**Master Thesis**

on

# The Design of Modal Proof Theories: the case of $S5$

by

Finiki Stouppa

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| Overseeing professor: | Prof. Dr. rer. nat. habil. Steffen Hölldobler |
| Supervisor: | Dr. Charles Stewart |

TECHNISCHE UNIVERITÄT DRESDEN

FAKULTÄT INFORMATIK

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**Abstract**

The sequent calculus does not seem to be capable of supporting cut-admissible formulations for S5. Through a survey on existing cut-admissible systems for this logic, we investigate the solutions proposed to overcome this defect. Accordingly, the systems can be divided into two categories: in those which allow semantic-oriented formulae and those which allow formulae in positions not reachable by the usual systems in the sequent calculus. The first solution is not desirable because it is conceptually impure, that is, these systems express concepts of frame semantics in the language of the logic.

Consequently, we focus on the systems of the second group for which we define notions related to *deep inference* - the ability to apply rules deep inside structures - as well as other desirable properties *good* systems should enjoy. We classify these systems accordingly and examine how these properties are affected in the presence of deep inference. Finally, we present a cut-admissible system for $S5$ in a formalism which makes explicit use of deep inference, the calculus of structures, and give reasons for its effectiveness in providing good modal formulations.

# Contents

# 1 Introduction

Modal logic has been widely applied in computer science as it provides a simple and intuitive language that has proven effective at capturing important concepts treated in such fields as knowledge representation and requirements engineering, especially of concurrent systems, and capturing these concepts in a computationally tractable manner. Moreover, the semantics of the elementary normal modal logic, as given in terms of Kripke frames, are characterized by simplicity and they are in a straightforward correspondence with the Hilbert-style axiomatization of its logics. However, its structural proof theory is more complex and less systematic. As a consequence, proof systems for its logics are provided in a wide variety of calculi. A study on the modal proof theory will benefit the current situation and provide more proof theoretical results. An important result is, for instance, a deeper understanding of the modal proof analysis, with positive consequences on existing decision procedures for modal logic, which are mainly given via tableau methods. Although in general proof theory underlies the design of tableau methods, modal tableau are instead usually guided by the frame semantics.

Many systems have been developed for the purpose of modal proof theory: systems in the sequent calculus, extended systems and systems in other sequential calculi. A presentation of most of these can be found in Wansing [27]. Their main concern is to provide formulations for some modal logics in a general enough framework, capable to handle a substantial number of logics. Since cut admissibility is essential for good proof theory and the sequent calculus fails to give cut-admissible formulations for two important modal logics, S5 and B, we concentrate in the cut-admissible systems so far proposed for those logics[1]. These are:

- eight systems for $S5$ and $B$ in calculi that allow indices or labels on formulae presented by Kanger [16], Mints [19], Orlowska [22], Simpson [26] and Braüner [6],
- one system for $S5$ in the higher arity sequent calculus presented by Sato [24],
- two systems for $S5$ and $B$ in the multiple sequent calculus presented by Indrzejczak [14, 15],
- one system for $S5$ in hypersequents presented by Avron [1] and two systems in calculi essentially analogous to hypersequents presented by Mints [20] and Pottinger [23],
- two systems for $S5$ and $B$ in display logic presented by Wansing [27] and
- two systems for $S5$ presented by Braüner [5] and Sato [25] which, though they are given in a sequent-style, they violate essential properties systems in the sequent calculus should enjoy. For this reason we call them extended sequent systems.

Observing how these systems allow cut-free proofs for the theorems of $S5$, we notice that nine of them use additional information on formulae other than their syntactic one. This information is related to Kripke-frame semantics - the semantics applied on normal modal logic - and denotes in which state (of a frame) the given formula holds. Such formulations

[1] The lack of a cut-admissible formulation in the literature does not prove that one could not be devised.

are obtained by extending the language of modal logic, and so they are not conceptually pure. In this thesis we are interested only in the properties of systems with conceptual purity, so that the above systems are not presented here.

The rest of the systems either imitate or enjoy implicitly deep inference, the ability of applying inference rules deep inside structures. The systems in the higher arity sequent calculus, multiple sequent calculus and hypersequents imitate deep inference by allowing deep rule applications at limited depth. On the other hand, the systems in display logic enjoy deep inference since deep applications are possible at any depth. Still, deep inference remains implicit in the systems, because of the shallow nature of their rules. However, this is not a real limitation of the calculus; display logic does not only give cut-admissible formulations for all logics of normal modal logic, but it also provides a technique for formulating rules out of the modal Hilbert-style axioms. This systematic formulation of the axiom rules (a property we call *systematicity*) seems to be strongly related to deep inference [2].

The calculus of structures is a generalization of the sequent calculus which makes explicit use of deep inference. It is explicit because its rules can be directly applied in substructures at any depth. This is denoted by the $S\{-\}$, which surrounds the premise and conclusion of a rule. Among the cut-admissible modal systems developed in this framework are systems for logics $K$, $K4$, $D$, $D4$, $M$, $S4$ and $S5$. All of them enjoy systematicity and, contrary to display logic, no special technique is needed for this.

The main achievements of this thesis are firstly a survey of systems and calculi for the proof theory of modal logic, with particular emphasis on the conceptually pure calculi. Second, an investigation of the properties responsible for providing *good* systems for modal logic. Third, a presentation of such a good system for logic $S5$ in the calculus of structures. In a descending order of importance, the main desiderata that we apply to determine which are the good systems for modal logic are: cut-admissibility, conceptual purity, generality of the calculus and systematicity for the modal rules. Deep inference seems to be unavoidable in achieving all of them. Moreover, its direct application results in systems with simple design and other desirable properties, such as locality. A system is said to be local when all of its rule applications require inspections only on formulae of bounded length. Although the system presented here is cut-admissible, the inclusion of restricted cuts results in a better upper bound on the length of the proofs. Whereas in general restricted cuts are defined over the set of subformulae of a given formula, locality ensures that such cuts are restricted to atoms.

The rest of the thesis is organized as follows: in Section 2 we give a short overview on normal modal logic and its systems in the sequent calculus. In Section 3 we present the systems for $S5$ (and $B$), give the definition of deep inference and its contextuals, and classify the systems according to some desirable properties. In Section 4 we introduce the calculus of structures and present a cut-admissible system for $S5$. Finally, Section 5 is dedicated in conclusions and work that need to be developed.

[2]This should not be a surprise considering that Hilbert-style systems for normal modal logic also enjoy deep inference.

# 2 Preliminaries

## 2.1 The Proof Theory of Sequential Calculi

Proof theory concerns the development and study of formal systems, called *calculi*, which are suitable for axiomatizing various logics. Its considerations include general properties on the *design* of the calculi, as well as properties on their *proofs*.

A *logic* $L$ consists of a set of well formed formulae $F$ and a consequence relation $\vdash$ over $F$ which assigns a truth value to each formula in $F$. A formula $\alpha$ which is true in logic $L$ is said to be a *theorem* of $L$ and is denoted as $\vdash_L \alpha$. The set of all theorems of $L$ is its *axiomatization.* A calculus provide mechanisms for generating the theorems of various logics. These include (i) a set of *axioms*, which are some basic theorems of the logic in consideration, and (ii) a set of *inference rules* which allow us to conclude a theorem under the assumption that some other theorems hold. Then, the set of all theorems is obtained by recursive rule applications on existent theorems. We call the set of axioms and inference rules that axiomatizes logic $L$ in calculus $C$, a *system for $L$ in $C$*.

A first division on calculi concerns their relevance in providing automating deductions for their theorems. A calculus is therefore called *proof search calculus* when it provides mechanisms for proving theorems deductively. This means that given a theorem of a certain logic, the construction of a proof for it is obtained by reasoning backwards: the theorem is provable if so are some other theorems. The most well-known proof search calculi are the calculus of *Natural Deduction* and the *Sequent Calculus.* In this thesis we are not going to present natural deduction. The sequent calculus is presented as formulated by Gentzen [10], and we refer to its systems as *Gentzen systems.* Moreover, other *sequential* calculi that have been developed based on its general ideas will also be presented.

Deductive calculi provide a meta-language applied on top of the language of a given logic, which allows inputs of a certain form, called *structures.* More specifically, in sequential calculi structures are built up from a set of connectives, called *structural connectives*, that are usually combined as logical connectives but are applied on formulae rather than on propositional variables. An inference rule takes the general form

$$\frac{P_1 \ \dots \ P_n}{C} \, (R)$$

where $R$ is the name of the rule, $n \geq 0$ and, $P_1, \ \dots \ , P_n$ and $C$ are schematic letters for structures called *premises* and *conclusion*, respectively. When the premise is empty, i.e. $n = 0$, the conclusion is an *axiom* of the system.

A rule application is an instance of a rule $R$ if all its premises and its conclusion match the corresponding ones of $R$. A finite sequent of rule applications is called a *derivation.* The premises of its first rule application are the *premises of the derivation.* Similarly, the conclusion of its last rule is the *conclusion of the derivation.* A *proof* is a derivation which starts with an axiom instance. The conclusion of a proof is called a *theorem.*

### The Sequent Calculus

Structures in the sequent calculus are called *Gentzen structures* and are given by the following simple definition:

**Definition 1** *Gentzen structures are recursively defined as follows:*

1. *Any formula $\alpha$ is a structure.*

2. *If $\Gamma_1$ and $\Gamma_2$ are structures then so is $\Gamma_1, \Gamma_2$.*

Structures are denoted with the letters $\Gamma, \Delta, \Gamma_1, \Delta_1 \dots$ and are closed under associativity and commutativity, so that, the comma can be seen as a separator of the elements of a (multi)set. For instance, $p \vee q, \neg q, p$ is a valid Gentzen structure. Structures are then build in *sequents*, which are of the form $\Gamma \vdash \Delta$, where $\Gamma$ and $\Delta$ are the *antecedent* and *succedent* of the sequent, respectively.

Gentzen systems are organized in a very clear, structural way. Each rule has two versions, the left and the right, one for each side of the turnstile. Moreover, there are two types of rules: the *logical* and the *structural* rules. The logical rules cover rules that introduce a logical connective to their conclusions. In each logical connective corresponds precisely a pair of rules (the left and right ones). In the structural rules belong those rules which manipulate structures only on their structural level (i.e. without changing any formulae).

The fundamental justification for Gentzen systems is the possibility of eliminating the cut-rule, which means that there is a procedure for eliminating cut-rule applications, whenever they occur in a proof. The cut-rule has the form

$$\frac{\Gamma_1 \vdash \Delta_1, \alpha \qquad \alpha, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,(\text{cut})$$

where $\alpha$ is the *cut-formula*. Reading the rule top-down in a computational context, the rule ensures the composition of two programs for which an output of the one is used as input to the other. In a mathematical context, while reading the rule bottom-up, it expresses the decomposition of a problem to smaller ones with the help of lemmas.

The absence of the cut-rule from a system is important both for implementions and theoretical reasons. Firstly, it ensures that for every conclusion, there are only finite choices of possible premises. This is expressed through the subformula property, which says that every formula in the premises of a rule is a subformula of the formulae in its conclusion. Secondly, it is good for proof analysis, as it allows properties on the structure of proofs to be exhibited. Moreover, it ensures that the meaning of the cut-rule is virtual: any lemmas used in a proof are real ones and can be ommited.

The system *LK* for propositional logic in the sequent calculus is presented in Figure 1. It is a representative system of *good* design and its calculus accomplishes all the properties that allow a calculus to be a *good* calculus. A list of properties have been presented by Wansing [27], Avron [1] and Indrzejczak [15]. Some of the most important ones follow.

### Good Proof Theoretical Properties

At a first level, the three most desirable properties a calculus and its systems should enjoy we consider to be the following:

**Cut-admissibility:** The cut-rule should either not be a rule of the systems - in this case we say that the system is cut-free - or the systems should be accompanied with a proof of cut-elimination.

$$\frac{}{\alpha \vdash \alpha}\,(Axiom) \qquad \frac{\Gamma_1 \vdash \Delta_1, \alpha \qquad \alpha, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,(Cut)$$

$$\frac{\Gamma \vdash \Delta}{\alpha, \Gamma \vdash \Delta}\,(W_l) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \alpha}\,(W_r)$$

$$\frac{\alpha, \alpha, \Gamma \vdash \Delta}{\alpha, \Gamma \vdash \Delta}\,(C_l) \qquad \frac{\Gamma \vdash \Delta, \alpha, \alpha}{\Gamma \vdash \Delta, \alpha}\,(C_r)$$

$$\frac{\alpha, \Gamma \vdash \Delta}{\alpha \wedge \beta, \Gamma \vdash \Delta}\,(\wedge \vdash) \qquad \frac{\beta, \Gamma \vdash \Delta}{\alpha \wedge \beta, \Gamma \vdash \Delta}\,(\wedge \vdash) \qquad \frac{\Gamma \vdash \Delta, \alpha \qquad \Gamma \vdash \Delta, \beta}{\Gamma \vdash \Delta, \alpha \wedge \beta}\,(\vdash \wedge)$$

$$\frac{\alpha, \Gamma \vdash \Delta \qquad \beta, \Gamma \vdash \Delta}{\alpha \vee \beta, \Gamma \vdash \Delta}\,(\vee \vdash) \qquad \frac{\Gamma \vdash \Delta, \alpha}{\Gamma \vdash \Delta, \alpha \vee \beta}\,(\vdash \vee) \qquad \frac{\Gamma \vdash \Delta, \beta}{\Gamma \vdash \Delta, \alpha \vee \beta}\,(\vdash \vee)$$

$$\frac{\alpha, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg\alpha}\,(\neg \vdash) \qquad \frac{\Gamma \vdash \Delta, \alpha}{\neg\alpha, \Gamma \vdash \Delta}\,(\vdash \neg)$$

$$\frac{\Gamma_1 \vdash \Delta_1, \alpha \qquad \beta, \Gamma_2 \vdash \Delta_2}{\alpha \supset \beta, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,(\supset\vdash) \qquad \frac{\alpha, \Gamma \vdash \Delta, \beta}{\Gamma \vdash \Delta, \alpha \supset \beta}\,(\vdash\supset)$$

Figure 1: System LK: a system for propositional logic

**Conceptual purity:** The systems should use only concepts provided by the language of a logic and not by its semantics.

**Generality:** The calculus should be general enough to cover systems for a wide variety of logics.

There are three more specific properties we would like to introduce. The first one is known as *Došen's principle* and has only sequential orientation. Došen's principle says that in a calculus, different systems should share the same logical rules and differ only on their structural rules. This principle has been the subject of discussion in philosophical logic, related to the meaning of the logical connectives which is, however, out of the scope of this thesis. The second property concerns the structure of the logical rules and it will be particularly useful later, in the modal systems:

**Definition 2** *A logical rule is said to be* ***focussed*** *when all its modifications are about a single formula in its conclusion, the primitive formula. Otherwise the rule is* ***unfocussed****.*

Moreover, a focussed rule is said to be *strongly* focussed when no logical connectives in other formulae (the side-formulae) are exhibited. Otherwise it is *weakly* focussed. For example, the rule

$$\frac{\Gamma \vdash \Delta, \alpha \qquad \Gamma \vdash \Delta, \beta}{\Gamma \vdash \Delta, \alpha \wedge \beta} \ (\vdash \wedge)$$

is strongly focussed, whereas the rules

$$\frac{\beta, \neg\alpha, \Gamma \vdash \Delta}{\neg\alpha, \Gamma \vdash \Delta, \neg\beta} \qquad \text{and} \qquad \frac{\beta, \Gamma \vdash \Delta, \alpha}{\neg\alpha, \Gamma \vdash \Delta, \neg\beta}$$

are weakly focussed and unfocussed, respectively.

The above definition is also expanded to systems. A system with all its logical rules being focussed is a *focussed* system. When additionally all the rules are strongly focussed, the system is a *strongly focussed* one, otherwise it is *weakly focussed.* Moreover, when there is at least one unfocussed rule, the system is an *unfocussed* one.

Finally, a property which strengthens the importance of cut admissibility is that of *analyticity.* A system is analytic when it is cut-admissible and none of its rules includes hidden cuts. For example, the rule

$$\frac{\Gamma_1 \vdash \Delta_1, \alpha, \beta \qquad \beta, \Gamma_2 \vdash \Delta_2, \alpha \qquad \alpha, \beta, \Gamma_3 \vdash \Delta_3}{\alpha \supset \beta, \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3} \ (\supset\vdash)$$

is not analytic as it is derivable in $LK$ using the rules of cut, contraction and left implication. Analiticity ensures that the system is suitable for proof analysis.

## 2.2 Normal modal logic

Normal modal logic is modal propositional logic with semantics that can be given in terms of *Kripke frames.* Modal propositional logic extends propositional logic with a unary connective, the *modality*, denoted as $\Box$ (box). The meaning often assigned to $\Box$ is that of *necessity*: for $P$ a modal formula, $\Box P$ is read as 'it is *necessary* that $P$' (or '$P$

*must* hold') and its negation as 'it is *possible* that not $P$' (or '$P$ *may* not hold'). This is why modal logic is often also called the *logic of necessity and possibility.* Accordingly, a variety of axioms can be formulated. For instance, axiom $T : \Box P \supset P$ expresses the desirable property 'whatever is necessary does actually hold'.

However, the term modal logic is used more widely and covers also logics with variant interpretations of the modality. In *epistemic logic*, for example, the modality is used for reasoning about *knowledge*. Here, $\Box P$ is instead read as 'the agent *knows* that $P$ holds' and its negation as 'the agent *believes* that not $P$ holds'. In this logic, axiom $T$ plays a vital role, since it expresses the consistency of the agent: if the agent knows that $P$ holds, then $P$ does actually hold. Although epistemic and other logics are of great importance in computer science, they will not be considered here, since they are not related to our topic. Formally, the language of modal logic is given by the following grammar:

**Definition 3** *Well formed formulae (wff) in modal propositional logic are generated as follows:*

$$P ::= \top \mid \bot \mid A \mid \neg P \mid P \wedge P \mid P \vee P \mid P \supset P \mid P \leftrightarrow P \mid \ \Box P \mid \ \diamond P$$

The set of all wff is denoted as $mF$. We use the schematic letters $A, B, \ldots$ for propositional variables and $P, Q, \ldots$ for any $mF$. The $\diamond$ (diamond) connective is again a modality related to the negation of the $\Box$ and is associated with the notion of *possibility.*

### The axiomatization

We start with the axiomatization of normal modal logic as given in the Hilbert-style calculus. We write $\vdash P$ to denote that P is a *theorem* of normal modal logic.

**Definition 4** *Normal modal logic is modal propositional logic which includes all the theorems generated as follows:*

**PC** *Tautologies of classical propositional logic are theorems of normal modal logic.*

**K** *Axiom* $K : \Box(P \supset Q) \supset (\Box P \supset \Box Q)$ *is a theorem of normal modal logic.*

**Def M** *The definition of diamond:* $\diamond P \leftrightarrow \neg \Box \neg P$ *is a theorem of normal modal logic.*

**US** *All theorems are closed under uniform substitution: if* $\vdash P$ *and* $P_1 \ldots P_n$ *variables occuring in* $P$*, then, for* $Q_1 \ldots Q_n \in mF$ , $\vdash P[P_1/Q_1 \ldots P_n/Q_n]$ *is again a theorem of normal modal logic.*

**N** *The rule of necessitation: If* $\vdash P$ *then* $\vdash \Box P$.

**MP** *The rule of Modus Ponens: If* $\vdash P$ *and* $\vdash P \supset Q$ *then* $\vdash Q$.

Thus, normal modal logic is a class of logics which contain the theorems generated by the above definition. The minimal logic is $K$ and contains exactly those theorems. We write $\vdash_K P$ to say that P is a theorem of $K$. Other logics are obtained by adding more axioms - and thus more theorems - to logic $K$. For example, logic $M$ is formulated by adding to $K$ the axiom $T : \Box P \supset P$.

Let us illustrate how theorems are generated by two simple though useful derived rules which will be later used:

1. **DR1** [3] If $\vdash_K P \supset Q$ then $\vdash_K \Box P \supset \Box Q$

   Proof. Applying the (N) rule in the given $\vdash_K P \supset Q$ yields $\vdash_K \Box(P \supset Q)$. Then, by (MP) to the latter and axiom $K$ (together with (US)): $\Box(P \supset Q) \supset (\Box P \supset \Box Q)$ we get $\vdash_K \Box P \supset \Box Q$, as required.

2. **DRt** If $\vdash_K P \supset Q$ then $\vdash_K \neg Q \supset \neg P$.

   Proof. The required result is easily obtained by (MP) application to the propositional tautology $\vdash_K (P \supset Q) \supset (\neg Q \supset \neg P)$ and the given $\vdash_K P \supset Q$.

Also, a property of normal modal logic which will be proved crucial later on, is given by the following lemma:

**Lemma 1** *Given $P$, $Q$ formulae in $mF$, if $\vdash P \supset Q$ is derivable in normal modal logic, then so is $\vdash F\{P\} \supset F\{Q\}$, where $F\{P\}$ is any formula which contains $P$ outside the scope of a negation, and $F\{Q\}$ is obtained from $F\{P\}$ by replacing its subformula $P$ with $Q$.*

*Proof.* Let $\vdash P \supset Q$ derivable in $K$. Then, by induction on the structure of $F$, we prove the lemma for logic $K$ and so for all its extensions:
Base Step. If $F$ is exactly $P$, then $\vdash_K P \supset Q$ holds by the assumption.
Induction Step. By induction hypothesis, given a formula $F'$, $\vdash_K F'\{P\} \supset F'\{Q\}$ (1) is derivable. There are 4 cases:

1. $F = \Box F'\{P\}$. Applying (DR1) to (1) yields $\vdash_K \Box F'\{P\} \supset \Box F'\{Q\}$ as required.

2. $F = \diamond F'\{P\}$. Applying (DRt) to (1) following by (DR1) yields $\vdash_K \Box\neg F'\{Q\} \supset \Box\neg F'\{P\}$ by which with (DRt) and (Def M) we get $\vdash_K \diamond F'\{P\} \supset \diamond F'\{Q\}$ as required.

3. $F = F'\{P\} \wedge Y$. Applying (MP) to (1) and the classical tautology $\vdash_K (F'\{P\} \supset F'\{Q\}) \supset ((F'\{P\} \wedge Y) \supset (F'\{Q\} \wedge Y))$, gives $\vdash_K (F'\{P\} \wedge Y) \supset (F'\{Q\} \wedge Y)$ as required.

4. $F = F'\{P\} \vee Y$. Similar to conjunction. □

Note that theorem (Def M) is an equivalence relation and so closed under the replacement theorem: whenever $\neg \Box \neg$ occurs in any part of a formula, it can be replaced with a $\diamond$ and vice versa. Some immediate consequences of (Def M) are the theorems:

$\vdash \Box P \leftrightarrow \neg \diamond \neg P$
$\vdash \neg \diamond P \leftrightarrow \Box \neg P$

[3] Its name is due to Hughes and Cresswell [13]

| | |
|---|---|
| **(DR1)** If $\vdash P \supset Q$ then $\vdash \Box P \supset \Box Q$ | **(PCt)** $\vdash (P \supset Q) \supset ((Q \supset R) \supset (P \supset R))$ |
| **(DRt)** If $\vdash P \supset Q$ then $\vdash \neg Q \supset \neg P$ | **(PCi)** $\vdash (P \supset Q) \supset ((Q \supset P) \supset (P \leftrightarrow Q))$ |
| **(Lem)** If $\vdash P \supset Q$ then $\vdash H\{P\} \supset H\{Q\}$ | **(PCe)** $\vdash (P \leftrightarrow Q) \supset ((Q \leftrightarrow R) \supset (P \leftrightarrow R))$ |

Figure 2: Some derived rules and propositional tautologies

$\vdash \neg \Box P \leftrightarrow \Diamond \neg P$

The last two we call *modal dualities.*

From now on, rules (DR1) and (DRt) will be used as normal rules and their proofs will be omited. The rules are summarized in Figure 2 together with some useful propositional tautologies and Lemma 1.

### The semantics

As mentioned above, the semantics of normal modal logic is usually given in terms of *Kripke frames.*

**Definition 5** *A **frame** is a pair $\langle W, R\rangle$, where $W$ is a non-empty set of states (also known as worlds) and $R \subseteq W \times W$ is a binary relation over the states.*

Intuitively, for $s, s' \in W$ , $sRs'$ denotes that state $s'$ is accessible from state $s$.

**Definition 6** *A **model** is a tuple $\langle W, R, V\rangle$, where $\langle W, R\rangle$ is a frame and $V$ is a binary relation over propositional variables and states. The relation $V$ is the valuation of every variable $A$ in a specific state $s$: $V(A, s) = 1$ or $V(A, s) = 0$, and is recursively extended on modal formulae as follows:*

1. $V(\top, s) = 1$ *and* $V(\bot, s) = 0$, *for every state.*
2. $V(\neg P, s) = 1$ *iff* $V(P, s) = 0$.
3. $V(P \wedge Q, s) = 1$ *iff* $V(P, s) = 1$ *and* $V(Q, s) = 1$.
4. $V(P \vee Q, s) = 1$ *iff* $V(P, s) = 1$ *or* $V(Q, s) = 1$.
5. $V(P \supset Q, s) = 1$ *iff* $V(P, s) = 0$ *or* $V(Q, s) = 1$.
6. $V(P \leftrightarrow Q, s) = 1$ *iff* $V(P, s) = V(Q, s)$.
7. $V(\Box P, s) = 1$ *iff for all* $s'$ *such that* $sRs'$, $V(P, s') = 1$.
8. $V(\Diamond P, s) = 1$ *iff there is a* $s'$ *such that* $sRs'$ *and* $V(P, s') = 1$.

**Example**

Take the frame $F = \langle W, R \rangle$ with $W = \{s_1, s_2, s_3\}$ and $R = \{(s_1, s_2), (s_1, s_3), (s_2, s_2)\}$ and a model $M = \langle F, V \rangle$ based on it with only positive valuations the $V(A, s_2)$ and $V(B, s_3)$. Then, by Definition 6 the following valuations holds:

| In $s_1$ | In $s_2$ | In $s_3$ |
|---|---|---|
| $V(\neg A, s_1) = 1$ | $V(\Box A, s_2) = 1$ | $V(\Box A, s_3) = 1$ |
| $V(\Diamond A, s_1) = 1$ | $V(\Diamond A, s_2) = 1$ | $V(\Diamond A, s_3) = 0$ |

Satisfiability and validity of a formula $P$ with respect to a frame $F$ or a class of frames $\mathcal{F}$ is defined as follows:

**Definition 7**

1. *Given a frame $\langle W, R \rangle$ and a formula $P$*
   *(i) $P$ is* satisfiable *iff there is a model $\langle W, R, V \rangle$ and a state $s \in W$, with $V(P, s) = 1$ and*
   *(ii) $P$ is* valid *iff for every model $\langle W, R, V \rangle$ and every state $s \in W$, $V(P, s) = 1$.*

2. *Given a class of frames $\mathcal{F}$ and a formula $P$*
   *(i) $P$ is* satisfiable *iff it is satisfiable in all frames $\langle W, R \rangle \in \mathcal{F}$ and*
   *(ii) $P$ is* valid *iff it is valid in all frames $\langle W, R \rangle \in \mathcal{F}$.*

All theorems of normal modal logic are satisfiable with respect to the class of all frames, denoted by $\mathcal{K}$, and all theorems of logic $K$ are additionally valid w.r.t $\mathcal{K}$. Validity for other logics is obtained by restricting the class of frames to include only frames which satisfy certain conditions on their accessibility relation $R$. For instance, logic $M$ is sound and complete with respect to the class of reflexive frames, that is frames $\langle W, R \rangle$ which satisfy the following condition: $\forall s.\ sRs$, with $s \in W$. In this sense, every logic corresponds to a certain class of frames.

### The cube

A semantic-based subclass of normal modal logic includes those logics for which their frame accessibility relation can be given by a set of *geometric sequents*, called a *geometric theory*. A geometric sequent is a first order sequent of the form:

$$\forall \overline{x}.\phi \supset \psi$$

where $\phi$ and $\psi$ are first-order formulae built up from atomic formulae using only the connectives in $\{\bot, \vee, \wedge, \exists\}$ (for a complete definition see Simpson [26]). Such logics we call *geometric modal logics*.

All the logics we are going to deal with are geometric ones. As an example of a non geometric logic, consider the logic which extends $K$ with the Löb axiom: $\Box(\Box p \supset p) \supset \Box p$. This logic is complete with respect to the class of frames with transitive and finite-path[4] relation, and is not definable in first order logic (the proof for it can be found in

[4]The finite-path property says that all paths starting from any state are finite.

textbook [3]). We can further restrict geometric modal logics into those with axioms (other than the $K$) of the form:

$$G: \ \diamond^h \Box^i P \supset \Box^j \diamond^k P$$

with $h, i, j, k \geq 0$, called *Scott-Lemmon logics*. An axiom which do not follow the $G$ pattern, though its frame condition is geometric, is, for instance, axiom $\Box M: \ \Box(\Box P \supset P)$ with frame condition $\forall wv.(wRv \supset vRv)$. All the axioms of the logics we are interested in match the general axiom $G$. These logics are obtained by adding to logic $K$ a number of the axioms in $\{D, T, 4, 5, B\}$, which are presented in Figure 3. All the logics obtained in this way can be then displayed in a complete lattice, known as *the cube* of normal modal logic. In this cube, a connection from logic $L_1$ to logic $L_2$ (with $L_1$ under or on the left-hand side of $L_2$) denotes that $L_1$ is subsumed by $L_2$, in other words all theorems of $L_1$ are also theorems of $L_2$. Here we list the main logics of the cube:

$$\begin{aligned} \text{Logic } D &: \text{extends logic } K \text{ with axiom } D \\ \text{Logic } M &: \text{extends logic } K \text{ with axiom } T \\ \text{Logic } S4 &: \text{extends logic } M \text{ with axiom } 4 \\ \text{Logic } S5 &: \text{extends logic } M \text{ with axiom } 5 \\ \text{Logic } B &: \text{extends logic } M \text{ with axiom } B \end{aligned}$$

The names of the rest of the logics are concatenations of the names of their axioms. For instance, logic $K45$ is the result of extending logic $K$ with the axioms 4 and 5. The weakest logic is logic $K$ and the strongest one is logic $S5$.

An important result in modal logic is the *Scott-Lemmon theorem* [18]) which states the correspondence between axioms of the $G$ form and the condition on frames.

**Theorem 1** *(Lemmon and Scott, 1977) The accessibility relation $R$ of a logic which extends $K$ with an axiom of the $G$ form is given:*

$$\text{(hijk-Convergence) } \forall wvu.((wR^h v \wedge wR^j u) \supset \exists x.(vR^i x \wedge uR^k x))$$

*where $R^n$ is the result of the composition of $R$ with itself $n$-times, $R^0 beingtheidentity$.*

Note that the sequent is a basic geometric one. Then, the frame condition of a Scott-Lemmon logic is the set of frame conditions that correspond to its axioms. For instance, logic $S4$ corresponds to the transitive, reflexive frames. The correspondence between frame conditions and axioms is shown in Figure 3.

### 2.3 Modal Systems in the Sequent Calculus

When dealing with modal systems we are additionally interested in two more properties[5]:

**Systematicity:** There is a clear technique for formulating the modal rules out of the modal axioms.

**Modularity:** Each axiom is captured by a finite number of rules.

[5] As defined by Wansing [27].

| Axioms | | Frame Condition | |
|---|---|---|---|
| $D:$ | $\Box P \supset \Diamond P$ | $\forall x.\exists y.xRy$ | $(seriality)$ |
| $T:$ | $\Box P \supset P$ | $\forall x.xRx$ | $(reflexivity)$ |
| $4:$ | $\Box P \supset \Box\Box P$ | $\forall xyz.(xRy \wedge yRz \supset xRz)$ | $(transitivity)$ |
| $5:$ | $\Diamond P \supset \Box\Diamond P$ | $\forall xyz.(xRy \wedge xRz \supset yRz)$ | $(Euclideanness)$ |
| $B:$ | $P \supset \Box\Diamond P$ | $\forall xy.(xRy \supset yRx)$ | $(symmetry)$ |

Figure 3: Axioms and their frame conditions

Some of the modal systems in the sequent calculus are neither systematic nor modular. Systems for logics $K$, $D$, $M$ and $S4$ have been presented by Ohnishi and Matsumoto [21] and are obtained by adding to the system for propositional logic $LK$ a combination of the following pairs of logical rules for the modalities (the rules are generalized versions of the original ones):

$$(1) \qquad \frac{\alpha, \Gamma \vdash \Delta}{\Diamond\alpha, \Box\Gamma \vdash \Diamond\Delta}(\Diamond \vdash) \qquad \frac{\Gamma \vdash \Delta, \alpha}{\Box\Gamma \vdash \Diamond\Delta, \Box\alpha}(\vdash \Box)$$

$$(2) \qquad \frac{\alpha, \Box\Gamma \vdash \Diamond\Delta}{\Diamond\alpha, \Box\Gamma \vdash \Diamond\Delta}(\Diamond \vdash) \qquad \frac{\Box\Gamma \vdash \Diamond\Delta, \alpha}{\Box\Gamma \vdash \Diamond\Delta, \Box\alpha}(\vdash \Box)$$

$$(3) \qquad \frac{\alpha, \Gamma \vdash \Delta}{\Box\alpha, \Gamma \vdash \Delta}(\Box \vdash) \qquad \frac{\Gamma \vdash \Delta, \alpha}{\Gamma \vdash \Delta, \Diamond\alpha}(\vdash \Diamond)$$

$$(4) \qquad \frac{\alpha, \Gamma \vdash \Delta}{\Box\alpha, \Box\Gamma \vdash \Diamond\Delta}(\Box \vdash) \qquad \frac{\Gamma \vdash \Delta, \alpha}{\Box\Gamma \vdash \Diamond\Delta, \Diamond\alpha}(\vdash \Diamond)$$

The notations $\Box\Gamma$ and $\Diamond\Gamma$ are abbreviations for the sets $\{\Box\alpha \mid \alpha \in \Gamma\}$ and $\{\Diamond\alpha \mid \alpha \in \Gamma\}$, respectively. The systems extend system $LK$ with the above rules as follows:

System for $K$ : the rules in (1)
System for $D$ : the rules in (1) and (4)
System for $M$ : the rules in (1) and (3)
System for $S4$ : the rules in (2) and (3)

Cut-elimination holds for all the systems. However, adding these systems to the sequent calculus causes failure of Došen's principle: the logical rules for the modalities vary depending on the logic. Moreover, as it has been already mentioned, some of these systems are not modular. The restriction of Gentzen systems on having at most a pair of rules for each connective, in combination with the way the modal systems are formulated, leads unavoidably to a situation where a pair of rules need to capture more than one axiom. This happens in logics which contain more than two axioms. For instance, the system for $S4$ presented above is not modular, since 3 axioms, the $K$, $T$ and 4, are captured by two pairs of rules. This situation makes also the formulation of the rules being random, since their design is based on providing cut-admissibility for the systems, through intuitive-based verifications on counterexamples. Furthermore, all of them are unfocussed except of the one for $S4$, which is weakly focussed.

A system for $S5$ is given by Ohnishi and Matsumoto [21] and is obtained by adding to $LK$ the logical rules in (3) and the following rules:

$$(5) \qquad \frac{\alpha, \diamond\Gamma_1, \Box\Delta_1 \vdash \diamond\Gamma_2, \Box\Delta_2}{\diamond\alpha, \diamond\Gamma_1, \Box\Delta_1 \vdash \diamond\Gamma_2, \Box\Delta_2} (\diamond \vdash) \qquad \frac{\diamond\Gamma_1, \Box\Delta_1 \vdash \diamond\Gamma_2, \Box\Delta_2, \alpha}{\diamond\Gamma_1, \Box\Delta_1 \vdash \diamond\Gamma_2, \Box\Delta_2, \Box\alpha} (\vdash \Box)$$

However, the cut-rule is not admissible in this system. Its failure is related to the weakly focussed rules in (5), since their side-conditions do not allow the introduction of the $\Box$-modality in certain cases, as it is desirable. For instance, the proof of axiom $B$ cannot be obtained without a cut-rule application:

$$\dfrac{\dfrac{\dfrac{p \vdash p}{p \vdash \diamond p} (\vdash \diamond) \qquad \dfrac{\diamond p \vdash \diamond p}{\diamond p \vdash \Box \diamond p} (\vdash \Box)}{p \vdash \Box \diamond p} (\text{cut})}{\vdash p \supset \Box \diamond p} (\vdash\supset)$$

As has been already observed (for instance by Avron [1]), it is sufficient to use a restricted form of the cut-rule, known as *analytic cut*. This cut-rule has additionally the condition that the cut-formula must be a subformula of the formulae in its conclusion. In this way, the technical advantages of cut-admissibility are saved. However, it does not solve the theoretical concerns, as it restricts significantly the strength of proof analysis. This is the reason we choose to call such rules *restricted* cut-rules rather than analytic ones. Moreover, having such rules in a system with no proof of their admissibility prohibits them from being real cut-rules, as there is no proof that the lemmas they introduce are nothing more than shortcuts (that they are real lemmas).

An alternative way for obtaining a cut-admissible system for $S5$, is to use the systems implemented in other frameworks for this logic. One suitable framework is the tableaux methods, for which it is known that its systems can be easily translated to systems in the sequent calculus. However, the tableaux system for $S5$ presented by Goré [11]) make also use of the cut-rule.

Consequently, the lack of a cut-admissible system for $S5$ (and also for $B$) leads to investigations of new calculi, presented in the next section.

## 3 The Systems for $S5$

Based on the way they have been obtained, cut-admissible systems for $S5$ are divided as follows:

1. Sequent-style systems which modify the usual Gentzen systems in a way that essential properties the systems in the sequent calculus should enjoy are violated. Such systems we call *extended sequent systems* and have been specifically developed and work only for $S5$.

2. Systems formulated in alternative calculi which modify in several systematic ways the sequent calculus. These calculi are still sequential.

We count the systems for $S5$ presented by Braüner [5] and Sato [25] among the extended sequent systems. Both make fundamental changes to the right logical rule for the $\Box$-modality. In Braüner's system, the logical rule for the modality is applicable only under a condition based on "the way the premises have been obtained"[6]. This condition is derived from the side condition of the right-hand side rule of the universal quantifier ($\vdash \forall$) in monadic predicate calculus. Accordingly, every formula carries (implicitly) information about the state it might be true, so that inside a sequent several states are exhibited. Consequently, Kripke-frame semantics are embedded in the system and so we do not examine the system here.

In Sato's system, a function is applied on a formula in the premise of the rule, which selectively falsifies its subformulae. This means that, besides introducing the $\Box$-modality, the rule also includes selective deep applications of the weakening rules. The use of functions in a sequent can be avoided by alternatively giving rules for their mappings. Such rules can see deeper in a formula as they exhibit more than one of its connectives. For instance, the rule

$$\frac{\Gamma \vdash \Delta, (+\alpha) \supset (-\beta)}{\Gamma \vdash \Delta, -(\alpha \supset \beta)}$$

could be added for the mapping of the function $(\alpha \supset \beta)^- = \alpha^+ \supset \beta^-$, where $()^-$ and $()^+$ are functions treated as logical connectives. This system is not presented here, as the one given in the higher arity sequent calculus (presented in the next section) follows the same idea and is more systematic.

For the new calculi, the following types are obtained, based on the way they extend the sequent calculus:

1. An *enrichment* of the sequent calculus: This is a calculus that extends the sequent calculus by building in it additional types. Such types are, for instance, connectives, judgements and formulae.

2. An *extension* of the sequent calculus: This is a calculus where sequents of the sequent calculus are built into a notion of a higher level. For instance, the calculus of hypersequents is an extension, since it deals with sets of sequents.

Moreover, we call a generalization or an abstraction of the sequent calculus, a calculus that identifies distinct notions of the sequent calculus. The calculus of structures presented in the next section is, for instance, a generalization calculus.

Some enrichment calculi with systems for $S5$ or $B$ have been presented by Kanger [16], Mints [19], Orlowska [22], Simpson [26] and Braüner [6][7]. All of them annotate indices onto formulae, which carry information based on Kripke-frame semantics[8], and as so they are rejected.

[6] the expression is due to Avron [1], where such rules are called "rules with non local nature"

[7] Although the hybrid system presented in this paper is not proved to be cut-admissible, "this can be done by a straightforward modification on the normalization results presented by Braüner [7] for natural deduction systems for hybrid logic" (private communication with the author).

[8] In hybrid logic indices are treated as normal connectives in the sense that they can appear wherever any other connective can do so. However, this does not solve our semantic concerns.

Other enrichments with systems for the same logics are higher arity sequent calculus, multiple sequent calculus and display logic, and are presented below together with two systems of an extension calculi, the hypersequents or tableaux calculi[9].

### 3.1 Higher Arity Sequent Calculus: $GS5^s$

Higher arity sequent calculi are calculi developed on augmented sequents of arity greater than two. This means that instead of the usual two-place consequence relation (with elements the antecedent and the succedent) a relation with more elements is now used.

Calculi of arity four have been independently studied in modal logic by Blamey and Humberstone [4] and Sato [24], in the latter of which a cut-admissible presentation of $S5$ is given. A four-place sequent takes the form $(\Gamma, \Pi, \Sigma, \Delta)$, where all the variables are Gentzen structures. Structures $\Gamma$ and $\Delta$ are the usual ones as for regular sequents, whereas $\Pi$ and $\Sigma$ are new. Following the notation introduced by Sato [24], however, such sequents are rather denoted as

$$\Gamma\ ; \Pi \vdash \Sigma;\ \Delta$$

When both $\Pi$ and $\Sigma$ are empty, the sequent takes the usual form $\Gamma \vdash \Delta$ and is called a *proper* sequent. In general, there are three types of rules:

1. The *internal* rules: they concern only the inner sets $\Pi$ and $\Sigma$ and are applied regardless the information in the other sets.

2. The *external* rules: these are the usual rules concerning the outer sets $\Gamma$ and $\Delta$. Again, no information about $\Pi$ and $\Sigma$ is exhibited.

3. The *interactive* rules: They are structural rules that move information from the innner sets to the outer ones and vice versa.

The system for $S5$, called $GS5^s$[10], is presented in the Appendix (Figure 9). Primitive connectives are only the ones in $\{\bot, \supset, \Box\}$. The rest are abbreviations of the following:

$$\top \equiv \bot \supset \bot \qquad \alpha \vee \beta \equiv (\alpha \supset \bot) \supset \beta$$
$$\neg\alpha \equiv \alpha \supset \bot \qquad \alpha \wedge \beta \equiv (\alpha \supset (\beta \supset \bot)) \supset \bot$$

Theorems of the system are only the provable *proper* sequents. Its design is a part of the completeness proof for some Kripke-type models, also presented in the same paper. Accordingly, a semantic proof for cut admissibility is given. Moreover, the judgment on the internal sets is given by the one on a proper sequent prefixed with the $\Box$-modality. So, given the translation of a proper sequent to a formula $\psi()$, the translation of an improper sequent of the form $\Gamma\ ; \Pi \vdash \Sigma;\ \Delta$ to a formula is the

$$\psi(\Gamma \vdash \Delta, \Box\psi(\Pi \vdash \Sigma)).$$

**Theorem 2** *If a proper sequent is provable in $GS5^s$ then it is provable without using the cut-rule.*

[9]The two calculi use different notations but represent the same concept.
[10]The original name of the system is $GS5$, but since it coincides with the one developed on hypersequents in section 3.3, $-^s$ (for Sato) is added to it.

A proof in the system for axiom B is obtained as follows:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{p \vdash p}\,(\text{ax})}{p \vdash p, \bot}\,(\text{ext: out}) \qquad \cfrac{\cfrac{}{\bot \vdash}\,(\bot \vdash)}{\bot \vdash p}\,(\text{ext: out}) \qquad \cfrac{\cfrac{}{\bot \vdash}\,(\bot \vdash)}{p, \bot \vdash}\,(\text{ext: out})}{p, p \supset \bot \vdash}\,(\supset\vdash\text{: out})}{p, p \supset \bot \vdash \bot;}\,(\text{ext: in})}{p, \Box(p \supset \bot) \vdash \bot;}\,(\Box \vdash\text{: out})}{p\ ; \Box(p \supset \bot) \vdash \bot;}\,(\text{enter} \vdash)}{p \vdash \Box(p \supset \bot) \supset \bot;}\,(\vdash\supset\text{: in})}{p \vdash \Box(\Box(p \supset \bot) \supset \bot)}\,(\vdash \text{exit})}{\vdash p \supset \Box(\Box(p \supset \bot) \supset \bot)}\,(\vdash\supset\text{: out})
$$

### 3.2 Multiple Sequent Calculus: *DSC*

Multiple sequent calculus (also known as "generalised sequent calculus") is an enrichment of Gentzen systems presented by Indrzejczak [14] and [15]. The main purpose of its developement is to give cut-admissible formulations for propositional modal logic with emphasis on *simplicity on proofs*. Its extentions concern both sequent and formula types. More specifically:

1. Sequents take the general forms $\Gamma \ \Box\vdash^i \ \Delta$ and $\Gamma \ \Diamond\vdash^j \ \Delta$, with $i, j \geq 0$, which abbreviate the judgments

$$\underbrace{\Box \ldots \Box}_{i \text{ times}}\vdash \qquad \text{and} \qquad \underbrace{\Diamond \ldots \Diamond}_{j \text{ times}}\vdash$$

   respectively. When $i$ or $j$ are set to zero, the sequents are the *classical* ones and are presented as usual: $\Gamma \ \vdash \ \Delta$. Otherwise, the sequents are called *non-classical* or *modal* sequents. When $i$ or $j$ are set to 1 they are omitted and the sequents take the form $\Gamma \ \Box\vdash \ \Delta$ and $\Gamma \ \Diamond\vdash \ \Delta$, respectively. Theorems of the systems are only proofs of classical sequents.

2. In addition to the set of modal formulae $MF$, the set of well formed formulae also includes the formulae in $MF$ prefixed with the special operator $-$. Such formulae are called *$S$-formulae*. The operator is used for shifting a formula from one side of a sequent to the other, and can only occur at the beginning of a formula. The formula $-(\alpha \vee \Box\beta)$ is, for instance, a valid well formed $S$-formula, whereas $\alpha \vee - \Box \beta$ is not.

In multiple sequent calculus are given cut-admissible presentations for the logics with axioms any of the $\{D, T, 4\}$, as well as for logics $KB$, $KDB$, $B$ and $S5$. However, the cut-rule is not a rule of the systems, which means that no proof of cut-elimination is given. Alternatively, completeness for the cut-free systems is obtained following methods that do not involve any forms of cut (such as Modus Ponens). The translation $\Upsilon()$ of a sequent to a formula is given as follows:

$$\Upsilon(\Gamma \underbrace{\Box\ldots\Box}_{i \text{ times}}\vdash \Delta) = \bigwedge \tau(\Gamma) \supset \underbrace{\Box\ldots\Box}_{i \text{ times}}(\bigvee \tau(\Delta))$$

$$\Upsilon(\Gamma \underbrace{\Diamond\ldots\Diamond}_{j \text{ times}}\vdash \Delta) = \underbrace{\Diamond\ldots\Diamond}_{j \text{ times}}(\bigwedge \tau(\Gamma)) \supset \bigvee \tau(\Delta)$$

where $\tau(\Gamma) = \{\tau(\alpha) \mid \alpha \in \Gamma\}$, $\tau(-\alpha) = \neg\alpha$ and $\tau(\alpha) = \alpha$.

The system for logic $S5$, called *DSC* (double sequent calculus), is presented in the Appendix (Figure 10) (for a complete presentation see Indrzejczak [14]). In this system, as its name reveals, only two types of sequents are exhibited: the classical one and the modal $\Gamma \Box\vdash\ \Delta$ ($i = 1$). The following notations are used:

1. $\Gamma\ (\Box)\vdash\ \Delta$ is used in an inference rule when the type of the sequent is unspecified. This means that the rule is applicable in both sequents.

2. The notations $\alpha^*$ and $(-\alpha)^*$ stand for $-\alpha$ and $\alpha$, respectively, where $\alpha$ is a formula in $MF$, and are used in shifting formulae in a sequent. Extending this to a set of formulae $\Gamma$, $\Gamma^*$ is the set of all $\alpha^*$, with $\alpha \in \Gamma$.

3. Similarly, $\neg\Gamma$, $\Box\Gamma$ and $\Diamond\Gamma$ denote the sets obtained by prefixing their formulae with the relevant connective.

4. $M\Gamma$ denotes the set obtained by prefixing each formula in $\Gamma$ with one of $\{\Box, -\Diamond\}$, the $\Box$-*formulae*, or one of $\{\Diamond, -\Box\}$, the $\Diamond$-*formulae*. A formula in $M\Gamma$ is called an $M$-*formula* and is accordingly denoted as $M\alpha$ (for $\alpha \in \Gamma$).

**Theorem 3** *System DSC (i.e. without any cut-rules) is sound and complete with respect to $S5$.*

To illustrate how proofs are obtained, consider the proof for axiom $B$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\ (AX)}{p \vdash \Diamond p}\ (\vdash \Diamond)}{p\ \Box\vdash\ \Diamond p}\ (NC)}{p \vdash \Box\Diamond p}\ (\vdash \Box)}{\vdash \Box\Diamond p, -p}\ (*_r)}{\vdash p \supset \Box\Diamond p}\ (\vdash\supset)$$

The systems for the rest of the logics are given in Indrzejczak [15]. Among all the systems, *DSC* enjoys the simplest design, due to the special properties of $S5$. Unsurprisingly, systems with axiom $B$ are the most complicated ones, since axiom $B$ is responsible for *symmetry*, the property which causes problems in the formulation of $B$ in a Gentzen system.

### 3.3 Hypersequents: $GS5$ and $LS5$

#### Hypersequents

The method of hypersequents is an extention of sequent calculus, where structures (called hypersequents) are finite (multi)sets of normal sequents. For instance,

$$\Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2 \mid \Gamma_3 \vdash \Delta_3$$

is a hypersequent of three sequents, seperated by $|$. The set has a disjunctive role: a hypersequent is provable if at least one of its sequents is provable. The schematic letters $\alpha, \beta, \ldots$ denote arbitrary formulae, $\Gamma, \Delta, \ldots$ (multi)sets of formulae and $G, H, \ldots$ (multi)sets of sequents.

Axioms are of the form $\alpha \vdash \alpha$. Logical rules and the cut-rule are formulated as in Gentzen systems, with the addition that they are applicable on sets of sequents. For instance, the rule for the introduction of negation to the right of the turnstile $(\vdash \neg)$ $\dfrac{\alpha, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg\alpha}\,(\vdash \neg)$ takes the form $\dfrac{G \mid \alpha, \Gamma \vdash \Delta \mid H}{G \mid \Gamma \vdash \Delta, \neg\alpha \mid H}\,(\vdash \neg)$ .

For the structural rules permutation, weakening and contraction there are two versions of rules:

1. the *internal* version, which is applied on *formulae* inside a sequent. These are the standard Gentzen rules, which are obtained in the same way logical rules are obtained, as described above.

2. the *external* version which is applied on *sequents*. For example, the external version of the contraction duplicates a sequent:

$$\frac{G \mid \Gamma \vdash \Delta \mid \Gamma \vdash \Delta \mid H}{G \mid \Gamma \vdash \Delta \mid H}\,(C\ external)$$

Since hypersequents is a more general framework than Gentzen systems, logics with problematic formulations in Gentzen systems can be successfully formulated here. This is obtained through new types of structural rules, that is rules without correspondence to any normal sequent ones. Two types of such rules are the ones for *splitting* and *shuffling* (see Avron [1]).

All modal systems with cut-admissible formulations in Gentzen systems can be formulated in hypersequents. Cut-admissible formulations for M and S4 are presented by Pottinger [23], for which their equivalence to the corresponding sequent systems is easily shown. Moreover, in Avron [1] and Pottinger [23] are presented cut-admissible formulations for S5. Here, we present the one in Avron [1], system GS5. This formulation makes use of the modalized splitting rule, a rule which allows us, under some structural conditions, to *split* a sequent.

More specifically, $GS5$ is obtained by adding to the system for propositional logic the modal rules for $S4$

$$\frac{\alpha, \Gamma \vdash \Delta}{\Box\alpha, \Gamma \vdash \Delta}\,(\Box \vdash) \qquad \frac{\Box\Gamma \vdash \alpha}{\Box\Gamma \vdash \Box\alpha}\,(\vdash \Box)$$

and the *modalized splitting* rule:

$$\frac{G \mid \Box\Gamma_1, \Gamma_2 \vdash \Box\Delta_1, \Delta_2 \mid H}{G \mid \Box\Gamma_1 \vdash \Box\Delta_1 \mid \Gamma_2 \vdash \Delta_2 \mid H}(MS)$$

When $\diamond$ is treated as primitive, an application of the $(MS)$ rule allows additionally splitting on formulae prefixed with $\diamond$ (and not only with $\Box$):

$$\frac{G \mid \Box\Gamma_1, \diamond\Gamma_2, \Gamma_3 \vdash \Box\Delta_1, \diamond\Delta_2, \Delta_3 \mid H}{G \mid \Box\Gamma_1, \diamond\Gamma_2 \vdash \Box\Delta_1, \diamond\Delta_2 \mid \Gamma_3 \vdash \Delta_3 \mid H}(MS)$$

System $GS5$ is presented in the Appendix (Figure 11). The rules for the propositional part are due to Gentzen system $LK$. It is equivalent to $S5$ and admits cut-elimination. We present the proof for axiom $B$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{p, \neg p \vdash}(\neg\vdash)}{p, \Box\neg p \vdash}(\Box\vdash)}{p \vdash \mid \Box\neg p \vdash}(MS)}{p \vdash \mid \vdash \neg\Box\neg p}(\vdash\neg)}{p \vdash \mid \vdash \Box\neg\Box\neg p}(\vdash\Box)}{p \vdash \Box\neg\Box\neg p \mid \vdash \Box\neg\Box\neg p}(W_r)}{p \vdash \Box\neg\Box\neg p \mid p \vdash \Box\neg\Box\neg p}(W_l)}{p \vdash \Box\neg\Box\neg p}(C\ external)}{\vdash p \supset \Box\neg\Box\neg p}(\vdash\supset)}{}$$

Completeness for $GS5$ is shown using the translation of a hypersequent G

$$G = \Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_n \vdash \Delta_n$$

to a formula of modal logic

$$\phi_G = \Box\psi_{\Gamma_1\vdash\Delta_1} \vee \ldots \vee \Box\psi_{\Gamma_n\vdash\Delta_n} ,$$

where $\psi_{\Gamma_i\vdash\Delta_i}(i = [1 \ldots n])$ is the usual translation of a sequent to a formula:

$$\psi_{\alpha_1,\ldots,\alpha_l\vdash\beta_1,\ldots,\beta_m} = \neg\alpha_1 \vee \ldots \vee \neg\alpha_l \vee \beta_1 \vee \ldots \beta_m .$$

The results of $GS5$ for its equivalence to $S5$ and its cut-admissibility follow:

**Proposition 1** $\vdash_{GS5} G$ *iff* $\vdash_{S5} \phi_G$ .

**Theorem 4** *System* $GS5$ *admits cut-elimination.*

We call $GS5^-$ the system obtained by removing the cut rule from $GS5$. The two systems are equivalent.

**Mints' system $LS5$**

A similar system for $S5$ has been presented by Mints [20], where a connective he calls *tableau*[11] is defined over lists of formulae. A tableau is denoted by $\langle\Gamma\rangle$, where $\Gamma$ is a non-empty set of formulae. As in system $GS5^s$, formulae are built up only from $\{\supset, \Box, \bot\}$. Moreover, for any formula $\alpha$, $\overline{\alpha}$ is again a formula and denotes that $\alpha$ occurs in the antecedent. For uniformity reasons, we treat tableau over two-sided sequents. Since the bar is only an indicator, this is only a presentational conversion and does not change the system as originally presented by Mints. Sets of formulae and sets of tableaux are denoted by the same letters as in hypersequents.

System $LS5$ is shown in the Appendix (Figure 12) and admits cut-elimination. The translation of a list of tableaux to a modal formula coincides to the one in $GS5$.

### 3.4 Display Logic: $DS5$

Display logic is an enrichment of Gentzen systems introduced by Belnap [2] and further developed by Wansing [27]. It extends Gentzen structures with two structural connectives: the unary connectives $*$ and $\bullet$. Formally,

**Definition 8** *Structures in display logic are built up as follows:*

$$X \ ::= \ \mathbf{f} \mid \mathbf{t} \mid \mathbf{I} \mid \alpha \mid \ \bullet X \ \mid \ * X \mid X \circ X$$

Letters $\mathbf{f}$ and $\mathbf{t}$ stand for falsity and truth, respectively, $\mathbf{I}$ for the empty structure, $X, Y, \ldots$ denote arbitrary structures and $\alpha, \beta, \ldots$ propositional formulae. The binary connective $\circ$ is the usual addition on structures and corresponds to the comma (,) of Gentzen structures. The $*$ shifts structures from one side of the turnstile to the other and the $\bullet$ marks a structure as intensional. As an example of a valid structure in the calculi, consider the structure $*\ (X \circ Y) \circ (\bullet\ Z)$. As usual, parenthesis are ommited when no confusion is possible. Note that the new connectives allow nested structures of arbitrary depth, and so we say that structures in display logic are *deep*, in contrast to the Gentzen structures which are *shallow*.

The main characteristic of display logic is the ability to move structures inside a sequent. This, in combination with other structural rules, allows any substructure to be *displayed* isolated to the left or to the right of the turnstile, without changing the meaning of the sequent. This is known as the *Display Theorem*:

**Theorem 5** *For every sequent $s$ and every antecedent (succedent) part $X$ of $s$ there exists a sequent $s'$ structurally equivalent to $s$ such that $X$ is the entire antecedent (succedent) of $s'$.*

In other words, every sequence is *deep*, in the sense that we can *reach* any of its substructures[12]. This corresponds to the structure context $S\{-\}$ of the calculus of structures, defined in section 4.1.

The basic structural rules in display logic are:

[11] This name is taken from tableau methods but the calculus is a deductive one

[12] Saying that we can reach a substructure implies that we do not lose any other information by doing so.

$$
\begin{array}{cccc}
(1) & (2) & (3) & (4) \\[1em]
\dfrac{X \circ Y \vdash Z}{X \vdash Z \circ *Y} & \dfrac{X \vdash Y \circ Z}{X \circ *Z \vdash Y} & \dfrac{X \vdash Y}{*Y \vdash *X} & \dfrac{\bullet X \vdash Y}{X \vdash \bullet Y} \\[1.5em]
\dfrac{X \vdash Z \circ *Y}{Y \vdash *X \circ Z} & \dfrac{X \circ *Z \vdash Y}{*Y \circ X \vdash Z} & \dfrac{*Y \vdash *X}{X \vdash **Y} &
\end{array}
$$

Double line abbreviates the rule obtained by reversing premise-conclusion. Thus, in all the rules presented above, their conclusion also entails their premise. Two sequents are called *structurally equivalent*, if both of them are derivable from each other. For example, the structures

$$X \vdash *Y \quad \text{and} \quad Y \vdash *X$$

are structurally equivalent. Note that, contrary to the deepness of the structures, the rules remain *shallow*. Kracht [17] shows that equivalent sequents are closed under congruence. Moreover, he shows that any structure is equivalent to a shallow structure in reduced normal form.

In Wansing [27] various modal systems are presented, including the normal modal and tense logics[13] of the cube. All of them are obtained in a systematic way: following the algorithm given by Kracht [17], each axiom $A$ can be mapped into a structural rule $A'$. Then, the logic extending $K$ with axioms $\{A, B, \ldots\}$ is formulated by adding to system $DK$ the corresponding rules $\{A', B', \ldots\}$. All the systems admit cut-elimination. The translation of a sequent to a formula is $\tau(X \vdash Y) = \tau_1(X) \supset \tau_2(Y)$, where $\tau_i$, $i \in \{1, 2\}$ are recursively defined as follows:

$$
\begin{aligned}
\tau_i(\alpha) &= \alpha \\
\tau_1(\mathbf{I}) &= \top \\
\tau_2(\mathbf{I}) &= \bot \\
\tau_i(*X) &= \neg\tau_i(X) \\
\tau_1(X \circ Y) &= \tau_1(X) \wedge \tau_1(Y) \\
\tau_2(X \circ Y) &= \tau_2(X) \vee \tau_2(Y) \\
\tau_1(\bullet X) &= \langle P \rangle \tau_1(X) \\
\tau_2(\bullet X) &= [F]\tau_2(X)
\end{aligned}
$$

The system for logic $K$, called $DK$, is shown in the Appendix, Figures 13 and 14. The systems for $S5$ and $B$ we call $DS5$ and $DB$ respectively and are obtained by extending system $DK$ with the structural rules that correspond to their axioms, the rules $(T')$, $(5')$ and $(B')$ presented also in the Appendix, Figure 15.

**Theorem 6** *System $DS5$ admits cut-elimination.*

[13] Tense or temporal logic is bimodal logic with modalities $[F], \langle F \rangle$ for the future (this coincides with the normal modality) and $[P], \langle P \rangle$ for the past.

In $DS5$, axiom $B$ has the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,(id)}{* \bullet * p \vdash \diamond p}\,(\vdash \diamond)}{\bullet * \bullet * p \vdash \diamond p}\,(5')}{* \bullet * p \vdash \Box \diamond p}\,(\vdash \Box)}{p \vdash \Box \diamond p}\,(T')}{\mathbf{I} \vdash p \supset \Box \diamond p}\,\vdash\supset$$

where the empty structure **I** can be added or removed in any place of a sequent.

## 3.5 A Characterization of the Systems

### Deep Inference

All of the systems presented in this section overcome the lack of cut-admissibility of the system for $S5$ in the sequent calculus, following the same technique: they exhibit formulae in a state not reachable by the usual Gentzen systems. More specifically, the rules in (5) presented in Section 2.3

$$\dfrac{\alpha, \diamond\Gamma_1, \Box\Delta_1 \vdash \diamond\Gamma_2, \Box\Delta_2}{\diamond\alpha, \diamond\Gamma_1, \Box\Delta_1 \vdash \diamond\Gamma_2, \Box\Delta_2}\,(\diamond \vdash) \qquad \dfrac{\diamond\Gamma_1, \Box\Delta_1 \vdash \diamond\Gamma_2, \Box\Delta_2, \alpha}{\diamond\Gamma_1, \Box\Delta_1 \vdash \diamond\Gamma_2, \Box\Delta_2, \Box\alpha}\,(\vdash \Box)$$

have the side-condition that all formulae must be prefixed with a modality. The new systems allow, directly or indirectly, partial applications of these rules, so that their side-conditions are avoided. Examples of partial applications of the $(\vdash \Box)$-rule are the following derivations in $GS5$ and $DS5$ respectively:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash \Box p}{p \vdash \ \mid\ \vdash \Box p}\,(MS)}{p \vdash \ \mid\ \vdash \Box \Box p}\,(\vdash \Box)}{p \vdash \Box \Box p \mid\ \vdash \Box \Box p}\,(W_r)}{p \vdash \Box \Box p \mid p \vdash \Box \Box p}\,(W_l)}{p \vdash \Box \Box p}\,(C\ external) \quad \text{and} \quad \dfrac{\dfrac{* \bullet * p \vdash p}{\bullet * \bullet * p \vdash p}\,(5')}{* \bullet * p \vdash \Box p}\,(\vdash \Box)$$

where their premises and conclusions matches the ones of the $(\vdash \Box)$-rule, without though satisfying the side-conditions. Note that such derivations do not always lead to proofs, since sequences such that $\vdash p \supset \Box p$ are not theorems of $S5$ and thus not provable in any of the systems. The systems then stay in $S5$ by changing the judgements of the consequence relation, as shown by the translations of their input to modal formulae.

Analytically, reading the rules bottom-up, the situation in each system is as follows:

1. In the higher arity system $GS5^s$ formula $\alpha$ is unconditionally placed to the inside set $\Sigma$ by the $(\vdash$ exit$)$-rule and is retrieved by the (enter)-rules only when it is prefixed with a modality.

2. In the multiple sequent system $DSC$ an application of the $(\vdash \Box)$-rule replaces the classical sequent with the modal one. The classical sequent is then reset only with an application under conditions of the $(NC)$-rule.

3. In the hypersequent system $GS5$, as can been seen in the above derivation, the sequent is first duplicated, so that all structures which do not satisfy the side-conditions of the $(\vdash \Box)$-rule will be safely removed by applications of weakening rules. Then, the two sequents are joined to a single sequent by an application of the conditional $(MS)$-rule.

4. In Mints' hypersequent system $LS5$ formula $\alpha$ is seperated to a new tableau by an application of the $(\vdash \Box)$-rule. The two tableaux are then joined under conditions with an application of the $(\Box_1 \vdash)$-rule.

5. In the display system $DS5$ an application of the $(\vdash \Box)$-rule marks the rest of the structure as intensional. The judgement is then reset either by an application of the $(5')$-rule or by applications of the introduction rules $(\Box \vdash)$ and $(\vdash \Diamond)$.

Consequently, roughly speaking, the systems provide techniques that allow, in certain cases, deep applications of the specific rules. Based on the depth of their structures the systems are splitted into two groups:

1. *Deep systems* are those with nested structures of infinite depth, called deep structures.

2. *Augmented systems* are systems with shallow structures of finite depth greater than the one of Gentzen structures.

All of the systems presented above are augmented systems of depth 1, except the ones in display logic which are deep. In $GS5$, for instance, structures have maximum depth 1, since the $|$ connective is only applied on sequents, whereas in $DS5$, the new connectives can be applied in any structure of the system and so structures are of arbitrary depth. The systems in the calculus of structures presented in the next section are also deep. Now we can define deep inference:

**Definition 9** ***Deep inference*** *is the ability to apply inference rules in any depth inside a structure, and so it is a property of deep systems. A system with such rules is said to* ***enjoy*** *deep inference. We say that a system only* ***imitates*** *deep inference if it is an augmented system of depth* $\mathbf{n}$ *with inference rules applied inside a structure till depth* $\mathbf{n}$.

The systems in display logic enjoy deep inference, as well as the systems in the calculus of structures. However, in contrast to the calculus of structures, in display logic deep inference is only *implicit*. This is because its systems are sequential with rules that are not applied directly inside structures. Instead, substructures are shifted so that a structurally equivalent sequent that matches the rule is obtained. For instance, the $(\vdash \vee)$-rule can be applied bottom-up to the substructure $p \vee q$ of the sequent $\mathbf{I} \vdash \bullet(p \vee q) \circ *(r \circ \bullet q)$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathbf{I} \vdash \bullet(p \circ q) \circ *(r \circ \bullet q)}{(r \circ \bullet q) \vdash \bullet(p \circ q)}\,(1)}{\bullet(r \circ \bullet q) \vdash p \circ q}\,(4)}{\bullet(r \circ \bullet q) \vdash p \vee q}\,(\vdash \vee)}{r \circ \bullet q \vdash \bullet(p \vee q)}\,(4)}{\mathbf{I} \vdash \bullet(p \vee q) \circ *(r \circ \bullet q)}\,(1)$$

In the above derivation, the conclusion sequent in its current form does not match the conclusion of the $(\vdash \vee)$-rule, though after equivalent transformations it does so.

The rest of the systems imitate deep inference, since they all have rules applicable both in depth 0 and 1. For instance, weakening in all systems has two versions, one for each depth.

### Other Properties

Only two of the above calculi satisfy Došen's principle, namely the multiple sequent calculus and display logic. Both of them give common logical rules for the modalities and different structural rules for the formulations of the specific logics[14]. In system $GS5^s$ and in both hypersequential systems $GS5$ and $LS5$, the logical rules for the modalities are more specific than the ones needed for logic $K$, meaning that those rules give more theorems than just the ones in $K$. Therefore, as long as those systems keep their current formulation, the addition of new systems to the calculi (such as a system for $K$) will cause failure of the principle.

As it can be observed from the systems which satisfy Došen's principle, its success in the cases of modal systems results complicated structural rules and, consequently, complicated proofs. Also, this principle is not applicable in the calculus of structures (presented in the next section). For these reasons, we are instead interested in the success of a similar notion, the one of modularity. This property, together with the properties being focussed and systematicity, has been presented in Sections 2.1 and 2.3. Figure 4 shows the systems for $S5$ already presented and some properties they enjoy, as well as the system for $S5$ in the calculus of structures, $KSg_{\{kt45\}}$. The calculi with modular systems are the multiple sequent calculus and display logic, and are the only ones with systems for logic $B$. The notion of *monotonicity* is defined as follows:

**Definition 10**
*1. A rule is of* ***increased (decreased) monotonicity*** *if the number of symbols in its premise with the most symbols is less (greater) than the number of symbols in its conclusion.*
*2. A set of rules is said to be* ***monotone*** *if it has rules only of the one type (and no rules of the other).*
*3. A system enjoys* ***monotonicity*** *if the set of all its rules is monotone.*

[14] although the logical rules in *DSC* differ from the ones in the other systems, they are only simplified versions of the original system and not new ones.

| *Systems* | *Properties* | | | |
|---|---|---|---|---|
| | *being focussed* | *systematicity* | *modularity* | *monotonicity* |
| $GS5^s$ | *weakly* | *fails* | *fails* | *subformula prop.* |
| $DSC$ | *weakly* | *fails* | *succeeds* | *subformula prop.* |
| $GS5$ | *weakly* | *fails* | *fails* | *subformula prop.* |
| $LS5$ | *strongly* | *fails* | *fails* | *subformula prop.* |
| $DS5$ | *strongly* | *succeeds* | *succeeds* | *fails* |
| $KSg_{\{kt45\}}$ | *n.a.* | *succeeds* | *succeeds* | *fails* |

Figure 4: Some properties of the systems for $S5$

Monotonicity is a notion similar to the subformula property which can be used in deep systems. In deep systems, the benefits of the subformula property are lost, as its presence does not ensure the existence of finitely many possible premises. For instance, in display logic a sequent is structurally equivalent with an infinite number of sequents, and so, there are always unboundedly many ways to apply a rule. Increased monotonicity ensures that this is never the case whereas decreased monotonicity works when the rules are applied top-down. Here we use this notion only for deep systems and check instead for the subformula property in the augmented ones.

Note that systematicity is used here only in relation with the axioms of each logic and not with other means. For instance, as mentioned by Sato [24], the system in the higher arity sequent calculus $GS5^s$ is obtained after close inspection of a completeness proof for a family of logics. However, this system is not considered to be systematic. Also, the property being focussed covers only logical rules and thus, it is not applicable in systems in the calculus of structures.

As it can be easily observed, the only systems with systematic and modular rules are the deep systems. Moreover, system $KSg_{\{kt45\}}$, the only system with explicit use of deep inference, satisfies additionally the following properties:

1. Simplicity. The rules are few and simple, and the connectives used the least possible. Also, systematicity is obtained in an obvious way without the use of special techniques.

2. No specific translation of its structures into formulae is needed, other than the direct one.

3. Locality. The rules can be restricted in such a way that the inspection of a formula of an arbitrary length is never necessary.

A short introduction to the calculus of structures and its modal systems follows.

# 4 The Calculus of Structures and $S5$

## 4.1 The Calculus of Structures

The calculus of structures is a proof theoretical framework introduced by Guglielmi [12]. It is a generalization of the sequent calculus, since there are no logical connectives or rules. Thus, all the connectives occuring in the structures are structural connectives which are applied only on atoms and their complements. *Modal structures* extend the propositional structures with modal connectives and equations. Similarly, modal systems extend the propositional one, system **SKSg**, presented by Brünnler [8].

**Definition 11** *Prestructures in modal KS systems are built up as follows:*

$$S ::= \mathit{ff} \mid \mathit{tt} \mid a \mid \overline{a} \mid \underbrace{[S, \ldots, S]}_{\geq 1} \mid \underbrace{(S, \ldots, S)}_{\geq 1} \mid \Box S \mid \Diamond S$$

The units **ff** and **tt** stand for *falsity* and *truth*, the schematic letters $a, b, \ldots$ and $\overline{a}, \overline{b}, \ldots$ for atoms and their complements (also called positive and negative literals respectively), $[S_1, \ldots, S_n]$ and $(S_1, \ldots, S_n)$ for disjunction and conjunction, and $\Box S$ and $\Diamond S$ for the usual modal operators. A prestructure context $S\{-\}$ is a prestructure in which an atom is replaced by $-$, *the hole*. $S\{R\}$ is a prestructure for which $-$ is replaced by the prestructure $R$, that is, a prestructure with $R$ as a subprestructure. Curly braces are omitted when the prestructure $R$ is precisely of the form $(S_1, \ldots, S_n)$ or $[S_1, \ldots, S_n]$. When modalities occur in prestructures or prestructure contexts, they are called *modal prestructures* or *modal prestructure contexts* respectively.

Since prestructures are defined in negation normal form we make use of their De Morgan dualities given by the following function $()^d$:

$$\begin{aligned} (a)^d &= \overline{a} \qquad (\overline{a})^d = a, \text{for any atom } a \text{ and its complement} \\ (\mathbf{tt})^d &= \mathbf{ff} \qquad (\mathbf{ff})^d = \mathbf{tt} \\ (R_1, \ldots, R_i)^d &= [R_1^d, \ldots, R_i^d] \\ [R_1, \ldots, R_i]^d &= (R_1^d, \ldots, R_i^d) \\ (\Box R)^d &= \Diamond R^d \\ (\Diamond R)^d &= \Box R^d \end{aligned}$$

Note that for $R = T$, $R^d = T^d$. Now we can define structures:

**Definition 12** *Structures $\mathcal{S}$ are defined modulo the following equations on prestructures:*

1. *Associativity:*

$$[R_1, \ldots, R_i, [T_1, \ldots, T_j], U_1, \ldots, U_k] = [R_1, \ldots, R_i, T_1, \ldots T_j, U_1, \ldots U_k] \qquad \text{and}$$
$$(R_1, \ldots, R_i, (T_1, \ldots, T_j), U_1, \ldots, U_k) = (R_1, \ldots, R_i, T_1, \ldots T_j, U_1, \ldots U_k)$$

2. *Commutativity:* $\quad [R, T] = [T, R] \qquad$ *and* $\qquad (R, T) = (T, R)$

3. *Identity:* $R = [R, ff]$, $R = (R, tt)$, $tt = \Box tt$ *and* $ff = \Diamond ff$.

*and are closed under the replacement theorem: If* $R = T$ *then* $S\{R\} = S\{T\}$*, for any prestructure context* $S\{-\}$*.*

Accordingly, a *structure context* $\mathcal{S}\{-\}$ extends the notion of substructure context onto structures as follows: $\mathcal{S}\{-\}$ is the structure defined over the prestructures of $\mathcal{S}$ in which an atom is replaced by $-$. Note that all prestructures of a structure share the same atoms and their complements. Similarly, $\mathcal{S}\{\mathcal{R}\}$ is the structure defined over the prestructures $S_i\{R_j\}$, with $i = [1 \ldots n]$ and $j = [1 \ldots m]$ for $S_1, \ldots, S_n, R_1, \ldots, R_m$ the prestructures in $\mathcal{S}$ and $\mathcal{R}$, respectively.

**Definition 13** *An inference rule is of the form* $\pi \dfrac{\mathcal{S}\{\mathcal{R}\}}{\mathcal{S}\{\mathcal{T}\}}$ ,

where $\pi$ is its name, and it is *deep* in the sense that it is applicable in any structure $\mathcal{S}\{\mathcal{R}\}$, no matter how deep inside $\mathcal{S}$ the prestructures of $\mathcal{R}$ occur. However, when we formulate or apply rules, we refer to a structure by one of its prestructures. A rule application $\pi \frac{\rho}{\tau}$ is such that, given a structure context $\mathcal{S}\{-\}$, $\rho = \mathcal{S}\{\mathcal{R}\}$ and $\tau = \mathcal{S}\{\mathcal{T}\}$. A *derivation* is a finite sequent of applications of inference rules. A *proof* is a derivation starting with the structure $\mathbf{tt}$.

In the symmetric, non cut-free systems in the calculus of structures, every inference rule has also its dual one. The latter is obtained by reversing the position of premise/conclusion and applying negation on them. For instance, the dual of the rule in Definition 13 is $\pi \dfrac{\mathcal{S}\{\overline{\mathcal{T}}\}}{\mathcal{S}\{\overline{\mathcal{R}}\}}$ , where $\overline{\mathcal{R}}$ is the structure obtained over the output prestructures of function $()^d$ when applying on the prestructures of $\mathcal{R}$. Note that for $\mathcal{R} = \mathcal{T}$ being two structures which share the same prestructures, $\overline{\mathcal{R}} = \overline{\mathcal{T}}$. When the resulted rule is different from the initial one, its name is extended by an up arrow $\uparrow$, while the name of the initial rule gets a down arrow $\downarrow$. So, the two rules are distinguished still sharing the same name, denoting in this way their duality. Consequently, the notion of duality is extended on derivations as follows: the dual of a derivation is its reverse derivation, with every structure negated. The rules of the symmetric system for classical propositional logic **SKSg** are:

Interaction and cut rules:

$$i\downarrow \frac{S\{\mathbf{tt}\}}{S[R, \overline{R}]} \qquad i\uparrow \frac{S(R, \overline{R})}{S\{\mathbf{ff}\}}$$

The switch rule:

$$s \frac{S([R, T], U)}{S[(R, U), T]}$$

The weakening rules:

$$w\downarrow \frac{S\{\mathbf{ff}\}}{S\{R\}} \qquad w\uparrow \frac{S\{R\}}{S\{\mathbf{tt}\}}$$

$$
i\downarrow \frac{S\{\mathbf{tt}\}}{S[R,\overline{R}]} \qquad s\,\frac{S([R,T],U)}{S[(R,U),T]} \qquad w\downarrow \frac{S\{\mathbf{ff}\}}{S\{R\}} \qquad c\downarrow \frac{S[R,R]}{S\{R\}}
$$

Figure 5: System **KSg**

And the contraction rules:

$$
c\downarrow \frac{S[R,R]}{S\{R\}} \qquad c\uparrow \frac{S\{R\}}{S(R,R)}
$$

In this system, prestructures are built up as in Definition 11 without the modal constructors and structures are closed under the same equations modal structures are, except of the modal equations of identity. Note that the dual of a derivation is again a valid derivation of the system. For instance, the dual of the derivation:

$$
s\,\frac{w\downarrow \dfrac{(T,U)}{([R,T],U)}}{[(R,U),T]} \quad \text{is the derivation} \quad w\uparrow \frac{s\,\dfrac{([\overline{R},\overline{U}],\overline{T})}{[(\overline{R},\overline{T}),\overline{U}]}}{[\overline{T},\overline{U}]}
$$

**Proposition 2** *System **SKSg** is equivalent to the classical propositional logic.*

The rules for interaction, cut and weakening can be restricted to be applied only on atoms (rather than on arbitrary structures). By adding to the system the medial rule

$$
m\,\frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}
$$

the same is also true for the contraction rule. In this way we get the *local* system **SKS**. It is local in the sense that none of its rules need to deal with structures of unbounded size. Note that dualization of the switch and the medial rules gives the same rules.

**Proposition 3** *Systems **SKSg** and **SKS** are strongly equivalent.*

Two systems are strongly equivalent if they share the same derivations. Cut-elimination for both **SKSg** and **SKS** is given by the following theorem and is a consequence of the admissibility of all their up rules - and so the cut rule.

**Theorem 7** *Systems **SKSg** and **SKS** admit cut-elimination.*

The cut-free systems obtained by removing the up rules are called **KSg** and **KS** respectively. System **KSg** is shown in Figure 5.

$$k\downarrow \frac{S\{\Box[R,T]\}}{S[\Box R,\Diamond T]} \qquad k\uparrow \frac{S(\Box R,\Diamond T)}{S\{\Diamond(R,T)\}}$$

$$t\downarrow \frac{S\{\Box R\}}{S\{R\}} \qquad t\uparrow \frac{S\{R\}}{S\{\Diamond R\}}$$

$$4\downarrow \frac{S\{\Diamond\Diamond R\}}{S\{\Diamond R\}} \qquad 4\uparrow \frac{S\{\Box R\}}{S\{\Box\Box R\}}$$

$$5\downarrow \frac{S\{\Diamond\Box R\}}{S\{\Box R\}} \qquad 5\uparrow \frac{S\{\Diamond R\}}{S\{\Box\Diamond R\}}$$

Figure 6: System $SKSg_{\{kt45\}}$: the new rules

### 4.2 $SKSg_{\{kt45\}}$: A system for $S5$

System $SKSg_{\{kt45\}}$ is obtained by extending system $SKSg$ with the rules $\{k\downarrow, t\downarrow, 4\downarrow, 5\downarrow\}$ and their duals (see Figure 6). System $SKSg_{\{k\}}$ is the system for logic $K$. All new rules are deep and local. Their correspondence to the Hilbert Axioms arises easily, as, for each axiom $A : R \supset T$ with the subformulae $R$ and $T$ in negation normal form, either the up or the down rule is of the form $a\,\frac{S\{R\}}{S\{T\}}$. The rules are then oriented so that the up ones will be admissible. Theorems of the system are its proofs, that is, all derivations starting with $\mathtt{tt}$. For instance, a proof of axiom $B$ is:

$$\begin{array}{l} i\downarrow \dfrac{\mathtt{tt}}{[\Diamond\Box\bar{R},\Box\Diamond R]} \\ 5\downarrow \dfrac{}{[\Box\bar{R},\Box\Diamond R]} \\ t\downarrow \dfrac{}{[\bar{R},\Box\Diamond R]} \end{array}$$

#### Equivalence to $S5$

Proving that system $SKSg_{\{kt45\}}$ is indeed a system for $S5$, requires at a first step a way to move between the two calculi, so that translations of theorems from the one calculi to the other will be possible. Function $()^s$ maps a modal formula to a structure and is recursively defined as follows (A is a propositional variable):

$$\top^s = \mathbf{tt}, \quad \bot^s = \mathbf{ff}, \quad A^s = a^s$$
$$(\neg P)^s = \overline{P^s}$$
$$(\Box P)^s = \Box P^s$$
$$(\Diamond P)^s = \Diamond P^s$$
$$(P \vee Q)^s = [P^s, Q^s]$$
$$(P \wedge Q)^s = (P^s, Q^s)$$
$$(P \supset Q)^s = [\overline{P^s}, Q^s]$$
$$(P \leftrightarrow Q)^s = ([\overline{P^s}, Q^s], [\overline{Q^s}, P^s])$$

Using the reverse function $()^{s^{-1}}$ we can translate a structure to a formula. We call this function $()^h$. Since there are arbitrary formulae that a structure can be mapped onto, we simply choose one of them. The following properties hold:

**Proposition 4** *For $F$ a formula of modal logic and $\mathcal{V}$ a structure of modal $KS$, the following holds*

1. $\vdash F \leftrightarrow F^{hs}$ *is derivable in normal modal logic and*
2. $\mathcal{V} = \mathcal{V}^{sh}$ *in system* $SKSg_{\{k\}}$.

**Proof**.

1. By induction on the structure of $F$. We prove the theorem for logic $K$, so the result is extended in all the logics:

   Base Step. For $F$ a propositional variable $A$, $A^{hs} = A$ and $\vdash_K A \leftrightarrow A$ is a propositional tautology and so, derivable in $K$. Similar proofs are given for the cases $F = \top$ and $F = \bot$.

   Induction Step. By induction hypothesis we have $P$ and $Q$ modal formulae with

   $$(1) \vdash_K P \leftrightarrow P^{hs} \quad \text{and} \quad (2) \vdash_K Q \leftrightarrow Q^{hs}$$

   respectively. There are seven cases:

   (a) $F = \neg P$. Then, $(\neg P)^{hs} = \neg P^{hs}$. Applying (MP) to the tautology $\vdash_K (P \leftrightarrow P^{hs}) \supset (\neg P \leftrightarrow \neg P^{hs})$ and (1) yields $\vdash_K \neg P \leftrightarrow \neg P^{hs}$, as required.

   (b) $F = \Box P$. Then $(\Box P)^{hs} = \Box P^{hs}$. Applying (MP) to the tautology $\vdash_K (P \leftrightarrow P^{hs}) \supset (P \supset P^{hs})$ and (1) gives $\vdash_K P \supset P^{hs}$, from which by (DR1) we derive $\vdash_K \Box P \supset \Box P^{hs}$. Similarly we can derive $\vdash_K \Box P^{hs} \supset \Box P$. Finally, applying (MP) to the latter two theorems and the propositional tautology (PCi) yields $\vdash_K \Box P \leftrightarrow \Box P^{sh}$.

(c) $F = \diamond P$. Then $(\diamond P)^{hs} = \diamond P^{hs}$. Replacing in (b) the result of (a) with (1) yields $\vdash_K \Box\neg P \leftrightarrow \Box\neg P^{hs}$, from which by (DRt) and (Def M) we get $\vdash_K \diamond P \leftrightarrow \diamond P^{hs}$, as required.

(d) $F = P \vee Q$. Then, $(P \vee Q)^{hs} = P^{hs} \vee Q^{hs}$. Applying (MP) to the tautology $\vdash_K (P \leftrightarrow P^{hs}) \supset ((Q \leftrightarrow Q^{hs}) \supset ((P \vee Q) \leftrightarrow (P^{hs} \vee Q^{hs})))$ with (1) and (2) yields $\vdash_K (P \vee Q) \leftrightarrow (P^{hs} \vee Q^{hs})$, as required.

(e) $F = P \wedge Q$. Similar to the above.

(f) $F = P \supset Q$. Then, $(P \supset Q)^{hs} = \neg P^{hs} \vee Q^{hs}$. $\vdash_K (\neg P \vee Q) \leftrightarrow (\neg P^{hs} \vee Q^{hs})$ is derived as in (d) with (1) replaced by the result of (a) and $\vdash_K (P \supset Q) \leftrightarrow (\neg P \vee Q)$ is a tautology. Applying (MP) to the appropriate instance of the tautology (PCe) and the two theorems mentioned above, we get $\vdash_K P \supset Q \leftrightarrow \neg P^{hs} \vee Q^{hs}$, as required.

(g) $F = P \leftrightarrow Q$. Similar to the above.

2. Function $()^h$ maps $\mathcal{V}$ to a formula $R^h$ and is guided by a specific prestructure $R$ in $\mathcal{V}$, so that the reverse function $()^s$ will result a structure that has $R$ as a prestructure. Obviously, for all prestructures $T$ in $\mathcal{V}$, $R = T$ will hold again so that all $T$ will be in $\mathcal{V}^{sh}$. □

The following theorem establishes the equivalence (up to provability) of system $SKSg_{\{kt45\}}$ to $S5$.

**Theorem 8** *$\vdash_{SKSg_{\{kt45\}}} \mathcal{V}$ iff $\vdash_{S5} \mathcal{V}^h$.*

**Proof**. From right to left. Let $\mathcal{V}$ be a structure and $\mathcal{V}^h$ a theorem in $S5$. We need to show that $\mathcal{V}$ is provable in $SKSg_{\{kt45\}}$. Thus, we prove that every axiom in $S5$ is provable in $SKSg_{\{kt45\}}$ and that proofs in $SKSg_{\{kt45\}}$ are closed under substitution, necessitation and modus ponens:

**Classical tautologies** . They are theorems in $SKSg$, since it is equivalent to propositional logic, and so they are in $SKSg_{\{kt45\}}$.

**Axiom DM** . $\diamond R^h \leftrightarrow \neg\Box\neg R^h = (\diamond R^h \supset \neg\Box\neg R^h) \wedge (\neg\Box\neg R^h \supset \diamond R^h) = (\neg\diamond R^h \vee \neg\Box\neg R^h) \wedge (\neg\neg\Box\neg R^h \vee \diamond R^h)$. Then, $\mathcal{V} = ([\Box\overline{R}, \diamond R], [\Box\overline{R}, \diamond R])$. Applying to it twice the interaction rule $(i\downarrow)$ yields $\mathtt{tt}$ as required.

**Axiom K** . $\Box(R^h \supset T^h) \supset (\Box R^h \supset \Box T^h) = \neg \Box (\neg R^h \vee T^h) \vee (\neg \Box R^h \vee \Box T^h)$. Then, $\mathcal{V} = [\diamond(R, \overline{T}), \diamond\overline{R}, \Box T]$. Take the proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{tt}}{\Box[R,\overline{R}]}\ i\downarrow}{\Box([R,\overline{R}],[T,\overline{T}])}\ i\downarrow}{\Box[T,([R,\overline{R}],\overline{T})]}\ s}{\Box[(R,\overline{T}),\overline{R},T]}\ s}{[\diamond(R,\overline{T}),\Box[\overline{R},T]]}\ k\downarrow}{[\diamond(R,\overline{T}),\diamond\overline{R},\Box T]}\ k\downarrow
$$

**Axiom T** . $\Box R^h \supset R^h = \neg\Box R^h \vee R^h$. Then $\mathcal{V} = [\diamond\overline{R}, R]$. Take the proof $\dfrac{\dfrac{\mathsf{tt}}{[\diamond\overline{R},\Box R]}\ i\downarrow}{[\diamond\overline{R},R]}\ t\downarrow$ .

**Axiom 4** . $\Box R^h \supset \Box\Box R^h = \neg \Box R^h \vee \Box\Box R^h$. Then, $\mathcal{V} = [\diamond\overline{R}, \Box\Box R]$ with proof

$$
\dfrac{\dfrac{\mathsf{tt}}{[\diamond\diamond\overline{R},\Box\Box R]}\ i\downarrow}{[\diamond\overline{R},\Box\Box R]}\ 4\downarrow
$$

**Axiom 5** . $\diamond R^h \supset \Box\diamond R^h = \neg\diamond R^h \vee \Box\diamond R^h$. Then $\mathcal{V} = [\Box\overline{R}, \Box\diamond R]$. Take the proof

$$
\dfrac{\dfrac{\mathsf{tt}}{[\diamond\Box\overline{R},\Box\diamond R]}\ i\downarrow}{[\Box\overline{R},\Box\diamond R]}\ 5\downarrow
$$

**Necessitation** . Assume $\vdash_{SKSg_{\{kt45\}}} R$ with proof $\Pi$. Applying the same proof $\Pi$ a level deeper we get a proof of $\Box R$ since, by definition, $\Box\mathsf{tt} = \mathsf{tt}$.

**Modus Ponens** . Assume $\vdash_{SKSg_{\{kt45\}}} [\overline{R}, T]$ with proof $\Pi_1$ and $\vdash_{SKSg_{\{kt45\}}} R$ with proof $\Pi_2$. Then build the proof

$$
\dfrac{\dfrac{\begin{array}{c}\mathsf{tt}\\ \Big\Vert\,\Pi_1;\Pi_2\\ ([\overline{R},T],R)\end{array}}{[(R,\overline{R}),T]}\ s}{T}\ i\uparrow
$$

.

From left to right. We need to show that for every rule $\dfrac{S\{W\}}{S\{Z\}}$ and equation $R_1 = R_2$, $\vdash S\{W\}^h \supset S\{Z\}^h$ and $\vdash S\{R_1\}^h \leftrightarrow S\{R_2\}^h$ are derivable in $S5$. Then, for every proof $\Pi$ of $\mathcal{V}$ with a sequent of rule applications and equations (reading the proof top-down): $\mathsf{tt}, \mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_{n-1}, \mathcal{V}$ $(n \geq 1)$ we get a chain of $S5$ theorems $\vdash_{S5} \top \supset \mathcal{V}_1^h$ (or $\vdash_{S5} \top \leftrightarrow \mathcal{V}_1^h$), $\vdash_{S5} \mathcal{V}_1^h \supset \mathcal{V}_2^h$ (or $\vdash_{S5} \mathcal{V}_1^h \leftrightarrow \mathcal{V}_2^h$), $\ldots$, $\vdash_{S5} \mathcal{V}_{n-1}^h \supset \mathcal{V}^h$ (or $\vdash_{S5} \mathcal{V}_{n-1}^h \leftrightarrow \mathcal{V}^h$), to which we replace every equality $\vdash_{S5} \mathcal{V}_i^h \leftrightarrow \mathcal{V}_{i+1}^h$ with the theorem $\vdash_{S5} \mathcal{V}_i^h \supset \mathcal{V}_{i+1}^h$

(this is done by (MP) and the appropriate tautology). Finally, a chain of (PCt) applications yields $\vdash_{S5} \top \supset \mathcal{V}^h$, which is equivalent to $\vdash_{S5} \mathcal{V}^h$. We start by proving the equations:

**Equations**

By lemma 1, which shows that Hilbert axioms are applicable deep inside formulae, it suffices to prove $\vdash_{S5} R_1^h \leftrightarrow R_2^h$. The theorems given by the equations of associativity, commutativity and the two classical ones of identity are propositional tautologies. For the modal equations of identity we have:

$\mathbf{tt} = \Box\mathbf{tt}$ . Applying (MP) to the propositional tautology $\vdash_{S5} \Box\top \supset (\top \leftrightarrow \Box\top)$ and $\vdash_{S5} \Box\top$ (which is the result from applying necessitation to the propositional tautology $\vdash_{S5} \top$), yields $\vdash_{S5} \top \leftrightarrow \Box\top$, as required.

$\mathbf{ff} = \Diamond\mathbf{ff}$ . Applying (DRt) to the previous result together with (Def M) we get as expected $\vdash_{S5} \bot \leftrightarrow \Diamond\bot$.

**Rules**

We prove that, given a rule $\dfrac{S\{W\}}{S\{Z\}}$ , $\vdash S\{W\}^h \supset S\{Z\}^h$ is derivable in $S5$. Again, using lemma 1 it suffices to prove that $\vdash W^h \supset Z^h$ is derivable in $S5$. The theorems for the classical rules are propositional tautologies. For the modal rules:

**Down rules**

**Rule** $\{k\downarrow\}$ . Applying (DR1) to the tautology $\vdash_{S5} (R^h \vee T^h) \supset (\neg T^h \supset R^h)$ yields $\vdash_{S5} \Box(R^h \vee T^h) \supset \Box(\neg T^h \supset R^h)$, from which by (K) and (PCt) we obtain $\vdash_{S5} \Box(R^h \vee T^h) \supset (\Box\neg T^h \supset \Box R^h)$. Now, applying (PCt) to the latter and the tautology $\vdash_{S5} (\Box\neg T^h \supset \Box R^h) \supset (\Box R^h \vee \neg\Box\neg T^h)$ yields $\vdash_{S5} \Box(R^h \vee T^h) \supset (\Box R^h \vee \neg\Box\neg T^h)$, from which the desired theorem easily follows by (Def M).

**Rule** $\{t\downarrow\}$ . $\vdash_{S5} \Box R^h \supset R^h$ follows immediately from axiom $T$.

**Rule** $\{4\downarrow\}$ . $\vdash_{S5} \Diamond\Diamond R^h \supset \Diamond R^h$ is obtained by applying (DRt) to axiom 4.

**Rule** $\{5\downarrow\}$ . As in $4\downarrow$, $\vdash_{S5} \Diamond\Box R^h \supset \Box R^h$ is obtained by applying (DRt) to axiom 5.

**Up rules**

The proof for the up rules follows from the proof of their corresponding down rules: We know that $\vdash S\{W\}^h \supset S\{Z\}^h$ is derivable in the corresponding Hilbert System. By $\vdash (S\{W\}^h \supset S\{Z\}^h) \supset (\neg S\{Z\}^h \supset \neg S\{W\}^h)$, which is a propositional tautology, and Modus Ponens, we get that $\vdash \neg S\{Z\}^h \supset \neg S\{W\}^h$ is also derivable, which is precisely what do we need. $\Box$

**Completeness for $KSg_{\{kt45\}}$**

Cut-elimination for $SKSg_{\{kt45\}}$ follows from the elimination of all its up rules. This is obtained via translations of cut-free proofs from the hypersequential system $GS5$ to $KSg_{\{kt45\}}$. Since $GS5$ admits cut-elimination trivially follows that system $KSg_{\{kt45\}}$ is complete. For this, we are going to use the translation $\phi_{\_}$ on hypersequents, defined in Section 3.3. A recursive definition for $\phi_{\_}$ follows:

**Definition 14** *The translation of a sequent $S$ to a modal formula is recursively defined as follows:*

1. $\psi_S = \bot$, *if $S$ is empty.*
2. $\psi_S = \neg P \vee \psi_{\Gamma \vdash \Delta}$, *if $S$ is of the form $P, \Gamma \vdash \Delta$.*
3. $\psi_S = P \vee \psi_{\Gamma \vdash \Delta}$, *if $S$ is of the form $\Gamma \vdash \Delta, P$.*

*For homomorphic reasons, we use the letters $P, Q, \ldots$ to range over modal formulae instead of the $\alpha, \beta, \ldots$ used in the hypersequents.*

**Definition 15** *The translation of a hypersequent $G$ to a modal formula is recursively defined as follows:*

1. $\phi_G = \bot$, *if $G$ is empty.*
2. $\phi_G = \phi_{G_1} \vee \Box\psi_{\Gamma \vdash \Delta} \vee \phi_{H_1}$, *if $G$ is of the form $G_1|\Gamma \vdash \Delta|H_1$.*

Note: Given two hypersequents $G$ and $H$, $(\phi_{G|H})^s = [(\phi_G)^s, (\phi_H)^s]$, where $()^s$ is the translation of a modal formula to a structure in the calculus of structures.

**Lemma 2** *For a hypersequent $G$ and a structure $R$, the following rule is derivable in $KSg_{\{k5\}}$:* $g \downarrow \dfrac{S\{\Box[R, (\phi_G)^s]\}}{S[\Box R, (\phi_G)^s]}$

**Proof**. By induction on the length of a hypersequent $G$.

Base step. $G$ is empty. Then $(\phi_G)^s = \mathbf{ff}$ and $S\{\Box[R, \mathbf{ff}]\} = S\{\Box R\} = S[\Box R, \mathbf{ff}]$.

Induction step. Let $G = G_1|\Gamma \vdash \Delta|H_1$. Then $(\phi_G)^s = [\Box(\psi_{\Gamma\vdash\Delta})^s, (\phi_{G_1|H_1})^s]$.

By induction hypothesis we have $g \downarrow \dfrac{S\{\Box[R, (\phi_{G_1|H_1})^s]\}}{S[\Box R, (\phi_{G_1|H_1})^s]}$ and the derivation

$$
\begin{array}{r}
k \downarrow \dfrac{S\{\Box[R, (\phi_{G_1|H_1})^s, \Box(\psi_{\Gamma\vdash\Delta})^s]\}}{S[\Box[R, (\phi_{G_1|H_1})^s], \Diamond\Box(\psi_{\Gamma\vdash\Delta})^s]} \\
5 \downarrow \dfrac{}{S[\Box[R, (\phi_{G_1|H_1})^s], \Box(\psi_{\Gamma\vdash\Delta})^s]} \\
g \downarrow \dfrac{}{S[\Box R, (\phi_{G_1|H_1})^s, \Box(\psi_{\Gamma\vdash\Delta})^s]}
\end{array}
$$

proves the lemma. $\Box$

**Proposition 5** $\vdash_{GS5^-} G$ *implies* $\vdash_{KSg_{\{kt45\}}} (\phi_G)^s$ .

**Proof**. By induction on the length of a derivation $D$, we prove that for every derivation step in $GS5^-$, there is a proof of its conclusion in $KSg_{\{kt45\}}$, under the assumption that its premises are provable.

Base step. $G$ is an axiom. Then, $\phi_G = \Box\phi_{P\vdash P} = \Box(\neg P \vee P)$ and, for $(\phi_G)^s = \Box[\overline{P^s}, P^s]$,

$$i\downarrow \frac{= \dfrac{\mathtt{tt}}{\Box\mathtt{tt}}}{\Box[\overline{P^s}, P^s]}$$ is a proof in KSg.

Induction step. Here Lemma 1 is freely applied.

1. Weakening. For the internal versions by induction hypothesis we have $\vdash_{KSg_{\{kt45\}}} [\Box(\psi_{\Gamma\vdash\Delta})^s, (\phi_{G|H})^s]$ and $\Pi$ a proof of it. Then, build the proofs

$$\begin{array}{c}\mathtt{tt}\\ \Big\Vert \Pi\\ w\downarrow \dfrac{[\Box(\psi_{\Gamma\vdash\Delta})^s, (\phi_{G|H})^s]}{[\Box[\overline{P^s}, (\psi_{\Gamma\vdash\Delta})^s], (\phi_{G|H})^s]}\end{array} \quad \text{and} \quad \begin{array}{c}\mathtt{tt}\\ \Big\Vert \Pi\\ w\downarrow \dfrac{[\Box(\psi_{\Gamma\vdash\Delta})^s, (\phi_{G|H})^s]}{[\Box[P^s, (\psi_{\Gamma\vdash\Delta})^s], (\phi_{G|H})^s]}\end{array}$$

   for left and right weakening, respectively. For external weakening, by induction hypothesis we have $\vdash_{KSg_{\{kt45\}}} (\phi_{G|H})^s$ with proof $\Pi$. Build

$$\begin{array}{c}\mathtt{tt}\\ \Big\Vert \Pi\\ w\downarrow \dfrac{(\phi_{G|H})^s}{[\Box(\psi_{\Gamma\vdash\Delta})^s, (\phi_{G|H})^s]}\end{array}$$

2. Contraction. Similar to weakening.

3. Modalized Splitting. By induction hypothesis $\vdash_{KSg_{\{kt45\}}} [\Box[\Diamond\overline{\Gamma_1^s}, \Box\Delta_1^s, (\psi_{\Gamma_2\vdash\Delta_2})^s], (\phi_{G|H})^s]$

holds with proof Π. Build the proof

$$
\begin{array}{r}
= \dfrac{\mathbf{tt}}{\Box\mathbf{tt}} \\
\Big\Vert \Pi \\
k\downarrow \dfrac{\Box[\Box[\diamond\overline{\Gamma_1^s}, \Box\Delta_1^s, (\psi_{\Gamma_2\vdash\Delta_2})^s], (\phi_{G|H})^s]}{\Box[\diamond\diamond\overline{\Gamma_1^s}, \Box[\Box\Delta_1^s, (\psi_{\Gamma_2\vdash\Delta_2})^s], (\phi_{G|H})^s]} \\
k\downarrow \dfrac{}{\Box[\diamond\diamond\overline{\Gamma_1^s}, \diamond\Box\,\Delta_1^s, \Box(\psi_{\Gamma_2\vdash\Delta_2})^s, (\phi_{G|H})^s]} \\
4\downarrow \dfrac{}{\Box[\diamond\overline{\Gamma_1^s}, \diamond\Box\,\Delta_1^s, \Box(\psi_{\Gamma_2\vdash\Delta_2})^s, (\phi_{G|H})^s]} \\
5\downarrow \dfrac{}{\Box[\diamond\overline{\Gamma_1^s}, \Box\Delta_1^s, \Box(\psi_{\Gamma_2\vdash\Delta_2})^s, (\phi_{G|H})^s]} \\
g\downarrow \dfrac{}{[\Box[\diamond\overline{\Gamma_1^s}, \Box\Delta_1^s, \Box(\psi_{\Gamma_2\vdash\Delta_2})^s], (\phi_{G|H})^s]} \\
k\downarrow \dfrac{}{[\Box[\diamond\overline{\Gamma_1^s}, \Box\Delta_1^s], \diamond\Box\,(\psi_{\Gamma_2\vdash\Delta_2})^s, (\phi_{G|H})^s]} \\
5\downarrow \dfrac{}{[\Box[\diamond\overline{\Gamma_1^s}, \Box\Delta_1^s], \Box(\psi_{\Gamma_2\vdash\Delta_2})^s, (\phi_{G|H})^s]}
\end{array}
$$

Note: $\diamond\Gamma, \Box\Delta$ are *abbreviations* for $[\diamond\Gamma_1, \ldots, \diamond\Gamma_n]$ and $[\Box\Delta_1, \ldots, \Box\Delta_m]$, for $\Gamma_1, \ldots, \Gamma_n$ elements of $\Gamma$ (similar for $\Delta$). The $k\downarrow$ rule is applied as many times as the elements in $\Gamma/\Delta$.

4. Conjuction. For the left introduction rule for conjuction, by induction hypothesis $\vdash_{KSg_{\{kt45\}}} [\Box[\overline{P^s}, (\psi_{\Gamma\vdash\Delta})^s], (\phi_{G|H})^s]$ holds. Applying $w\downarrow$ yields $\vdash_{KSg_{\{kt45\}}} [\Box[\overline{P^s}, \overline{Q^s}, (\psi_{\Gamma\vdash\Delta})^s], (\phi_{G|H})^s]$, as it is needed.
   For the right rule, by induction hypothesis $\vdash_{KSg_{\{kt45\}}} [\Box[P^s, (\psi_{\Gamma\vdash\Delta})^s], (\phi_{G|H})^s]$ and $\vdash_{KSg_{\{kt45\}}} [\Box[Q^s, (\psi_{\Gamma\vdash\Delta})^s], (\phi_{G|H})^s]$ hold with proofs $Pi_1$ and $Pi_2$ respectively. Build the proof

$$
\begin{array}{r}
= \dfrac{\mathbf{tt}}{\Box(\mathbf{tt}, \mathbf{tt})} \\
\Big\Vert \Pi_1; \Pi_2 \\
s \dfrac{\Box([\Box[P^s, (\psi_{\Gamma\vdash\Delta})^s], (\phi_{G|H})^s], [\Box[Q^s, (\psi_{\Gamma\vdash\Delta})^s], (\phi_{G|H})^s])}{\Box[(\Box[P^s, (\psi_{\Gamma\vdash\Delta})^s], [\Box[Q^s, (\psi_{\Gamma\vdash\Delta})^s], (\phi_{G|H})^s]), (\phi_{G|H})^s]} \\
s \dfrac{}{\Box[(\Box[P^s, (\psi_{\Gamma\vdash\Delta})^s], \Box[Q^s, (\psi_{\Gamma\vdash\Delta})^s]), (\phi_{G|H})^s, (\phi_{G|H})^s]} \\
c\downarrow \dfrac{}{\Box[(\Box[P^s, (\psi_{\Gamma\vdash\Delta})^s], \Box[Q^s, (\psi_{\Gamma\vdash\Delta})^s]), (\phi_{G|H})^s]} \\
t\downarrow \dfrac{}{\Box[([P^s, (\psi_{\Gamma\vdash\Delta})^s], \Box[Q^s, (\psi_{\Gamma\vdash\Delta})^s]), (\phi_{G|H})^s]} \\
t\downarrow \dfrac{}{\Box[([P^s, (\psi_{\Gamma\vdash\Delta})^s], [Q^s, (\psi_{\Gamma\vdash\Delta})^s]), (\phi_{G|H})^s]} \\
s \dfrac{}{\Box[([P^s, (\psi_{\Gamma\vdash\Delta})^s], Q^s), (\psi_{\Gamma\vdash\Delta})^s, (\phi_{G|H})^s]} \\
s \dfrac{}{\Box[(P^s, Q^s), (\psi_{\Gamma\vdash\Delta})^s, (\psi_{\Gamma\vdash\Delta})^s, (\phi_{G|H})^s]} \\
c\downarrow \dfrac{}{\Box[(P^s, Q^s), (\psi_{\Gamma\vdash\Delta})^s, (\phi_{G|H})^s]} \\
g\downarrow \dfrac{}{[\Box[(P^s, Q^s), (\psi_{\Gamma\vdash\Delta})^s], (\phi_{G|H})^s]}
\end{array}
$$

5. Disjunction and implication. These cases are similar to conjuction.

$$
i\downarrow\frac{S\{\mathsf{tt}\}}{S[R,\overline{R}]}\qquad w\downarrow\frac{S\{\mathsf{ff}\}}{S\{R\}}\qquad c\downarrow\frac{S[R,R]}{S\{R\}}
$$

$$
s\frac{S([R,T],U)}{S[(R,U),T]}\qquad k\downarrow\frac{S\{\Box[R,T]\}}{S[\Box R,\Diamond T]}
$$

$$
t\downarrow\frac{S\{\Box R\}}{S\{R\}}\qquad 4\downarrow\frac{S\{\Diamond\Diamond R\}}{S\{\Diamond R\}}\qquad 5\downarrow\frac{S\{\Diamond\Box R\}}{S\{\Box R\}}
$$

Figure 7: System $KSg_{\{kt45\}}$

6. Negation. Here the results coincide with the induction hypothesis.
7. Rules for $\Box$. By induction hypothesis we have $\vdash_{KSg_{\{kt45\}}} [\Box[\overline{P^s},(\psi_{\Gamma\vdash\Delta})^s],(\phi_{G|H})^s]$ with proof $Pi_1$ and $\vdash_{KSg_{\{kt45\}}} [\Box[\Diamond\overline{\Gamma^s},P^s],(\phi_{G|H})^s]$ with proof $\Pi_2$. Build the proofs

$$
\begin{array}{c}
= \dfrac{\mathsf{tt}}{\Box\mathsf{tt}} \\
\Big\| \Pi_1 \\
k\downarrow \dfrac{\Box[\Box[\overline{P^s},(\psi_{\Gamma\vdash\Delta})^s],(\phi_{G|H})^s]}{\dfrac{\Box[\Diamond\overline{P^s},\Box(\psi_{\Gamma\vdash\Delta})^s,(\phi_{G|H})^s]}{\dfrac{\Box[\Diamond\overline{P^s},(\psi_{\Gamma\vdash\Delta})^s,(\phi_{G|H})^s]}{[\Box[\Diamond\overline{P^s},(\psi_{\Gamma\vdash\Delta})^s],(\phi_{G|H})^s]}\,g\downarrow}\,t\downarrow}
\end{array}
\quad\text{and}\quad
\begin{array}{c}
= \dfrac{\mathsf{tt}}{\Box\mathsf{tt}} \\
\Big\| \Pi_2 \\
k\downarrow \dfrac{\Box[\Box[\Diamond\overline{\Gamma^s},P^s],(\phi_{G|H})^s]}{\dfrac{\Box[\Diamond\Diamond\overline{\Gamma^s},\Box P^s,(\phi_{G|H})^s]}{\dfrac{\Box[\Diamond\overline{\Gamma^s},\Box P^s,(\phi_{G|H})^s]}{[\Box[\Diamond\overline{\Gamma^s},\Box P^s],(\phi_{G|H})^s]}\,g\downarrow}\,4\downarrow}
\end{array}
$$

   for the left and right introduction rules for $\Box$, respectively. $\Box$

**Theorem 9** $\vdash_{S5} F$ *implies* $\vdash_{KSg_{\{kt45\}}} F^s$ .

**Proof**. Since $S5$ and $GS5$ are equivalent $\vdash_{GS5} \vdash F$ and so, $\vdash_{GS5^-} \vdash F$, which implies $\vdash_{KSg_{\{kt45\}}} \Box(\psi_{\vdash F})^s$ (Proposition 5). Since $F$ is a single formula, we have $\vdash_{KSg_{\{kt45\}}} \Box F^s$. Applying $t\downarrow$ yields $\vdash_{KSg_{\{kt45\}}} F^s$, as needed. $\Box$

**Corollary 1** *Systems* $KSg_{\{kt45\}}$ *and* $S5$ *are equivalent.*

**Corollary 2** *System* $SKSg_{\{kt45\}}$ *admits elimination of its up-rules.*

The cut-free system $KSg_{\{kt45\}}$, is shown in Figure 7.

**System $SKS_{\{kt45\}}$: A local system**

Locality for $SKSg_{\{kt45\}}$ can be obtained by extending the results for $SKSg$ to the modal cases. We remind that a system is called *local* if all of its rules have local nature, meaning that no inspection of structures of unbounded length is required. Locality is obtained by restricting all the non-local rules to their atomic forms. In general, the atomic form of a rule is the result of applying uniform substitutions on the arbitrary structures occuring in the rule, with atoms. For instance, the atomic form of the interaction rule $\{i\downarrow\}$

$$i\downarrow \frac{S\{\mathbf{tt}\}}{S[R,\overline{R}]} \quad \text{is} \quad ai\downarrow \frac{S\{\mathbf{tt}\}}{S[a,\overline{a}]}$$

Notice that their names carry an $a$ (for *atomic*) in front of the general name. The restriction of the non-local rules to their atomic ones has been done in a way similar to the case for **SKSgq**, the system for predicate logic (see Brünnler [8]). The non-local rules are again identity, weakening, contraction and their duals. By adding to $SKSg_{\{kt45\}}$ the local rules

$$l\downarrow \frac{S[\Box R,\Box T]}{S\{\Box[R,T]\}} \qquad l\uparrow \frac{S\{\Diamond(R,T)\}}{S(\Diamond R,\Diamond T)}$$

$$j\downarrow \frac{S[\Diamond R,\Diamond T]}{S\{\Diamond[R,T]\}} \qquad j\uparrow \frac{S\{\Box(R,T)\}}{S(\Box R,\Box T)}$$

(which are derivable in $SKSg_{\{k\}}$) and restricting the non-local rules to atoms, we obtain a local system which is weakly equivalent to the general one. This means that they share the same theorems, but not necessarily the same derivations.

**Proposition 6** *The rules* $\{l\downarrow, l\uparrow, j\downarrow, j\uparrow\}$ *are derivable in* $SKSg_{\{k\}}$.

**Proof**. For $l\downarrow$ take the derivation:

$$c\downarrow \frac{w\downarrow \dfrac{w\downarrow \dfrac{S[\Box R,\Box T]}{S[\Box R,\Box[R,T]]}}{S[\Box[R,T],\Box[R,T]]}}{S\{\Box[R,T]\}} \ .$$

For $j\downarrow$ the proof is similar. For the up rules take the dual of the down rule proofs. □

The rule $uw\downarrow \dfrac{S\{\mathbf{ff}\}}{S\{\Box\mathbf{ff}\}}$ is obviously local and derivable in $SKSg_{\{k\}}$, since it is an instance of weakening. By adding $uw\downarrow$ and its dual to the local system for $S5$, we obtain strong equivalence to its general system. The local system for $SKSg_{\{kt45\}}$, called $SKS_{\{kt45\}}$, is shown in Figure 8.

**Theorem 10** *Systems* $SKSg_{\{kt45\}}$ *and* $SKS_{\{kt45\}}$ *are strongly equivalent.*

$$ai\downarrow \frac{S\{\mathtt{tt}\}}{S[a,\bar{a}]} \qquad ai\uparrow \frac{S(a,\bar{a})}{S\{\mathtt{ff}\}}$$

$$aw\downarrow \frac{S\{\mathtt{ff}\}}{S\{a\}} \qquad aw\uparrow \frac{S\{a\}}{S\{\mathtt{tt}\}}$$

$$s\,\frac{S([R,T],U)}{S[(R,U),T]}$$

$$uw\downarrow \frac{S\{\mathtt{ff}\}}{S\{\Box\mathtt{ff}\}} \qquad uw\uparrow \frac{S\{\diamond\mathtt{tt}\}}{S\{\mathtt{tt}\}}$$

$$ac\downarrow \frac{S[a,a]}{S\{a\}} \qquad ac\uparrow \frac{S\{a\}}{S(a,a)}$$

$$k\downarrow \frac{S\{\Box[R,T]\}}{S[\Box R,\diamond T]} \qquad k\uparrow \frac{S(\Box R,\diamond T)}{S\{\diamond(R,T)\}}$$

$$m\,\frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

$$l\downarrow \frac{S[\Box R,\Box T]}{S\{\Box[R,T]\}} \qquad l\uparrow \frac{S\{\diamond(R,T)\}}{S(\diamond R,\diamond T)}$$

$$j\downarrow \frac{S[\diamond R,\diamond T]}{S\{\diamond[R,T]\}} \qquad j\uparrow \frac{S\{\Box(R,T)\}}{S(\Box R,\Box T)}$$

$$t\downarrow \frac{S\{\Box R\}}{S\{R\}} \qquad t\uparrow \frac{S\{R\}}{S\{\diamond R\}}$$

$$4\downarrow \frac{S\{\diamond\diamond R\}}{S\{\diamond R\}} \qquad 4\uparrow \frac{S\{\Box R\}}{S\{\Box\Box R\}}$$

$$5\downarrow \frac{S\{\diamond\Box R\}}{S\{\Box R\}} \qquad 5\uparrow \frac{S\{\diamond R\}}{S\{\Box\diamond R\}}$$

Figure 8: System $SKS_{\{kt45\}}$

**Proof**. It is enough to prove the theorem for the $K$ systems, since all modal rules are local and do not change in the local system. Proposition 6 shows that all the rules in $SKS_{\{k\}}$ are derivable in $SKSg_{\{k\}}$. For the other direction, we extend theorem 3.23 presented in Brünnler [8] to the modal cases:

By induction hypothesis rules $i\downarrow$, $w\downarrow$ and $c\downarrow$ are applicable to structure $T$:

$$i\downarrow\frac{S\{\mathtt{tt}\}}{S[T,\overline{T}]}\quad,\qquad w\downarrow\frac{S\{\mathtt{ff}\}}{S\{T\}}\quad and\quad c\downarrow\frac{S[T,T]}{S\{T\}}\quad.$$

Then, the following derivations are built up only from rules of the local system and the induction hypothesis. For $R$ a modal extention of $T$, we have the following cases:

1. $R = \Box T$.

$$k\downarrow\frac{i\downarrow\dfrac{=\dfrac{S\{\mathtt{tt}\}}{S\{\Box\mathtt{tt}\}}}{S\{\Box[T,\overline{T}]\}}}{S[\Box T,\Diamond\overline{T}]}\quad,\qquad w\downarrow\frac{uw\downarrow\dfrac{S\{\mathtt{ff}\}}{S\{\Box\mathtt{ff}\}}}{S\{\Box T\}}\quad and\quad c\downarrow\frac{l\downarrow\dfrac{S[\Box T,\Box T]}{S\{\Box[T,T]\}}}{S\{\Box T\}}\quad.$$

2. $R = \Diamond T$.

$$k\downarrow\frac{i\downarrow\dfrac{=\dfrac{S\{\mathtt{tt}\}}{S\{\Box\mathtt{tt}\}}}{S\{\Box[T,\overline{T}]\}}}{S[\Diamond T,\Box\overline{T}]}\quad,\qquad w\downarrow\frac{=\dfrac{S\{\mathtt{ff}\}}{S\{\Diamond\mathtt{ff}\}}}{S\{\Diamond T\}}\quad and\quad c\downarrow\frac{j\downarrow\dfrac{S[\Diamond T,\Diamond T]}{S\{\Diamond[T,T]\}}}{S\{\Diamond T\}}\quad.$$

The derivations for the up rules are the duals of the above. □

The above theorem holds also for systems $KSg_{\{kt45\}}$ and $KS_{\{kt45\}}$. As a result, we obtain cut-elimination for the local system.

### 4.3 Other normal modal systems

As it has been already mentioned, the system for logic $K$ in the calculus of structures is system $SKSg_{\{k\}}$ and is formulated by adding to the system for propositional logic $SKSg$ the pair of rules $k\downarrow$ and $k\uparrow$. Accordingly, systems for $K4$, $D$, $D4$, $M$ and $S4$ are obtained by extending $SKSg$ with the rules that correspond to their axioms. The systems are then called $SKSg_{\{k4\}}$, $SKSg_{\{kd\}}$, $SKSg_{\{kd4\}}$, $SKSg_{\{kt\}}$ and $SKSg_{\{kt4\}}$, respectively. All their rules except the $d$ rule have been presented earlier in this section (for the modal rules see Figure 6). The $d$ rule is formulated as follows:

$$d\,\frac{S\{\Box R\}}{S\{\Diamond R\}}$$

Note that the rule is self-dual, which means that the down and up rules are identical.

Equivalence of the systems to the relevant logics is shown in the same way equivalence of system $SKSg_{\{kt45\}}$ to $S5$ has been proved in Section 4.2 (see Theorem 8). For the systems without the $d$ rule, that is systems $SKSg_{\{k\}}$, $SKSg_{\{k4\}}$, $SKSg_{\{kt\}}$ and $SKSg_{\{kt4\}}$, their equivalence is included in that proof. For systems $SKSg_{\{kd\}}$ and $SKSg_{\{kd4\}}$, we additionally need to show that (i) axiom $D$ is provable in the systems and (ii) the $d$ rule is derivable in logic $D$:

**Axiom D** . $\Box R \supset \Diamond R = \neg \Box R \vee \Diamond R$. Then $Q = [\Diamond \overline{R}, \Diamond R]$. Take the proof $i\downarrow \dfrac{\dfrac{\mathtt{tt}}{[\Diamond \overline{R}, \Box R]}}{[\Diamond \overline{R}, \Diamond R]} d\downarrow$

**Rule** $\{d\}$ . $\vdash_D \Box R \supset \Diamond R$ follows immediately from axiom $D$.

**Theorem 11** *Systems $KSg_{\{k\}}$, $KSg_{\{k4\}}$, $KSg_{\{kd\}}$, $KSg_{\{kd4\}}$, $KSg_{\{kt\}}$ and $KSg_{\{kt4\}}$ are complete for $K$, $K4$, $D$, $D4$, $M$ and $S4$, respectively.*

For the systems $KSg_{\{k\}}$, $KSg_{\{kd\}}$, $KSg_{\{kt\}}$ and $KSg_{\{kt4\}}$ the above theorem follows from the cut-elimination results for their symmetric systems. This is shown following the same technique used for proving cut-elimination for system $SKSg_{\{kt45\}}$, that is, by translating the proofs of a known cut-admissible system to proofs of the system in consideration, in which proofs only down or self-dual rules occur. The cut-admissible systems used in these proofs are the systems in the sequent calculus presented in Section 2.3. Furthermore, for all the systems (and so, also for systems $KSg_{\{k4\}}$ and $KSg_{\{kd4\}}$) a semantic completeness proof is given. Both techniques have been developed and applied by Brünnler to the system for propositional logic $SKSg$. The first result can be found in Brünnler [8], whereas the second one is still unpublished [9].

Lastly, all of the systems can be trivially restricted to their local ones $(S)KS_{\{k\}}$, $(S)KS_{\{k4\}}$, $(S)KS_{\{kd\}}$, $(S)KS_{\{kd4\}}$, $(S)KS_{\{kt\}}$ and $(S)KS_{\{kt4\}}$ respectively, since the proof for locality for system $(S)KSg_{\{kt45\}}$ presented earlier is actually a proof for locality for all the systems that extend system $(S)KSg_{\{k\}}$ with local rules.

## 5 Conclusions and Future Directions

Through a study on the current systems for $S5$, we have verified the importance of deep inference in providing good formulations for normal modal logic. Moreover, we have presented a cut-admissible system for $S5$ in the calculus of structures, a calculus which makes explicit use of deep inference. The system is additionally distinguished by a simple design and a direct way for obtaining the modal rules out of the modal axioms. Also, the system is local, which means that any application of its rules requires the inspection only of formulae of bounded length.

However, the proof of cut-elimination presented here is indirect, since the cuts are eliminated externally in another system with known cut-elimination results. Proving these results internally is a matter of future work, which will also provide (i) completeness for system $KSg_{\{kt5\}}$, a system for $S5$ with minimal rules and (ii) a way for proving completeness for system $KSg_{\{ktb\}}$, a cut-admissible system for logic $B$. The b rules are again systematic and correspond precisely to axiom $B$:

$$b\downarrow \frac{S\{\Diamond \Box R\}}{S\{R\}} \qquad b\uparrow \frac{S\{R\}}{S\{\Box \Diamond R\}}$$

Notice that systems $KSg_{\{kt45\}}$ and $KSg_{\{kt45b\}}$ are strongly equivalent, since the $b\downarrow$ rule is derivable in $KSg_{\{kt45\}}$. Furthermore, since monotonicity fails, another issue of future work is the implementation of techniques for obtaining decidability results for all the systems.

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## A Appendix: The $S5$ systems

$$\frac{}{\alpha \vdash \alpha}\ (\text{ax}) \qquad\qquad \frac{}{\bot \vdash}\ (\bot \vdash)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma', \Gamma \vdash \Delta, \Delta'}\ (\text{extension: out}) \qquad\qquad \frac{\Gamma\ ; \Pi \vdash \Sigma;\ \Delta}{\Gamma\ ; \Pi', \Pi \vdash \Sigma, \Sigma';\ \Delta}\ (\text{extension: in})$$

$$\frac{\Gamma \vdash \Delta, \alpha \qquad \alpha, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ (\text{cut}) \qquad\qquad \frac{\Gamma\ ; \vdash \alpha;\ \Delta}{\Gamma \vdash \Box\alpha, \Delta}\ (\vdash\ \text{exit})$$

$$\frac{\Gamma, \Box\alpha\ ; \Pi \vdash \Sigma;\ \Delta}{\Gamma\ ; \Box\alpha, \Pi \vdash \Sigma;\ \Delta}\ (\text{enter}\ \vdash) \qquad\qquad \frac{\Gamma\ ; \Pi \vdash \Sigma;\ \Box\, \alpha, \Delta}{\Gamma\ ; \Pi \vdash \Sigma, \Box\alpha;\ \Delta}\ (\vdash\ \text{enter})$$

$$\frac{\Gamma_1 \vdash \Delta_1, \alpha, \beta \qquad \beta, \Gamma_2 \vdash \Delta_2, \alpha \qquad \alpha, \beta, \Gamma_3 \vdash \Delta_3}{\alpha \supset \beta, \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3}\ (\supset\vdash\text{: out})$$

$$\frac{\alpha, \Gamma \vdash \Delta, \beta}{\Gamma \vdash \Delta, \alpha \supset \beta}\ (\vdash\supset\text{: out}) \qquad\qquad \frac{\Gamma\ ; \alpha, \Pi \vdash \Sigma, \beta;\ \Delta}{\Gamma\ ; \Pi \vdash \Sigma, \alpha \supset \beta;\ \Delta}\ (\vdash\supset\text{: in})$$

$$\frac{\Gamma\ ; \Pi_1 \vdash \Sigma_1, \alpha, \beta;\ \Delta \qquad \Gamma\ ; \beta, \Pi_2 \vdash \Sigma_2, \alpha;\ \Delta \qquad \Gamma\ ; \alpha, \beta, \Pi_3 \vdash \Sigma_3;\ \Delta}{\Gamma\ ; \alpha \supset \beta, \Pi_1, \Pi_2, \Pi_3 \vdash \Sigma_1, \Sigma_2, \Sigma_3;\ \Delta}\ (\supset\vdash\text{: in})$$

$$\frac{\alpha, \Gamma \vdash \Delta}{\Box\alpha, \Gamma \vdash \Delta}\ (\Box \vdash\text{: out}) \qquad\qquad \frac{\Box\Gamma \vdash \Box\Delta, \alpha}{\Box\Gamma \vdash \Box\Delta, \Box\alpha}\ (\vdash \Box\text{ : out})$$

Figure 9: System $GS5^s$: A system for $S5$

$$\frac{}{\alpha \vdash \alpha}\ (AX) \qquad \frac{\Gamma\ \Box\!\vdash\ \Delta}{\Delta^*\ \Box\!\vdash\ \Gamma^*}\ (TR)$$

$$\frac{\Gamma \vdash \Delta, \alpha}{\alpha^*, \Gamma \vdash \Delta}\ (*_l) \qquad \frac{\alpha, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \alpha^*}\ (*_r)$$

$$\frac{\Gamma\ (\Box)\!\vdash\ \Delta}{\alpha, \Gamma\ (\Box)\!\vdash\ \Delta}\ (W_l) \qquad \frac{\Gamma\ (\Box)\!\vdash\ \Delta}{\Gamma\ (\Box)\!\vdash\ \Delta, \alpha}\ (W_r)$$

$$\frac{\alpha, \beta, \Gamma\ (\Box)\!\vdash\ \Delta}{\alpha \wedge \beta, \Gamma\ (\Box)\!\vdash\ \Delta}\ (\wedge \vdash) \qquad \frac{\Gamma_1\ (\Box)\!\vdash\ \Delta_1, \alpha \qquad \Gamma_2\ (\Box)\!\vdash\ \Delta_2, \beta}{\Gamma_1, \Gamma_2\ (\Box)\!\vdash\ \Delta_1, \Delta_2, \alpha \wedge \beta}\ (\vdash \wedge)$$

$$\frac{\alpha, \Gamma_1\ (\Box)\!\vdash\ \Delta_1 \qquad \beta, \Gamma_2\ (\Box)\!\vdash\ \Delta_2}{\alpha \vee \beta, \Gamma_1, \Gamma_2\ (\Box)\!\vdash\ \Delta_1, \Delta_2}\ (\vee \vdash) \qquad \frac{\Gamma\ (\Box)\!\vdash\ \Delta, \alpha, \beta}{\Gamma\ (\Box)\!\vdash\ \Delta, \alpha \vee \beta}\ (\vdash \vee)$$

$$\frac{-\alpha, \Gamma\ (\Box)\!\vdash\ \Delta}{\neg\alpha, \Gamma\ (\Box)\!\vdash\ \Delta}\ (\neg \vdash) \qquad \frac{\Gamma\ (\Box)\!\vdash\ \Delta, -\alpha}{\Gamma\ (\Box)\!\vdash\ \Delta, \neg\alpha}\ (\vdash \neg)$$

$$\frac{-\alpha, \Gamma_1\ (\Box)\!\vdash\ \Delta_1 \qquad \beta, \Gamma_2\ (\Box)\!\vdash\ \Delta_2}{\alpha \supset \beta, \Gamma_1, \Gamma_2\ (\Box)\!\vdash\ \Delta_1, \Delta_2}\ (\supset\vdash) \qquad \frac{\Gamma\ (\Box)\!\vdash\ \Delta, -\alpha, \beta}{\Gamma\ (\Box)\!\vdash\ \Delta, \alpha \supset \beta}\ (\vdash\supset)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma\ \Box\!\vdash\ \Delta}\ (NC), \text{ where } \Gamma \text{ or } \Delta \text{ either contains only } M\text{-formulae or is empty}$$

$$\frac{\alpha, \Gamma\ (\Box)\!\vdash\ \Delta}{\Box\alpha, \Gamma\ (\Box)\!\vdash\ \Delta}\ (\Box \vdash) \qquad \frac{\Gamma\ \Box\!\vdash\ M\Delta, \alpha}{\Gamma \vdash M\Delta, \Box\alpha}\ (\vdash \Box)$$

$$\frac{\alpha, M\Gamma\ \Box\!\vdash\ \Delta}{\diamond\alpha, M\Gamma \vdash \Delta}\ (\diamond \vdash) \qquad \frac{\Gamma\ (\Box)\!\vdash\ \Delta, \alpha}{\Gamma\ (\Box)\!\vdash\ \Delta, \diamond\alpha}\ (\vdash \diamond)$$

Figure 10: System DSC: a system for $S5$

$$\frac{}{\alpha \vdash \alpha}\,(Axiom) \qquad \frac{G \mid\ \Box\,\Gamma_1, \Gamma_2 \vdash \Box\Delta_1, \Delta_2 \mid H}{G \mid\ \Box\,\Gamma_1 \vdash \Box\Delta_1 \mid \Gamma_2 \vdash \Delta_2 \mid H}\,(MS)$$

$$\frac{G \mid \Gamma \vdash \Delta \mid H}{G \mid \alpha, \Gamma \vdash \Delta \mid H}\,(W_l) \qquad \frac{G \mid \Gamma \vdash \Delta \mid H}{G \mid \Gamma \vdash \Delta, \alpha \mid H}\,(W_r)$$

$$\frac{G \mid \alpha, \alpha, \Gamma \vdash \Delta \mid H}{G \mid \alpha, \Gamma \vdash \Delta \mid H}\,(C_l) \qquad \frac{G \mid \Gamma \vdash \Delta, \alpha, \alpha \mid H}{G \mid \Gamma \vdash \Delta, \alpha \mid H}\,(C_r)$$

$$\frac{G \mid H}{G \mid \Gamma \vdash \Delta \mid H}\,(W\ external) \qquad \frac{G \mid \Gamma \vdash \Delta \mid \Gamma \vdash \Delta \mid H}{G \mid \Gamma \vdash \Delta \mid H}\,(C\ external)$$

$$\frac{G_1 \mid \Gamma_1 \vdash \Delta_1, \alpha \mid H_1 \qquad G_2 \mid \alpha, \Gamma_2 \vdash \Delta_2 \mid H_2}{G_1 \mid G_2 \mid \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \mid H_1 \mid H_2}\,(Cut)$$

$$\frac{G \mid \alpha, \Gamma \vdash \Delta \mid H}{G \mid \alpha \wedge \beta, \Gamma \vdash \Delta \mid H}\,(\wedge \vdash) \qquad \frac{G \mid \Gamma \vdash \Delta, \alpha \mid H}{G \mid \Gamma \vdash \Delta, \alpha \vee \beta \mid H}\,(\vdash \vee)$$

$$\frac{G \mid \Gamma \vdash \Delta, \alpha \mid H \qquad G \mid \Gamma \vdash \Delta, \beta \mid H}{G \mid \Gamma \vdash \Delta, \alpha \wedge \beta \mid H}\,(\vdash \wedge)$$

$$\frac{G \mid \beta, \Gamma \vdash \Delta \mid H}{G \mid \alpha \wedge \beta, \Gamma \vdash \Delta \mid H}\,(\wedge \vdash) \qquad \frac{G \mid \Gamma \vdash \Delta, \beta \mid H}{G \mid \Gamma \vdash \Delta, \alpha \vee \beta \mid H}\,(\vdash \vee)$$

$$\frac{G \mid \alpha, \Gamma \vdash \Delta \mid H \qquad G \mid \beta, \Gamma \vdash \Delta \mid H}{G \mid \alpha \vee \beta, \Gamma \vdash \Delta \mid H}\,(\vee \vdash)$$

$$\frac{G \mid \alpha, \Gamma \vdash \Delta \mid H}{G \mid \Gamma \vdash \Delta, \neg\alpha \mid H}\,(\neg \vdash) \qquad \frac{G \mid \Gamma \vdash \Delta, \alpha \mid H}{G \mid \neg\alpha, \Gamma \vdash \Delta \mid H}\,(\vdash \neg)$$

$$\frac{G_1 \mid \Gamma_1 \vdash \Delta_1, \alpha \mid H_1 \qquad G_2 \mid \beta, \Gamma_2 \vdash \Delta_2 \mid H_2}{G_1 \mid G_2 \mid \alpha \supset \beta, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \mid H_1 \mid H_2}\,(\supset\vdash) \qquad \frac{G \mid \alpha, \Gamma \vdash \Delta, \beta \mid H}{G \mid \Gamma \vdash \Delta, \alpha \supset \beta \mid H}\,(\vdash\supset)$$

$$\frac{G \mid \alpha, \Gamma \vdash \Delta \mid H}{G \mid\ \Box\,\alpha, \Gamma \vdash \Delta \mid H}\,(\Box \vdash) \qquad \frac{G \mid\ \Box\,\Gamma \vdash \alpha \mid H}{G \mid\ \Box\,\Gamma \vdash \Box\alpha \mid H}\,(\vdash \Box)$$

Figure 11: System $GS5$

$$\frac{}{\langle \alpha \vdash \alpha \rangle}\,(Axiom) \qquad \frac{}{\langle \bot \vdash \ \rangle}\,(\bot \vdash)$$

$$\frac{G,\ \langle \Gamma_1 \vdash \Delta_1, \alpha \rangle \qquad H,\ \langle \alpha, \Gamma_2 \vdash \Delta_2 \rangle}{G,\ H,\ \langle \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \rangle}\,(Cut)$$

$$\frac{G,\ \langle \Gamma \vdash \Delta \rangle}{G,\ \langle \alpha, \Gamma \vdash \Delta \rangle}\,(W_l) \qquad \frac{G,\ \langle \Gamma \vdash \Delta \rangle}{G,\ \langle \Gamma \vdash \Delta, \alpha \rangle}\,(W_r)$$

$$\frac{G,\ \langle \alpha, \alpha, \Gamma \vdash \Delta \rangle}{G,\ \langle \alpha, \Gamma \vdash \Delta \rangle}\,(C_l) \qquad \frac{G,\ \langle \Gamma \vdash \Delta, \alpha, \alpha \rangle}{G,\ \langle \Gamma \vdash \Delta, \alpha \rangle}\,(C_r)$$

$$\frac{G}{G,\ \langle \Gamma \vdash \Delta \rangle}\,(W\ external) \qquad \frac{G,\ \langle \Gamma \vdash \Delta \rangle,\ \langle \Gamma \vdash \Delta \rangle}{G,\ \langle \Gamma \vdash \Delta \rangle}\,(C\ external)$$

$$\frac{G,\ \langle \Gamma \vdash \Delta, \alpha \rangle \qquad G,\ \langle \beta, \Gamma \vdash \Delta \rangle}{G,\ \langle \alpha \supset \beta, \Gamma \vdash \Delta \rangle}\,(\supset\vdash) \qquad \frac{G,\ \langle \alpha, \Gamma \vdash \Delta, \beta \rangle}{G,\ \langle \Gamma \vdash \Delta, \alpha \supset \beta \rangle}\,(\vdash\supset)$$

$$\frac{G,\ \langle \alpha, \Gamma_1 \vdash \Delta_1 \rangle,\ \langle \Gamma_2 \vdash \Delta_2 \rangle}{G,\ \langle \Box\alpha, \Gamma_2 \vdash \Delta_2 \rangle,\ \langle \Gamma_1 \vdash \Delta_1 \rangle}\,(\Box_1 \vdash)$$

$$\frac{G,\ \langle \alpha, \Gamma \vdash \Delta \rangle}{G,\ \langle \Box\alpha, \Gamma \vdash \Delta \rangle}\,(\Box_2 \vdash) \qquad \frac{G,\ \langle \Gamma \vdash \Delta \rangle,\ \langle \alpha \rangle}{G,\ \langle \Gamma \vdash \Delta, \Box\alpha \rangle}\,(\vdash \Box)$$

Figure 12: System $LS5$

$$\frac{}{\alpha \vdash \alpha}\ (id) \qquad \frac{X \vdash \alpha \quad \alpha \vdash Y}{X \vdash Y}\ (cut)$$

$$\frac{}{\mathbf{f} \vdash \mathbf{I}}\ (\mathbf{f} \vdash) \qquad \frac{X \vdash \mathbf{I}}{X \vdash \mathbf{f}}\ (\vdash \mathbf{f})$$

$$\frac{\mathbf{I} \vdash X}{\mathbf{t} \vdash X}\ (\mathbf{t} \vdash) \qquad \frac{}{\mathbf{I} \vdash \mathbf{t}}\ (\vdash \mathbf{t})$$

$$\frac{\alpha \circ \beta \vdash X}{\alpha \wedge \beta \vdash X}\ (\wedge \vdash) \qquad \frac{X \vdash \alpha \quad Y \vdash \beta}{X \circ Y \vdash \alpha \wedge \beta}\ (\vdash \wedge)$$

$$\frac{\alpha \vdash X \quad \beta \vdash Y}{\alpha \vee \beta \vdash X \circ Y}\ (\vee \vdash) \qquad \frac{X \vdash \alpha \circ \beta}{X \vdash \alpha \vee \beta}\ (\vdash \vee)$$

$$\frac{*\alpha \vdash X}{\neg\alpha \vdash X}\ (\neg \vdash) \qquad \frac{X \vdash *\alpha}{X \vdash \neg\alpha}\ (\vdash \neg)$$

$$\frac{X \vdash \alpha \quad \beta \vdash Y}{\alpha \supset \beta \vdash *X \circ Y}\ (\supset\vdash) \qquad \frac{X \circ \alpha \vdash \beta}{X \vdash \alpha \supset \beta}\ (\vdash\supset)$$

$$\frac{\alpha \vdash X}{\Box\alpha \vdash \bullet X}\ (\Box \vdash) \qquad \frac{\bullet X \vdash \alpha}{X \vdash \Box\alpha}\ (\vdash \Box)$$

$$\frac{* \bullet *\alpha \vdash X}{\diamond\alpha \vdash X}\ (\diamond \vdash) \qquad \frac{X \vdash \alpha}{* \bullet *X \vdash \diamond\alpha}\ (\vdash \diamond)$$

Figure 13: System $DK$: the logical, the axiom and the (cut) rules.

$$\frac{\mathbf{I}\circ X\vdash Z}{X\vdash Z}\,(\mathbf{I}-)_l \qquad \frac{X\vdash Z\circ\mathbf{I}}{X\vdash Z}\,(\mathbf{I}-)_r$$

$$\frac{\mathbf{I}\vdash X}{Z\vdash X}\,(\mathbf{I}\ ex) \qquad \frac{X\vdash\mathbf{I}}{X\vdash Z}\,(ex\ \mathbf{I})$$

$$\frac{\mathbf{I}\vdash X}{*\mathbf{I}\vdash X}\,(\mathbf{I}^*)_l \qquad \frac{X\vdash\mathbf{I}}{X\vdash *\mathbf{I}}\,(\mathbf{I}^*)_r$$

$$\frac{X_1\vdash Z}{X_1\circ X_2\vdash Z}\,(M)_l \qquad \frac{Z\vdash X_1}{Z\vdash X_1\circ X_2}\,(M)_r$$

$$\frac{X\circ X\vdash Z}{X\vdash Z}\,(C)_l \qquad \frac{Z\vdash X\circ X}{Z\vdash X}\,(C)_r$$

$$\frac{X_1\circ X_2\vdash Z}{X_2\circ X_1\vdash Z}\,(P)_l \qquad \frac{Z\vdash X_1\circ X_2}{Z\vdash X_2\circ X_1}\,(P)_r$$

$$\frac{X_1\circ(X_2\circ X_3)\vdash Z}{(X_1\circ X_2)\circ X_3\vdash Z}\,(A)_l \qquad \frac{Z\vdash X_1\circ(X_2\circ X_3)}{Z\vdash(X_1\circ X_2)\circ X_3}\,(A)_r$$

$$\frac{X\vdash\mathbf{I}}{X\vdash\bullet\mathbf{I}}\,(MN)_l \qquad \frac{\mathbf{I}\vdash X}{\mathbf{I}\vdash\bullet X}\,(MN)_r$$

Figure 14: Systems $DK$: additional structural rules

$$\frac{*\bullet *X\vdash Y}{X\vdash Y}\,(T') \qquad \frac{*\bullet *X\vdash Y}{\bullet *\bullet *X\vdash Y}\,(5') \qquad \frac{*\bullet *(X\circ *\bullet *Y)\vdash Z}{Y\circ *\bullet *X\vdash Z}\,(B')$$

Figure 15: Rules for the axioms: additional structural rules for $S5$ and $B$

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**Statement of Academic Honesty**

I hereby declare that I have not used any auxiliary means for my thesis work other than that what has been cited in my thesis.

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Finiki Stouppa
Matr. No: 2894180