \wedge \text{married}(y_m, y_f)$$

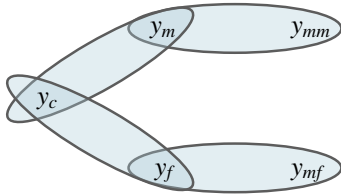


↪ cyclic query

## Example: Acyclic CQs

“Is there a child whose parents are married with someone?”

$$\exists y_c, y_m, y_f, y_{mm}, y_{mf}. \text{mother}(y_c, y_m) \wedge \text{father}(y_c, y_f) \wedge \text{married}(y_m, y_{mm}) \wedge \text{married}(y_{mf}, y_f)$$

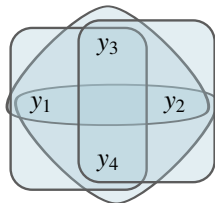
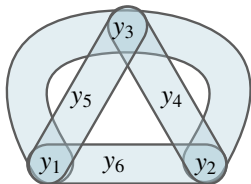
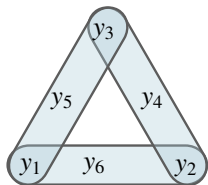


↪ acyclic query

# Defining Acyclic Queries

Queries in general are hypergraphs

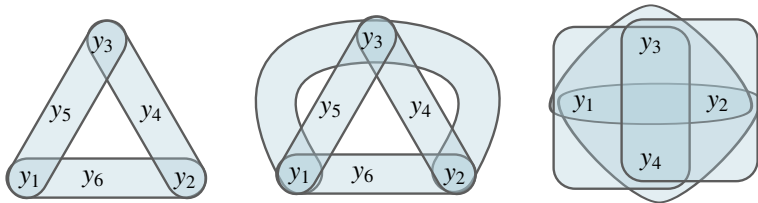
→ What does “acyclic” mean?



# Defining Acyclic Queries

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~> What does “acyclic” mean?



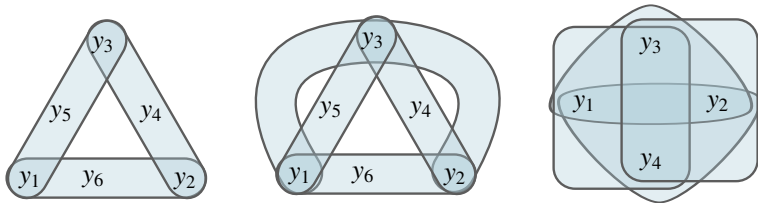
View hypergraphs as graphs to check acyclicity?

- **Primal graph:** same vertices; edges between each pair of vertices that occur together in a hyperedge
- **Incidence graph:** vertices and hyperedges as vertices, with edges to mark incidence (bipartite graph)

# Defining Acyclic Queries

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- **Incidence graph**: vertices and hyperedges as vertices, with edges to mark incidence (bipartite graph)

However: both graphs have cycles in almost all cases

# Acyclic Hypergraphs

GYO-reduction algorithm to check acyclicity:

(after Graham [1979] and Yu & Özsoyoğlu [1979])

Input: hypergraph  $H = \langle V, E \rangle$  (we don't need relation labels here)

Output: GYO-reduct of  $H$

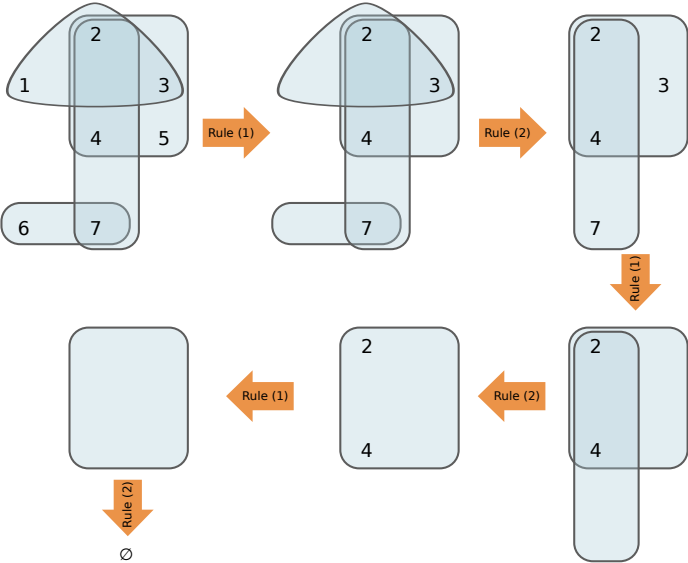
Apply the following simplification rules as long as possible:

- (1) Delete all vertices that occur in at most one hyperedge
- (2) Delete all hyperedges that are empty or that are contained in other hyperedges

**Definition 6.7:** A hypergraph is **acyclic** if its GYO-reduct is  $\langle \emptyset, \emptyset \rangle$ .

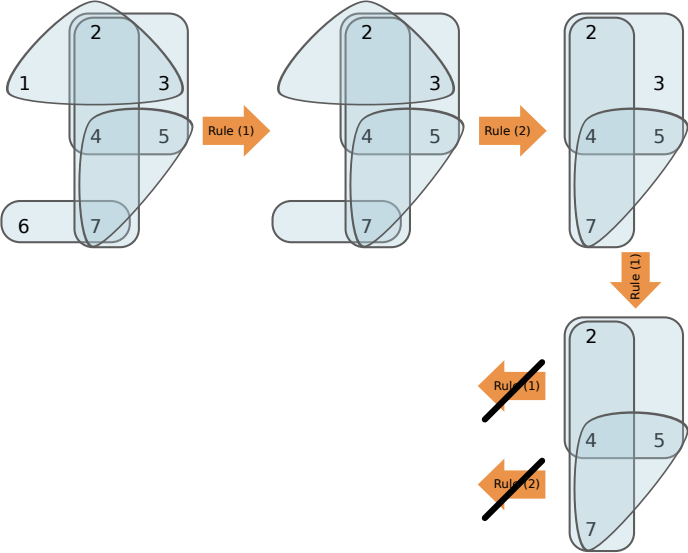
A CQ is **acyclic** if its associated hypergraph is.

# Example 1: GYO-Reduction





# Example 2: GYO-Reduction

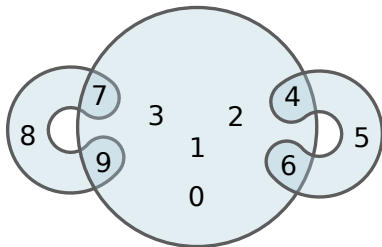


# Alternative Version of GYO-Reduction

An **ear** of a hypergraph  $\langle V, E \rangle$  is a hyperedge  $e \in E$  that satisfies one of the following:

- (1) there is an edge  $e' \in E$  such that  $e \neq e'$  and every vertex of  $e$  is either only in  $e$  or also in  $e'$ , or
- (2)  $e$  has no intersection with any other hyperedge.

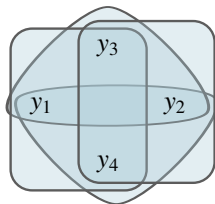
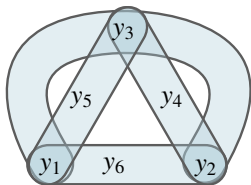
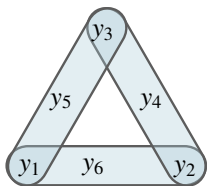
Example:



$\leadsto$  edges  $\langle 4, 5, 6 \rangle$  and  $\langle 7, 8, 9 \rangle$  are ears

# Examples

Any ears?



# GYO'-Reduction

Input: hypergraph  $H = \langle V, E \rangle$

Output: GYO'-reduct of  $H$

Apply the following simplification rule as long as possible:

- Select an ear  $e$  of  $H$
- Delete  $e$
- Delete all vertices that only occurred in  $e$

**Theorem 6.8:** The GYO-reduct is  $\langle \emptyset, \emptyset \rangle$  if and only if the GYO'-reduct is  $\langle \emptyset, \emptyset \rangle$

↪ alternative characterization of acyclic hypergraphs

# Join Trees

Both GYO algorithms can be implemented in linear time

Open question: what benefit does BCQ acyclicity give us?

# Join Trees

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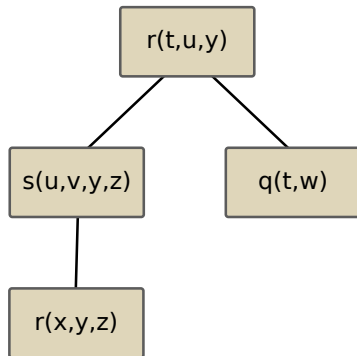
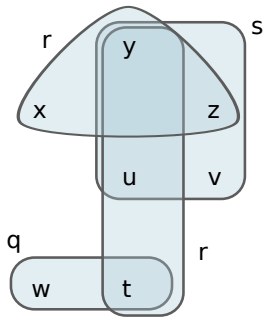
Fact: if a BCQ is acyclic, then it has a join tree

**Definition 6.9:** A **join tree** of a (B)CQ is an arrangement of its query atoms in a tree structure  $T$ , such that for each variable  $x$ , the atoms that refer to  $x$  are a connected subtree of  $T$ .

A (B)CQ that has a join tree is called a **tree query**.

# Example: Join Tree

$$\exists x, y, z, t, u, v, w. (r(x, y, z) \wedge r(t, u, y) \wedge s(u, v, y, z) \wedge q(t, w))$$



# Processing Join Trees Efficiently

Join trees can be processed in polynomial time

Key ingredient: the semijoin operation

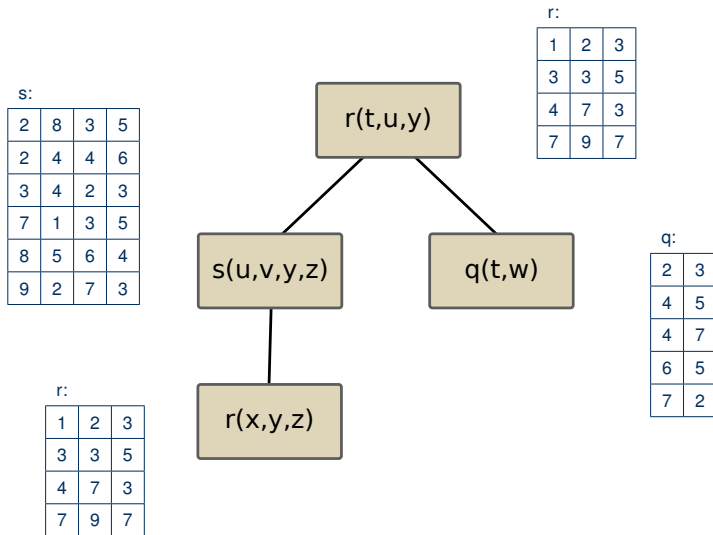
**Definition 6.10:** Given two relations  $R[U]$  and  $S[V]$ , the **semijoin**  $R^I \bowtie S^I$  is defined as  $\pi_U(R^I \bowtie S^I)$ .

Join trees can now be processed by computing semijoins bottom-up

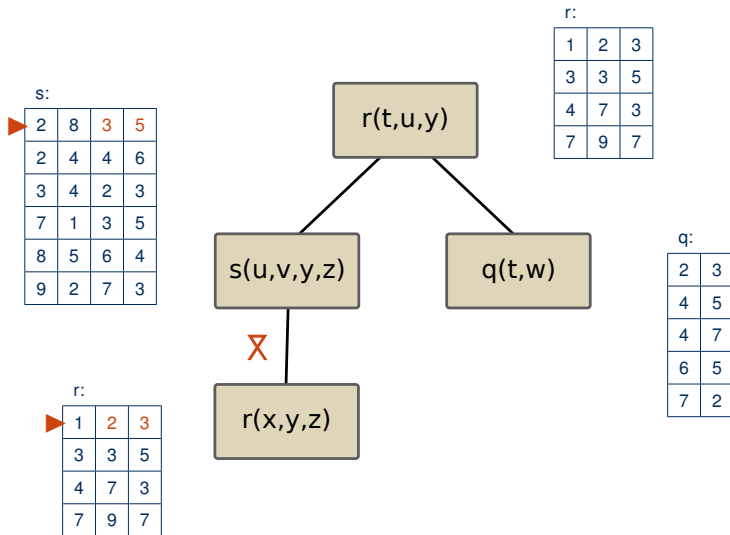
→ Yannakakis' Algorithm



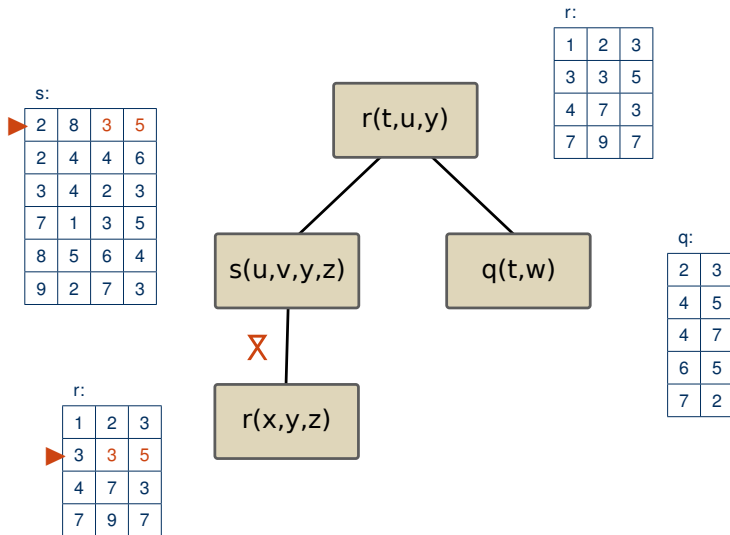
# Yannakakis' Algorithm by Example



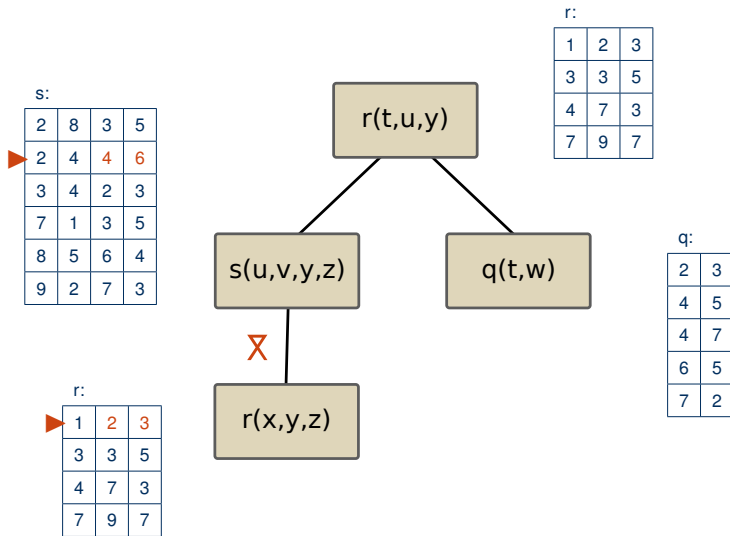
# Yannakakis' Algorithm by Example



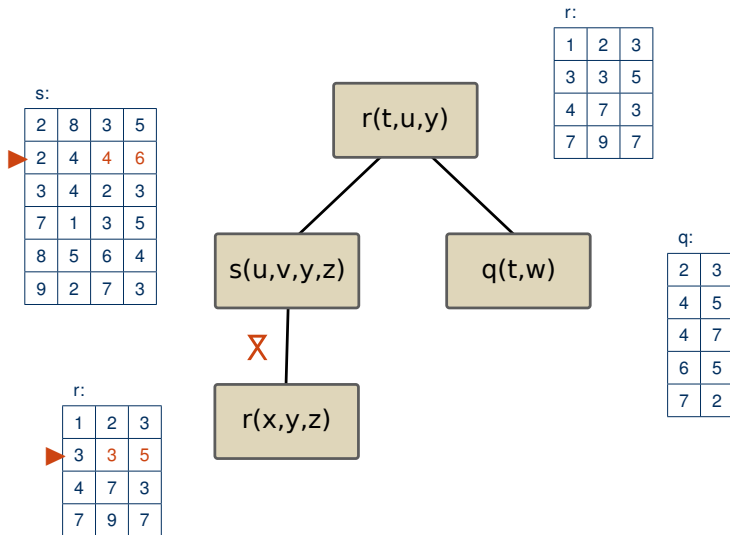
# Yannakakis' Algorithm by Example



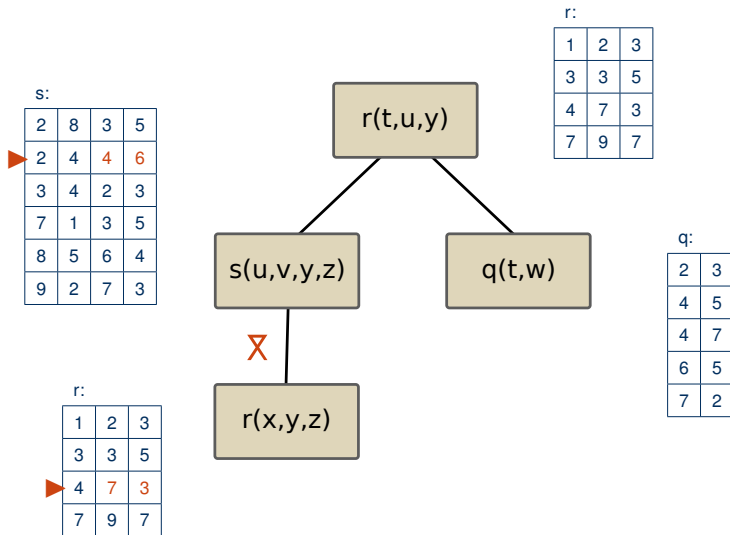
# Yannakakis' Algorithm by Example



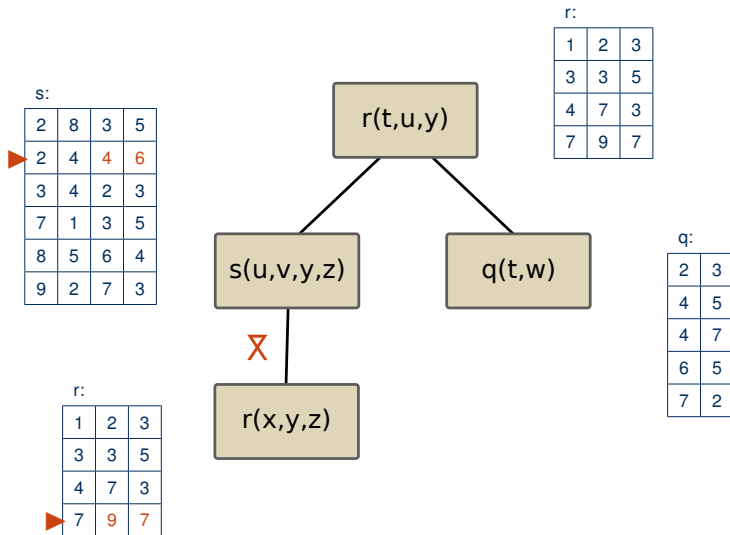
# Yannakakis' Algorithm by Example



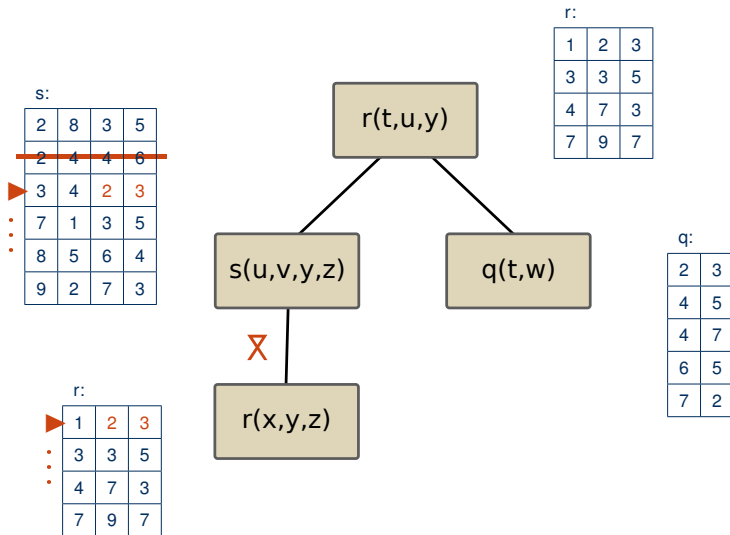
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# Yannakakis' Algorithm by Example

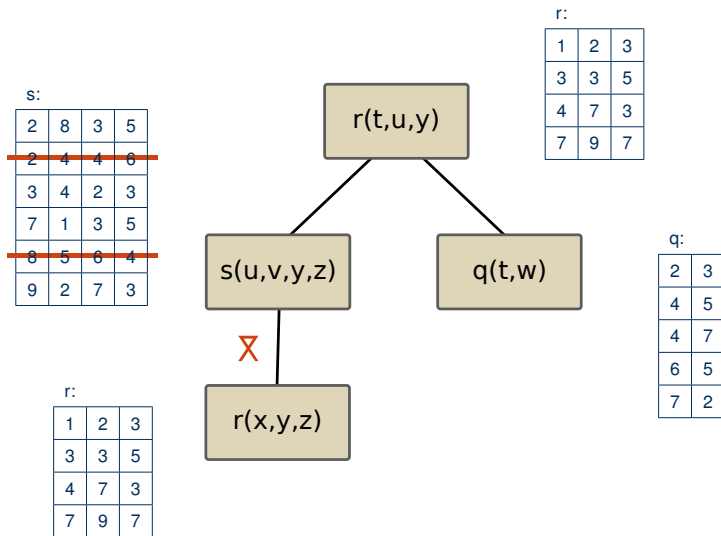


# Yannakakis' Algorithm by Example

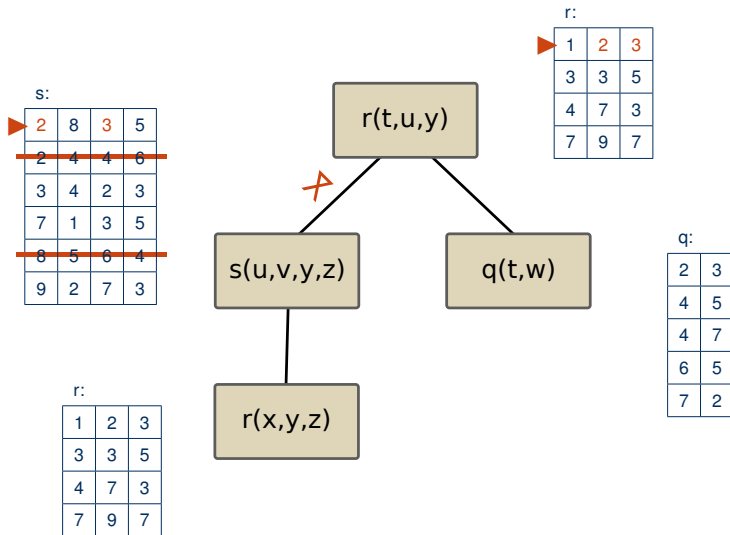




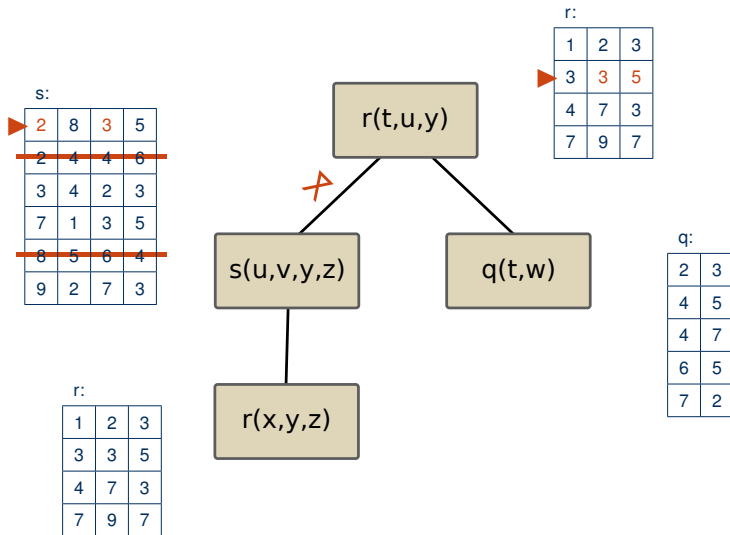
# Yannakakis' Algorithm by Example



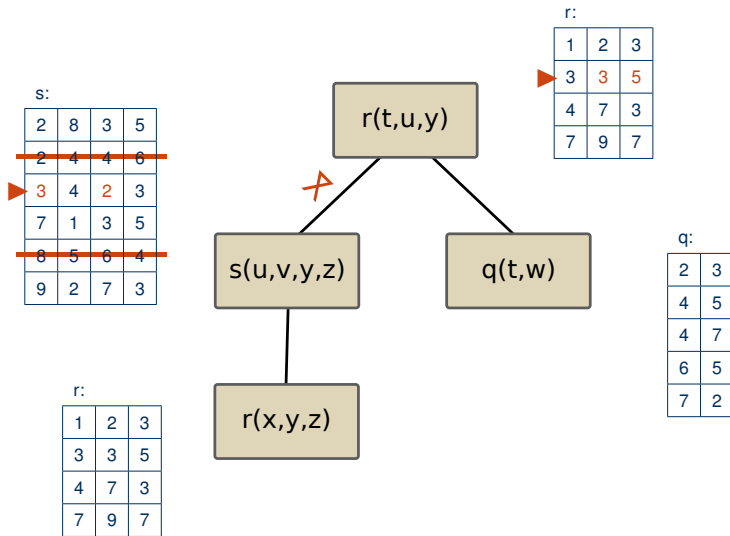
# Yannakakis' Algorithm by Example



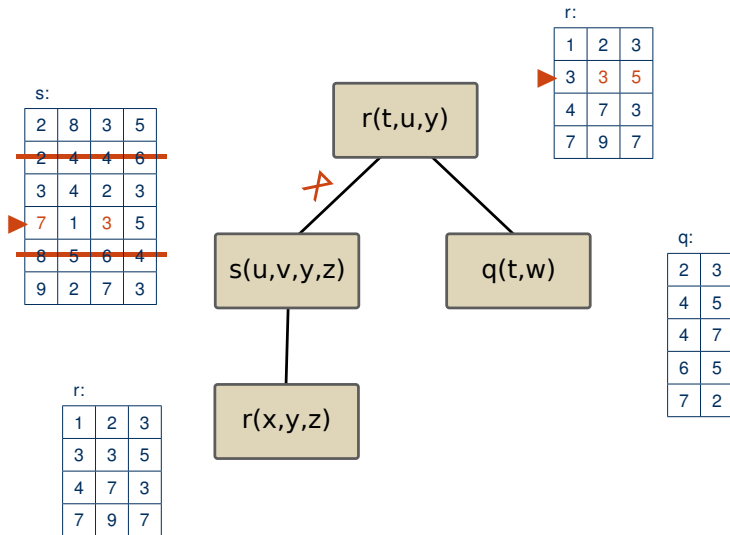
# Yannakakis' Algorithm by Example



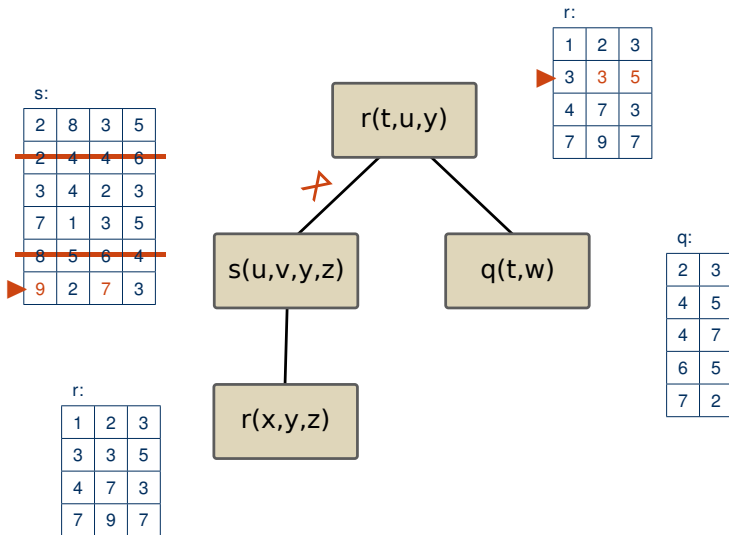
# Yannakakis' Algorithm by Example



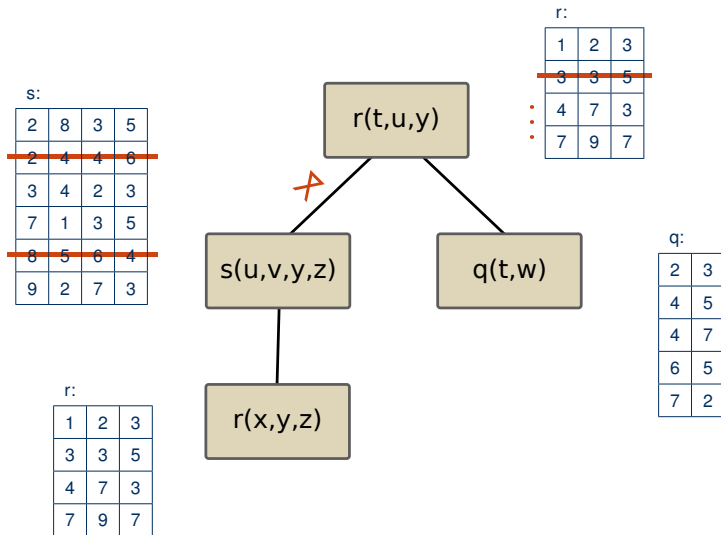
# Yannakakis' Algorithm by Example



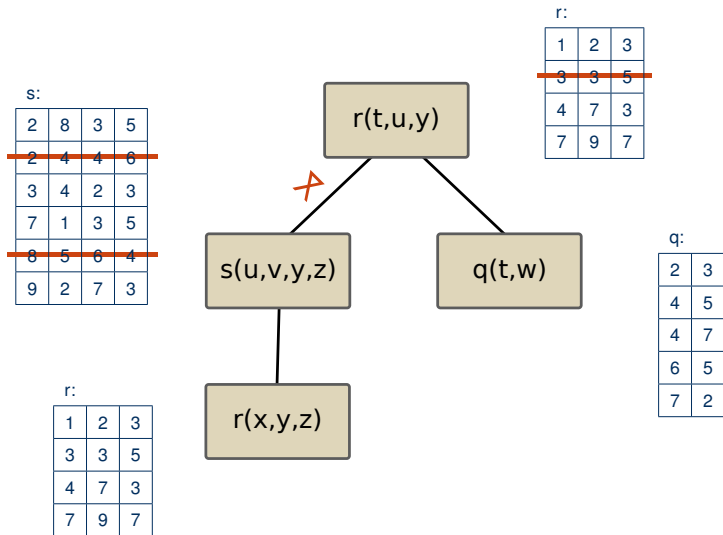
# Yannakakis' Algorithm by Example



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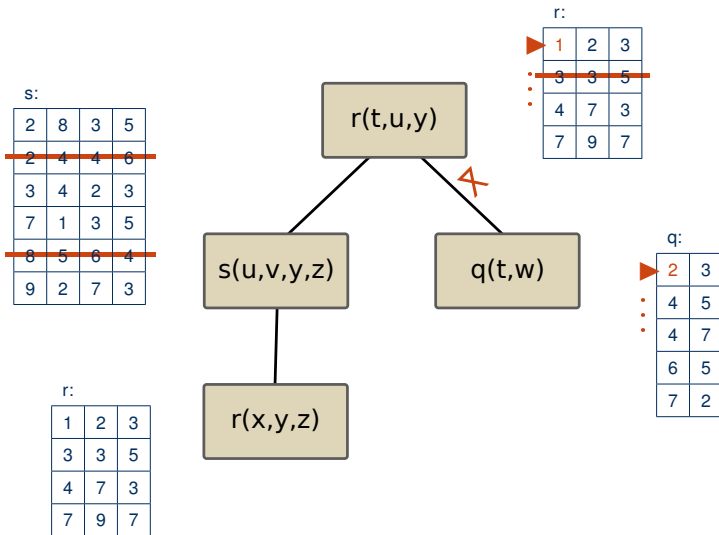


# Yannakakis' Algorithm by Example

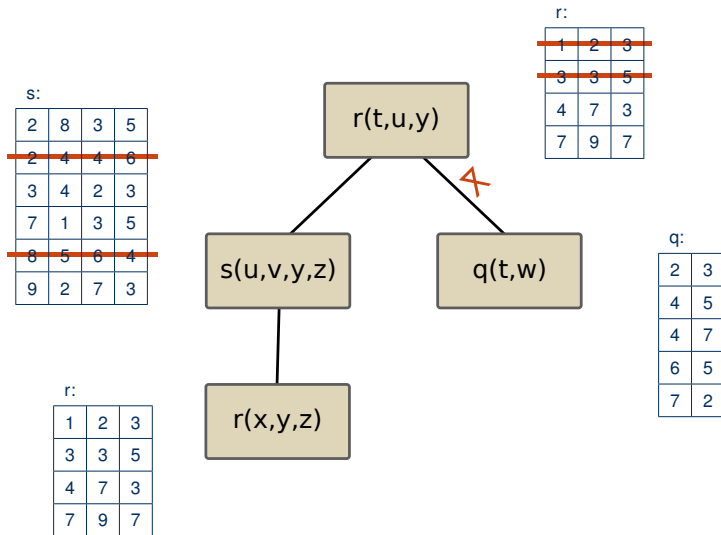




# Yannakakis' Algorithm by Example



# Yannakakis' Algorithm by Example



# Yannakakis' Algorithm: Summary

Polynomial time procedure for answering BCQs

Does not immediately compute answers in the version given here

~> modifications needed

Even tree queries can have exponentially many results,  
but each can be computed (not just checked) in P

~> **output-polynomial** computation of results

# Summary and Outlook

Conjunctive queries (CQs) are an important special case of FO queries

Boolean CQ answering, the homomorphism problem and constraint satisfaction problems are equivalent and NP-complete

CQ answering is simpler, namely in P, when CQs are tree queries

- Check acyclicity with GYO algorithm
- Evaluate query using Yannakakis' Algorithm

## Open questions:

- Tree queries are rather special. Are there more general conditions for good queries?
- What about query optimisation?