Agenda

- Basic Idea of the Tableau Calculus
- Propositional Example
- Transformation into Negation Normal Form
- Satisfiability of $ALC$ Concepts
- Correctness and Termination
- Summary
Agenda

- Basic Idea of the Tableau Calculus
- Propositional Example
- Transformation into Negation Normal Form
- Satisfiability of $\mathcal{ALC}$ Concepts
- Correctness and Termination
- Summary
Automation

- by now: ad hoc arguments about satisfiability of DL axioms
- a concept is satisfiable, if it has a model
  ~ idea: constructive decision procedure that tries to build models
- analog: truth tables in propositional logic
Automation

- by now: ad hoc arguments about satisfiability of DL axioms
- a concept is satisfiable, if it has a model
  -> idea: constructive decision procedure that tries to build models
- analog: truth tables in propositional logic

\[(p \lor q) \rightarrow (\neg p \lor \neg q)\]
Automation

- by now: ad hoc arguments about satisfiability of DL axioms
- a concept is satisfiable, if it has a model
  \[ \sim \text{idea: constructive decision procedure that tries to build models} \]
- analog: truth tables in propositional logic

\[(p \lor q) \rightarrow (\neg p \lor \neg q)\]

negation in front of complex expressions and non-atomic operators difficult to handle, thus reformulate:
Automation

- by now: ad hoc arguments about satisfiability of DL axioms
- a concept is satisfiable, if it has a model
  \[ \sim \text{ idea: constructive decision procedure that tries to build models} \]
- analog: truth tables in propositional logic

\[(p \lor q) \rightarrow (\neg p \lor \neg q)\]

negation in front of complex expressions and non-atomic operators difficult to handle, thus reformulate:

\[\neg(p \lor q) \lor (\neg p \lor \neg q)\]
Automation

- by now: ad hoc arguments about satisfiability of DL axioms
- a concept is satisfiable, if it has a model
  → idea: constructive decision procedure that tries to build models
- analog: truth tables in propositional logic

\[(p \lor q) \rightarrow (\neg p \lor \neg q)\]

negation in front of complex expressions and non-atomic operators difficult to handle, thus reformulate:

\[\neg(p \lor q) \lor (\neg p \lor \neg q)\]

\[\neg(p \land \neg q) \lor (\neg p \lor \neg q)\]
Automation

- by now: ad hoc arguments about satisfiability of DL axioms
- a concept is satisfiable, if it has a model
  → idea: constructive decision procedure that tries to build models
- analog: truth tables in propositional logic

\[(p \lor q) \rightarrow (\neg p \lor \neg q)\]

negation in front of complex expressions and non-atomic operators difficult to handle, thus reformulate:

\(- (p \lor q) \lor (\neg p \lor \neg q)\)

\((\neg p \land \neg q) \lor (\neg p \lor \neg q)\)

\((\neg p \land \neg q) \lor \neg p \lor \neg q\)
Agenda

• Basic Idea of the Tableau Calculus
• Propositional Example
• Transformation into Negation Normal Form
• Satisfiability of $\mathcal{ALC}$ Concepts
• Correctness and Termination
• Summary
Simple Tableau

\[(p \land \lnot q) \lor \lnot p \lor \lnot q\]
Simple Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

- disjunctions lead to branches in the tableau
- tableau: finite set of tableau branches
Simple Tableau

\( (\neg p \land \neg q) \lor \neg p \lor \neg q \)

- \( \neg p \land \neg q \)
- \( \neg p \)
- \( \neg q \)

- disjunctions lead to branches in the tableau
- tableau: finite set of tableau branches
Simple Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

- \(\neg p\)
- \(\neg q\)

- disjunctions lead to branches in the tableau
- tableau: finite set of tableau branches
- compare: truth table

<table>
<thead>
<tr>
<th>(I(p))</th>
<th>(I(q))</th>
<th>(I(\neg p))</th>
<th>(I(\neg q))</th>
<th>(I(p \lor q))</th>
<th>(I(\neg p \lor \neg q))</th>
<th>(I((p \lor q) \rightarrow (\neg p \lor \neg q)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>t</td>
<td>f</td>
<td>t</td>
<td>f</td>
<td>t</td>
<td>f</td>
</tr>
<tr>
<td>t</td>
<td>f</td>
<td>f</td>
<td>t</td>
<td>t</td>
<td>t</td>
<td>t</td>
</tr>
<tr>
<td>f</td>
<td>t</td>
<td>t</td>
<td>f</td>
<td>t</td>
<td>t</td>
<td>t</td>
</tr>
<tr>
<td>f</td>
<td>f</td>
<td>t</td>
<td>t</td>
<td>f</td>
<td>t</td>
<td>t</td>
</tr>
</tbody>
</table>

TU Dresden Deduction Systems
Simple Tableau with Contradiction

$$(\neg p \lor q) \land p \land \neg q$$
Simple Tableau with Contradiction

\[(\neg p \lor q) \land p \land \neg q\]

\[\neg p \lor q\]

\[p\]

\[\neg q\]
Simple Tableau with Contradiction

$(\neg p \lor q) \land p \land \neg q$

$\neg p \lor q$

$p$

$\neg q$

$\neg p$ $q$
Simple Tableau with Contradiction

\[(\neg p \lor q) \land p \land \neg q\]

\[
\neg p \lor q
\]

\[p\]

\[\neg q\]

\[
\neg p
\]

\[q\]
Simple Tableau with Contradiction

\[(\neg p \lor q) \land p \land \neg q\]

\[-p \lor q\]

\[p\]

\[\neg q\]

\[-p\]

\[q\]

\[\bot\]

- if a branch contains an atomic contradiction (clash), we call this branch closed
Simple Tableau with Contradiction

\[(\neg p \lor q) \land p \land \neg q\]

- \(\neg p \lor q\)
- \(p\)
- \(\neg q\)

\(-p\)
\(q\)

\(\bot\)  \(\bot\)

- if a branch contains an atomic contradiction (clash), we call this branch closed
- a tableau is closed, if all its branches are
Simple Tableau with Contradiction

\[(\neg p \lor q) \land p \land \neg q\]

- \(\neg p \lor q\)
  - \(p\)
  - \(\neg q\)

\[\neg p \quad q\]

\[\bot \quad \bot\]

- if a branch contains an atomic contradiction (clash), we call this branch closed
- a tableau is closed, if all its branches are
- a complete tableau without open branches shows the formula’s unsatisfiability
Constructing a Model from the Tableau

\((\neg p \land \neg q) \lor \neg p \lor \neg q\)

- \(\neg p \land \neg q\)
- \(\neg p\)
- \(\neg q\)

- given an open branch, we can construct a model
Constructing a Model from the Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

- given an open branch, we can construct a model
- let I(p)=false and let I(q)=false
Constructing a Model from the Tableau

\[ (\neg p \land \neg q) \lor \neg p \lor \neg q \]

- given an open branch, we can construct a model
- let \( I(p) = \text{false} \) and let \( I(q) = \text{false} \)
- let \( I(p) = \text{false} \) (\( I(q) \) is irrelevant since not in the branch, default assignment \( \text{false} \))
Constructing a Model from the Tableau

\[(\neg p \land \neg q) \lor \neg p \lor \neg q\]

- given an open branch, we can construct a model
- let \(I(p) = \text{false}\) and let \(I(q) = \text{false}\)
- let \(I(p) = \text{false}\) (\(I(q)\) is irrelevant since not in the branch, default assignment false)
- let \(I(q) = \text{false}\) (\(I(p)\) is irrelevant since not in the branch, default assignment false)
Propositional Tableau

- not always exponentially many combinations have to be checked (as opposed to truth table method)
- branches can be built one after the other \(\Rightarrow\) only polynomial space needed
- if we care about satisfiability we can stop after constructing the first complete open branch
Construction with only one Branch in Memory

\[(\neg p \lor q) \land p \land q\]
Construction with only one Branch in Memory

\[(\neg p \lor q) \land p \land q\]

\[\neg p^{1a} \lor q^{1b}\]

\[p\]

\[q\]

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked
Construction with only one Branch in Memory

\[(\neg p \lor q) \land p \land q\]

\[\neg p^{1a} \lor q^{1b}\]

\[p\]

\[q\]

\[\neg p^{1a}\]

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked
Construction with only one Branch in Memory

\[(\neg p \lor q) \land p \land q\]

\[\neg p^{1a} \lor q^{1b}\]

\[p\]

\[q\]

\[\neg p^{1a}\]

\[\bot^{1a}\]

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked
- when encountering a contradiction caused by a choice, remove marked formulae and try next choice
Construction with only one Branch in Memory

\[(\neg p \lor q) \land p \land q\]

\[\neg p^{1a} \lor q^{1b}\]

- when encountering a disjunction we assign so-called choice points
- all extensions of the branch based on such a choice are also marked
- when encountering a contradiction caused by a choice, remove marked formulae and try next choice
From Propositional Tableau to Tableau for DLs

How can the tableaux be extended for checking satisfiability of $\mathcal{ALC}$ concepts? NB: initially, we assume no underlying knowledge base, thus unsatisfiability means that the concept is contradictory “by itself”.

- tableau represents an element of the domain (plus its “environment”)
From Propositional Tableau to Tableau for DLs

How can the tableaux be extended for checking satisfiability of $\mathcal{ALC}$ concepts? NB: initially, we assume no underlying knowledge base, thus unsatisfiability means that the concept is contradictory “by itself”.

- tableau represents an element of the domain (plus its “environment”)
- tableau branch: finite set of propositions of the form $C(a), r(a, b)$
- for existential quantifiers, new domain elements are introduced
- universal quantifiers propagate formulae (=concept expressions) to neighboring elements
Agenda

- Basic Idea of the Tableau Calculus
- Propositional Example
- Transformation into Negation Normal Form
- Satisfiability of $\mathcal{ALC}$ Concepts
- Correctness and Termination
- Summary
Propositional Logic – Some Logical Equivalences

- We aim at negations being present only in front of atomic concepts

$$\varphi \land \psi \equiv \psi \land \varphi$$
$$\varphi \lor \psi \equiv \psi \lor \varphi$$
$$\varphi \rightarrow \psi \equiv \neg \varphi \lor \psi$$
$$\varphi \leftrightarrow \psi \equiv (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$$

$$\varphi \land (\psi \land \omega) \equiv (\varphi \land \psi) \land \omega$$
$$\varphi \lor (\psi \lor \omega) \equiv (\varphi \lor \psi) \lor \omega$$
$$\neg (\varphi \land \psi) \equiv \neg \varphi \lor \neg \psi$$
$$\neg (\varphi \lor \psi) \equiv \neg \varphi \land \neg \psi$$

$$\varphi \land \varphi \equiv \varphi$$
$$\varphi \lor \varphi \equiv \varphi$$
$$\neg \neg \varphi \equiv \varphi$$

$$\varphi \land (\psi \lor \varphi) \equiv \varphi$$
$$\varphi \lor (\psi \land \varphi) \equiv \varphi$$

$$\varphi \lor (\psi \land \omega) \equiv (\varphi \lor \psi) \land (\varphi \lor \omega)$$
$$\varphi \land (\psi \lor \omega) \equiv (\varphi \land \psi) \lor (\varphi \land \omega)$$
Further Logical Equivalences

\[ \neg(C \cap D) \iff \neg C \cup \neg D \]
\[ \neg(D \cup D) \iff \neg C \cap \neg D \]
\[ \neg \neg C \iff C \]
\[ \neg(\forall r. C) \iff \exists r. (\neg C) \]
\[ \neg(\exists r. C) \iff \forall r. (\neg C) \]
\[ \neg(\leq n \cdot s. C) \iff \geq n + 1 \cdot s. C \]
\[ \neg(\geq n \cdot s. C) \iff \leq n - 1 \cdot s. C, \quad n \geq 1 \]
\[ \neg(\geq 0 \cdot s. C) \iff \bot \]

- apply these rules iteratively until none can be applied any more
- \( \iff \) equivalent concept in negation normal form
NNF Transformation

recursive definition of an NNF transformation:

if $C$ atomic:

\[ NNF(C) := C \quad \text{NNF}(\neg C) := \neg C \]

otherwise:

\[ NNF(\neg\neg C) := NNF(C) \]
\[ NNF(C \sqcap D) := NNF(C) \sqcap NNF(D) \quad \text{NNF}(\neg(C \sqcap D)) := NNF(\neg C) \sqcup NNF(\neg D) \]
\[ NNF(C \sqcup D) := NNF(C) \sqcup NNF(D) \quad \text{NNF}(\neg(C \sqcup D)) := NNF(\neg C) \sqcap NNF(\neg D) \]
\[ NNF(\forall r.C) := \forall r.(NNF(C)) \quad \text{NNF}(\neg(\forall r.C)) := \exists r.(NNF(\neg C)) \]
\[ NNF(\exists r.C) := \exists r.(NNF(C)) \quad \text{NNF}(\neg(\exists r.C)) := \forall r.(NNF(\neg C)) \]
\[ NNF(\leq n \ s. C) := \leq n \ s.(NNF(C)) \quad \text{NNF}(\neg(\leq n \ s. C)) := \geq n + 1 \ s.(NNF(C)) \]
\[ NNF(\geq n \ s. C) := \geq n \ s.(NNF(C)) \quad \text{NNF}(\neg(\geq n \ s. C)) := \leq n - 1 \ s.(NNF(C)) \]
\[ \text{if } n \geq 1 \]
\[ NNF(\geq 0 \ s. C) := \top \quad \text{NNF}(\neg(\geq 0 \ s. C)) := \bot \quad \text{otherwise} \]
NNF Transformation – Example

\[
\begin{align*}
\text{NNF}(\neg(\neg C \cap (\neg D \cup E))) \\
= \text{NNF}(\neg\neg C) \cup \text{NNF}(\neg(\neg D \cup E)) \\
= \text{NNF}(C) \cup \text{NNF}(\neg(\neg D \cup E)) \\
= C \cup \text{NNF}(\neg(\neg D \cup E)) \\
= C \cup (\text{NNF}(\neg \neg D) \cap \text{NNF}(\neg E)) \\
= C \cup (\text{NNF}(D) \cap \text{NNF}(\neg E)) \\
= C \cup (D \cap \text{NNF}(\neg E)) \\
= C \cup (D \cap \neg E)
\end{align*}
\]
Agenda

• Basic Idea of the Tableau Calculus
• Propositional Example
• Transformation into Negation Normal Form
• Satisfiability of $\mathcal{ALC}$ Concepts
• Correctness and Termination
• Summary
Tableau for $\mathcal{ALC}$ Concepts

- tableau for a propositional formulal $\alpha$: one element, labeled with subformulae of $\alpha$
- tableau for an $\mathcal{ALC}$ concept $C$: graph (more precisely: tree) where the nodes are labeled with subformulae of $C$
  - root labeled with $C$
  - represents model for $C$ (if complete and clash-free)
  - non-root nodes are enforced by existential quantifiers

Definition

Let $C$ be an $\mathcal{ALC}$ concept, $\text{SF}(C)$ the set of all subformulae of $C$ and $\text{Rol}(C)$ the set of all roles occurring in $C$. A tableau for $C$ is a tree $G = \langle V, E, L \rangle$, with nodes $V$, edges $E \subseteq V \times V$ and a labeling function $L$ with $L: V \rightarrow 2^{\text{SF}(C)}$ and $L: V \times V \rightarrow 2^{\text{Rol}(C)}$. 
Properties of the $\mathcal{ALC}$ Tableau Algorithm

- the algorithm is specified as a set of rules
- every rule breaks down a complex concept into its parts
- rules applicable in any order
- the algorithm is non-deterministic (due to disjunction)
- check for atomic contradictions

Tableau algorithm for checking satisfiability of $\mathcal{ALC}$ concepts

**Input:** an $\mathcal{ALC}$ concept in NNF

**Output:**
- `true` if there is a clash-free tableau where no more rules can be applied
- `false` otherwise (tableau closed)
Tableau Rules for $\mathcal{ALC}$ Concepts

$\Box$-rule: For an arbitrary $v \in V$ mit $C \cap D \in L(v)$ and
$\{C, D\} \not\subseteq L(v)$, let $L(v) := L(v) \cup \{C, D\}$.

$\bigvee$-rule: For an arbitrary $v \in V$ with $C \cup D \in L(v)$ and
$\{C, D\} \cap L(v) = \emptyset$, choose $X \in \{C, D\}$ and let
$L(v) := L(v) \cup \{X\}$.

$\exists$-rule: For an arbitrary $v \in V$ with $\exists r. C \in L(v)$ such that
there is no $r$-successor $v'$ of $v$ with $C \in L(v')$,
let $V = V \cup \{v'\}$, $E = E \cup \{(v, v')\}$, $L(v') := \{C\}$ and
$L(v, v') := \{r\}$ for $v'$ a new node.

$\forall$-rule: For arbitrary $v, v' \in V$, $v'$ $r$-neighbor of $v$,
$\forall r. C \in L(v)$ and $C \not\in L(v')$, let $L(v') := L(v') \cup \{C\}$.

- a node $v'$ is an $r$-neighbor of a node $v$ if $\langle v, v' \rangle \in E$ and $r \in L(v, v')$
Tableau Rules for $\mathcal{ALC}$ Concepts

$\square$-rule: For an arbitrary $v \in V$ mit $C \sqcap D \in L(v)$ and 
$\{C, D\} \not\subseteq L(v)$, let $L(v) := L(v) \cup \{C, D\}$.

$\square$-rule: For an arbitrary $v \in V$ with $C \sqcup D \in L(v)$ and
$\{C, D\} \cap L(v) = \emptyset$, choose $X \in \{C, D\}$ and let
$L(v) := L(v) \cup \{X\}$.

$\exists$-rule: For an arbitrary $v \in V$ with $\exists r.C \in L(v)$ such that
there is no $r$-successor $v'$ of $v$ with $C \in L(v')$,
let $V = V \cup \{v'\}$, $E = E \cup \{(v, v')\}$, $L(v') := \{C\}$ and
$L(v, v') := \{r\}$ for $v'$ a new node.

$\forall$-rule: For arbitrary $v, v' \in V$, $v'$ $r$-neighbor of $v$,
$\forall r.C \in L(v)$ and $C \not\in L(v')$, let $L(v') := L(v') \cup \{C\}$.

- a node $v'$ is an $r$-neighbor of a node $v$ if $\langle v, v' \rangle \in E$ and $r \in L(v, v')$
- rule application order: “don’t care” non-determinism
Tableau Rules for $\mathcal{ALC}$ Concepts

\(\Box\)-rule: For an arbitrary \(v \in V\) with \(C \cap D \in L(v)\) and \(\{C, D\} \not\subseteq L(v)\), let \(L(v) := L(v) \cup \{C, D\}\).

\(\bigcirc\)-rule: For an arbitrary \(v \in V\) with \(C \cup D \in L(v)\) and \(\{C, D\} \cap L(v) = \emptyset\), choose \(X \in \{C, D\}\) and let \(L(v) := L(v) \cup \{X\}\).

\(\exists\)-rule: For an arbitrary \(v \in V\) with \(\exists r.C \in L(v)\) such that there is no \(r\)-successor \(v'\) of \(v\) with \(C \in L(v')\), let \(V = V \cup \{v'\}\), \(E = E \cup \{(v, v')\}\), \(L(v') := \{C\}\) and \(L(v, v') := \{r\}\) for \(v'\) a new node.

\(\forall\)-rule: For arbitrary \(v, v' \in V\), \(v'\) \(r\)-neighbor of \(v\), \(\forall r.C \in L(v)\) and \(C \notin L(v')\), let \(L(v') := L(v') \cup \{C\}\).

- a node \(v'\) is an \(r\)-neighbor of a node \(v\) if \((v, v') \in E\) and \(r \in L(v, v')\)
- rule application order: “don’t care” non-determinism
- choice of disjunction: “don’t know” non-determinism
Tableau Algorithmus Example

\[ C = \exists r. (A \cup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \]

\[ L(u) = \{ C \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \cup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \]

\[ L(u) = \{ C, \exists r. (A \cup \exists r. B), \exists r. \neg A, \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \cup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \]

\[ L(u) = \{ C, \exists r. (A \cup \exists r. B), \exists r. \neg A, \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \} \]

\[ L(v) = \{ A \cup \exists r. B \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \cup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \]

\[
\begin{align*}
L(u) &= \{ C, \exists r. (A \cup \exists r. B), \\
&\quad \exists r. \neg A, \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \} \\
L(v) &= \{ A \cup \exists r. B \} \\
L(w) &= \{ \neg A \}
\end{align*}
\]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[ L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \} \]

\[ L(v) = \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A) \} \]

\[ L(w) = \{ \neg A \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[
L(u) = \{C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A))\}
\]

\[
L(v) = \{A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A)\}
\]

\[
L(w) = \{\neg A, \forall r. (\neg B \sqcup A)\}
\]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[
\begin{align*}
L(u) &= \{ C, \exists r. (A \sqcup \exists r. B), \\
&\quad \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \} \\
L(v) &= \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), A \} \\
L(w) &= \{ \neg A, \forall r. (\neg B \sqcup A) \}
\end{align*}
\]
Tableau Algorithmus Example

$$C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A))$$

$L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \}$

$L(v) = \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), \times \}$

$L(w) = \{ \neg A, \forall r. (\neg B \sqcup A) \}$
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r.B) \land \exists r. \neg A \land \forall r. (\neg A \land \forall r. (\neg B \sqcup A)) \]

\[ L(u) = \{ C, \exists r. (A \sqcup \exists r.B), \exists r. \neg A, \forall r. (\neg A \land \forall r. (\neg B \sqcup A)) \} \]

\[ L(v) = \{ A \sqcup \exists r.B, \neg A, \forall r. (\neg B \sqcup A), \times, \exists r.B \} \]

\[ L(w) = \{ \neg A, \forall r. (\neg B \sqcup A) \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[ L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \} \]

\[ L(v) = \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), \times, \exists r. B \} \]

\[ L(w) = \{ \neg A, \forall r. (\neg B \sqcup A) \} \]

\[ L(x) = \{ B \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \cup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \]

\[ L(u) = \{ C, \exists r. (A \cup \exists r. B), \exists r. \neg A, \forall r. (\neg A \cap \forall r. (\neg B \cup A)) \} \]

\[ L(v) = \{ A \cup \exists r. B, \neg A, \forall r. (\neg B \cup A), \boxed{\times}, \exists r. B \} \]

\[ L(w) = \{ \neg A, \forall r. (\neg B \cup A) \} \]

\[ L(x) = \{ B, \neg B \cup A \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \sqcup A)) \]

\[
\begin{align*}
L(u) &= \{ C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r. (\neg A \cap \forall r. (\neg B \sqcup A)) \} \\
L(v) &= \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), \times, \exists r. B \} \\
L(w) &= \{ \neg A, \forall r. (\neg B \sqcup A) \} \\
L(x) &= \{ B, \neg B \sqcup A, \neg B \}
\end{align*}
\]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \cap \exists r. \neg A \cap \forall r. (\neg A \cap \forall r. (\neg B \sqcup A)) \]

\[
L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \\
\exists r. \neg A, \forall r. (\neg A \cap \forall r. (\neg B \sqcup A)) \} \\
L(v) = \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), \times, \exists r. B \} \\
L(w) = \{ \neg A, \forall r. (\neg B \sqcup A) \} \\
L(x) = \{ B, \neg B \sqcup A, \times B \} \]
Tableau Algorithmus Example

\[ C = \exists r. (A \sqcup \exists r. B) \sqcap \exists r. \neg A \sqcap \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \]

\[ L(u) = \{ C, \exists r. (A \sqcup \exists r. B), \exists r. \neg A, \forall r. (\neg A \sqcap \forall r. (\neg B \sqcup A)) \} \]

\[ L(v) = \{ A \sqcup \exists r. B, \neg A, \forall r. (\neg B \sqcup A), \times, \exists r. B \} \]

\[ L(w) = \{ \neg A, \forall r. (\neg B \sqcup A), \times, \exists r. B \} \]

\[ L(x) = \{ B, \neg B \sqcup A, \times, \exists r. B, \} \]
Tableau Algorithm Example

the model $\mathcal{I}$ constructed by the algorithm is the following:

$$\Delta^\mathcal{I} = \{u, v, w, x\}$$
$$A^\mathcal{I} = \{x\}$$
$$B^\mathcal{I} = \{x\}$$
$$r^\mathcal{I} = \{\langle u, v \rangle, \langle u, w \rangle, \langle v, x \rangle\}$$

Check that indeed $C^\mathcal{I} = \{u\}$, given the defined semantics of $\mathcal{ALC}$
Tableau Algorithm Properties

1. The model is finite: only finitely many elements in the domain.
2. The model is tree-shaped: the tableau is a labeled tree.

The algorithm will always construct finite trees:
- From a clash-free tableau, we can construct a finite model.
- If there is no clash-free tableau, there is no model.
Agenda

- Basic Idea of the Tableau Calculus
- Propositional Example
- Transformation into Negation Normal Form
- Satisfiability of $ALC$ Concepts
- Correctness and Termination
- Summary
Tableau Properties

- the depth (number of nested quantifiers) decreases in every node
- every node is labeled only with subformulae of $C$
- $C$ has only polynomially many subformulae
- if the output is true we can build a model out of the constructed tableau
- on the other hand, we can use a model of a satisfiable concept to construct a clash-free tableau for this concept
Theorem

1. the algorithm terminates for every input
2. if the output is \textit{true}, then the input concept is satisfiable
3. if the input concept is satisfiable, then the output is \textit{true}.
Tableau Algorithm for \textit{ALC} Concepts

**Theorem**

1. The algorithm terminates for every input.
2. If the output is \textit{true}, then the input concept is satisfiable.
3. If the input concept is satisfiable, then the output is \textit{true}.

**Corollary**

Every \textit{ALC} concept $C$ has the following properties:

1. **finite model property**: If $C$ has a model, then it has a finite one.
2. **tree model property**: If $C$ has a model, then it has a tree-shaped one.
Agenda

- Basic Idea of the Tableau Calculus
- Propositional Example
- Transformation into Negation Normal Form
- Satisfiability of $\mathcal{ALC}$ Concepts
- Correctness and Termination
- Summary
Summary

- we now have a constructive method for building model abstractions
- satisfiable $\mathcal{ALC}$ concepts always have a finite model that we can construct
- the algorithm is correct, complete and terminating
- serves as basis for practically implemented algorithms
- next: extension to knowledge bases