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# Horn Logics and Datalog

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# Recap

- Looked at using Propositional Logic (PL) for representing knowledge
- effectively implementable (SAT solvers), but lacks expressiveness

Want a KR language that can ...

1. Represent sets of objects
2. Represent relationships between objects
3. Write statements that are true for some or all objects satisfying certain conditions
4. Express everything we can express in propositional logic (*and*, *or*, *implies*, *not*, ...)

$\rightsquigarrow$ First-order logic (FOL)

However: FOL satisfiability is undecidable

$\rightsquigarrow$ cannot hope for implementations

# Propositional Horn Fragment

PL Horn Fragment: only allows the following kinds of formulas ($n > 0$):

$$P_1 \wedge \ldots \wedge P_n \to Q \qquad \textit{rules}$$
$$P \qquad \textit{facts}$$

With $P_i$, Q being atoms, and where Q can be $\bot$.

Horn Clauses: Clauses with at most one positive literal.

$$\neg P_1 \vee \ldots \vee \neg P_n \vee Q$$

> (Fact) Entailment. Input: set $\mathcal{H}$ of Horn formulas and atom $P$
> Answer: true if every model of $\mathcal{H}$ is also a model of $P$
> and false otherwise.

PL Horn entailment is solvable in polynomial time.

# Lifting PL Horn to FOL Horn

First-Order Horn Clauses: Clauses with at most one positive literal

But now, atoms can contain variables, constants, and function symbols.

Implicit assumption: all variables ∀-quantified, but quantifiers not written

Some examples of First-Order Horn clauses:

$$\neg \mathit{JuvArthritis}(x) \lor \mathit{Arthritis}(x)$$

$$\neg \mathit{Arthritis}(x) \lor \neg \mathit{JuvDisease}(x) \lor \mathit{JuvArthritis}(x)$$

$$\neg \mathit{Child}(x) \lor \neg \mathit{Adult}(x)$$

$$\neg \mathit{Affects}(x, y) \lor \mathit{Person}(y)$$

$$\neg \mathit{JuvDisease}(x) \lor \mathit{Affects}(x, f(x))$$

$$\mathit{JuvDisease}(\mathit{JRA})$$

# Horn Logics

Horn Formulas: FOL sentences whose CNF yield Horn clauses.

Horn Logics: Syntactic FOL fragments allowing only Horn Formulas.

Some examples of Horn formulas:

$$\forall x.(\mathit{Arthritis}(x) \wedge \mathit{JuvDisease}(x) \rightarrow \mathit{JuvArthritis}(x))$$

$$\forall x.(\mathit{Child}(x) \wedge \mathit{Adult}(x) \rightarrow \bot)$$

$$\forall x.(\forall y.(\mathit{Affects}(x,y) \rightarrow \mathit{Person}(y)))$$

$$\forall x.(\mathit{JuvDisease}(x) \rightarrow \exists y.(\mathit{Affects}(x,y) \wedge \mathit{Child}(y)))$$

$$\forall x.(\forall y.(\forall z.(\mathit{fatherOf}(x,y) \wedge \mathit{brotherOf}(x,z) \rightarrow \mathit{uncleOf}(z,y))))$$

$$\mathit{JuvDisease}(\mathit{JRA})$$

# Expressivity

We cannot express "disjunctive information":

- Covering statements:

$$\forall x.(Person(x) \rightarrow Adult(x) \vee Child(x) \vee Teenager(x))$$

- Negation on the left of implication

$$\forall x.(Person(x) \wedge \neg Woman(x) \rightarrow Man(x))$$

As well as many others ...

Note, however, that some formulas apparently "disjunctive" are Horn:

$$\forall x.(Adult(x) \vee Child(x) \vee Teenager(x) \rightarrow Person(x))$$

# Expressivity

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Note, however, that some formulas apparently "disjunctive" are Horn:

$$\forall x.(Adult(x) \vee Child(x) \vee Teenager(x) \rightarrow Person(x))$$

...because they can be rewritten into formulas that are obviously Horn:

$$\forall x.(Adult(x) \rightarrow Person(x))$$
$$\forall x.(Child(x) \rightarrow Person(x))$$
$$\forall x.(Teenager(x) \rightarrow Person(x))$$

# Existential Rules

$$\forall\vec{x}.\forall\vec{z}.(\varphi(\vec{x},\vec{z}) \to \exists\vec{y}.\psi(\vec{x},\vec{y})) \qquad \textit{Existential Rule}$$

$$\forall\vec{x}.\forall\vec{z}.(\varphi(\vec{x},\vec{z}) \to \bot) \qquad \bot\text{—}\textit{Rule}$$

$$P(\vec{a}) \qquad \textit{Fact}$$

$\varphi(\vec{x},\vec{z})$: conjunction of function-free atoms with vars $\vec{x}\cup\vec{z}$.

$\psi(\vec{x},\vec{y})$: conjunction of function-free atoms with vars $\vec{x}\cup\vec{y}$.

$$\forall x.(\mathit{Arthritis}(x) \wedge \mathit{JuvDisease}(x) \to \mathit{JuvArthritis}(x)) \qquad \textit{Rule}$$

$$\forall x.(\mathit{Child}(x) \wedge \mathit{Adult}(x) \to \bot) \qquad \bot\text{-}\textit{Rule}$$

$$\forall x.(\mathit{JuvDisease}(x) \to \exists y.(\mathit{Affects}(x,y) \wedge \mathit{Child}(y))) \qquad \textit{Rule}$$

$$\mathit{JuvDisease}(\mathit{JRA}) \qquad \text{Fact}$$

Examples of Horn formulas outside this logic:

$$\forall x.(\mathit{Adult}(x) \vee \mathit{Child}(x) \vee \mathit{Teenager}(x) \to \mathit{Person}(x))$$

# Reasoning with Existential Rules

Fact Entailment:
Input: pair $\langle \mathcal{R}, \mathcal{F} \rangle$ of rules and facts and a fact $P$.
Answer is true iff $\langle \mathcal{R}, \mathcal{F} \cup \{\neg P\} \rangle$ is unsatisfiable.

Resolution (inferencing calculus for FOL) can be optimised for Horn clauses.

General strategy: allow only certain kinds of resolution inferences:

- Need to show completeness
  Unsatisfiability must imply that the empty clause is derivable.
- No need to show soundness
  Still just resolution, which is sound.

# Recall FOL Resolution Rule

$$\frac{\alpha \vee \varphi \qquad \neg\beta \vee \psi}{(\varphi \vee \psi)MGU(\alpha, \beta)}$$

$\alpha, \beta$ are atoms
MGU($\alpha, \beta$) is Most General Unifier of $\alpha$ and $\beta$

Examples:

$$\frac{(\neg ArthritisPat(x) \vee Affects(f(x), x)) \qquad ArthritisPat(g(a))}{Affects(f(g(a)), g(a))} \qquad \{x \mapsto g(a)\}$$

$$\frac{Affects(x, John) \qquad \neg Affects(JRA, y)}{\square} \qquad \{x \mapsto JRA, y \mapsto John\}$$

$$\frac{JuvDisease(h(g(f(x), a))) \qquad \neg JuvDisease(h(g(y, y)))}{\textit{Rule not applicable}}$$

# Recall FOL Factoring Rule

$$\frac{\gamma \vee \delta \vee \psi}{(\gamma \vee \psi)MGU(\gamma,\delta)} \quad \gamma,\delta \text{ literals, same sign}$$

Examples:

$$\frac{ArthritisPat(x) \vee Affects(f(x),x) \vee ArthritisPat(g(a))}{Affects(f(g(a)),g(a)) \vee ArthritisPat(g(a))} \quad \{x \mapsto g(a)\}$$

$$\frac{Affects(x,John) \vee Affects(JRA,y)}{Affects(JRA,John)} \quad \{x \mapsto JRA, y \mapsto John\}$$

$$\frac{\neg JuvDisease(h(g(f(x),a))) \vee \neg JuvDisease(h(g(y,z)))}{\neg JuvDisease(h(g(f(x),a)))} \quad \{y \mapsto f(x), z \mapsto a\}$$

# Recall FOL Resolution Procedure

**procedure** Sat($\mathcal{S}$)
  **repeat**
    **for** all clauses $C_1$, $C_2$ in $\mathcal{S}$ **do**
      $\mathcal{S} := \mathcal{S} \cup \mathrm{resolve}(C_1, C_2)$
    **end for**
  **until** No new clause can be added to $\mathcal{S}$ or $\Box \in \mathcal{S}$
  **If** $\Box \in \mathcal{S}$ **return** false
  **return** true
**end procedure**

Function resolve($C_1, C_2$) applies FO resolution in all possible ways, and then applies factoring in all possible ways.

# Recall FOL Resolution Procedure

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    **until** No new clause can be added to $\mathcal{S}$ or $\square \in \mathcal{S}$
    **If** $\square \in \mathcal{S}$ **return** false
    **return** true
**end procedure**

Function resolve($C_1, C_2$) applies FO resolution in all possible ways, and then applies factoring in all possible ways.

Wait . . . in all possible ways?

# Resolution with Free Selection

Resolution with free selection: a complete strategy

- Calculus parameterised by a Selection Function $S$
- $S$ assigns to each Horn clause $C$ a non-empty subset of its literals:
  - $S(C)$ contains the single positive literal, OR
  - $S(C)$ contains a subset of negative literals
- Restrict resolution such that we only resolve on selected literals

We are free to design the selection function ourselves:

If we satisfy the basic constraints, completeness is guaranteed.

# Resolution with Free Selection

A reasonable selection function:

- Select the set of all negative literals in each clause
- If there is no negative literal, select the (unique) positive literal

As usual, we prepare for resolution (Skolemisation, CNF, clause form)

$$
\begin{aligned}
A(x) \to \exists y.R(x,y) \land B(y) &\quad\leadsto\quad \neg A(x) \lor R(x,f(x)) \\
&\quad\phantom{\leadsto}\quad \neg A(x) \lor B(f(x)) \\
B(x) \to C(x) &\quad\leadsto\quad \neg B(x) \lor C(x) \\
R(x,y) \land C(y) \to D(x) &\quad\leadsto\quad \neg R(x,y) \lor \neg C(y) \lor D(x) \\
A(a) &\quad\leadsto\quad A(a)
\end{aligned}
$$

We now want to see whether $D(a)$ follows ...

# Resolution with Free Selection

$$\neg A(x) \vee R(x, f(x)) \quad (1)$$
$$\neg A(x) \vee B(f(x)) \quad (2)$$
$$\neg B(x) \vee C(x) \quad (3)$$
$$\neg R(x, y) \vee \neg C(y) \vee D(x) \quad (4)$$
$$A(a) \quad (5)$$
$$\neg D(a) \quad (6)$$

With this selection, we don't need to resolve (1) and (4)

Observation: This strategy amounts to Unit Resolution

One of the premises of resolution must be a unit clause!

# Resolution with Free Selection

| | | |
|---|---|---|
| $\neg A(x) \vee R(x, f(x))$ | | (1) |
| $\neg A(x) \vee B(f(x))$ | | (2) |
| $\neg B(x) \vee C(x)$ | | (3) |
| $\neg R(x, y) \vee \neg C(y) \vee D(x)$ | | (4) |
| $A(a)$ | | (5) |
| $\neg D(a)$ | | (6) |
| $R(a, f(a))$ | (1) + (5) | (7) |
| $B(f(a))$ | (2) + (5) | (8) |
| $C(f(a))$ | (8) + (3) | (9) |
| $\neg C(f(a)) \vee D(a)$ | (7) + (4) | (10) |
| $D(a)$ | (9) + (10) | (11) |
| $\square$ | (11) + (6) | (12) |

# Resolution with Free Selection

We still have termination problems ...

$$A(x) \to \exists y.R(x,y) \land A(y) \quad \rightsquigarrow \quad \neg A(x) \lor R(x,f(x))$$

$$\neg A(x) \lor A(f(x))$$

$$A(a) \quad \rightsquigarrow \quad A(a)$$

$$\neg A(x) \lor R(x,f(x))$$
$$\neg A(x) \lor A(f(x))$$
$$A(a)$$
$$R(a,f(a))$$
$$A(f(a))$$
$$R(a,f(f(a)))$$
$$A(f(f(a)))$$
$$\ldots$$

# Resolution with Free Selection

We still have termination problems . . .

$$A(x) \to \exists y.R(x,y) \land A(y) \quad \rightsquigarrow \quad \neg A(x) \lor R(x,f(x))$$
$$\neg A(x) \lor A(f(x))$$
$$A(a) \quad \rightsquigarrow \quad A(a)$$

$$\neg A(x) \lor R(x,f(x))$$
$$\neg A(x) \lor A(f(x))$$
$$A(a)$$
$$R(a,f(a))$$
$$A(f(a))$$
$$R(a,f(f(a)))$$
$$A(f(f(a)))$$
$$\ldots$$

**Theorem**

Unsatisfiability and fact entailment over existential rules are undecidable (semi-decidable).

That is, as difficult as checking unsatisfiability in FOL.

# Datalog

To achieve decidability we need to sacrifice (even more) expressivity.

Datalog: The quintessential rule-based KR language

$$\forall \vec{x}.\forall \vec{z}.(\varphi(\vec{x},\vec{z}) \to \psi(\vec{x})) \qquad \textit{Rule}$$

$$\forall \vec{x}.\forall \vec{z}.(\varphi(\vec{x},\vec{z}) \to \bot) \qquad \bot\text{—}\textit{Rule}$$

$$P(\vec{a}) \qquad \textit{Fact}$$

$\varphi(\vec{x},\vec{z})$ and $\psi(\vec{x})$: conjunctions of function-free atoms

We can still express

$$\forall x.(\forall y.(\forall z.(\mathit{fatherOf}(x,y) \land \mathit{brotherOf}(x,z) \to \mathit{uncleOf}(z,y))))$$

$$\forall x.(\forall y.(\mathit{Affects}(x,y) \to \mathit{Person}(y)))$$

But, we can no longer express

$$\forall x.(\mathit{JuvDisease}(x) \to \exists y.(\mathit{Affects}(x,y) \land \mathit{Child}(y)))$$

# Decidability of Entailment

**Theorem**

Fact entailment in Datalog is decidable.

Decidability follows directly from Herbrand's theorem

- Our problem reduces to unsatisfiability of $\mathcal{S} = \mathcal{R} \cup \mathcal{F} \cup \{\neg P\}$
- $\mathcal{R} \cup \mathcal{F} \cup \{\neg P\}$ is a set of clauses without function symbols so Herbrand universe finite
- Gilmore's FOL unsatisfiability algorithm terminates.

# Decidability of Entailment

Our algorithm is an adaptation of Gilmore's when Herbrand universe is finite

1: **procedure** Datalog-Gil($\langle \mathcal{R}, \mathcal{F} \rangle, P$)
2: Compute Herbrand Universe $U$
3: $\mathcal{R}' := \text{ground}(\mathcal{R}, U)$
4: **return** Horn-Prop($\langle \mathcal{R}', \mathcal{F} \rangle, P$)
5: **end procedure**

Subroutine Horn-Prop solves entailment problem for Horn PL

# Complexity Considerations

$$\forall x.(\forall y.(\forall z.(fatherOf(x,y) \wedge brotherOf(x,z) \rightarrow uncleOf(z,y))))$$
$$fatherOf(John, Mary)$$
$$brotherOf(John, Peter)$$

Herbrand universe: constants in $\langle \mathcal{R}, \mathcal{F} \rangle$

$$U = \{John, Mary, Peter\}$$

Grounding leads to exponential size set of propositional clauses

$$fatherOf(John, John) \wedge brotherOf(John, John) \rightarrow uncleOf(John, John)$$
$$fatherOf(John, Mary) \wedge brotherOf(John, Mary) \rightarrow uncleOf(Mary, Mary)$$
$$fatherOf(John, Peter) \wedge brotherOf(John, Peter) \rightarrow uncleOf(Peter, Peter)$$

and so on

Size of the grounding grows as $\mathcal{O}(c^v)$, where

- $c$ is the max. number of constants in facts.
- $v$ is the max. number of variables in rules.

# Complexity Considerations

Propositional entailment in Horn PL can be decided in polynomial time.

Overall process takes exponential time (because of grounding).

**Theorem**

Fact entailment in Datalog is decidable in ExpTime.

In fact, the problem is also ExpTime-hard (beyond this course).

$\rightsquigarrow$ Naive grounding algorithm is worst-case optimal.

# Practical Considerations

From a practical point of view, we can do much better:

- Avoid computing the grounding upfront
- Instantiate variables to constants “on the fly”

We develop two resolution-based strategies:

1. Forward chaining:

   Start from facts and instantiate rules to derive new facts whenever possible until goal is derived

2. Backward chaining:

   Start from goal and proceed “backwards” to derive the empty clause

Both strategies can be seen as Resolution with Free Selection.

# Forward Chaining (Example)

Start from facts and instantiate rules to derive new facts whenever possible until goal (or $\Box$) is derived.

Example: Check if *Child*(*John*) follow from the following:

$$\forall x.(JuvArthritis(x) \to JuvDisease(x)) \tag{13}$$

$$\forall x.(\forall y.(JuvDisease(x) \land Affects(x, y) \to Child(y))) \tag{14}$$

$$JuvArthritis(JRA) \tag{15}$$

$$Affects(JRA, John) \tag{16}$$

Match existing facts to rule bodies to derive new facts.

From Fact (15) and Rule (13) we obtain the following by unit resolution

$$JuvDisease(JRA) \tag{17}$$

From Facts (17) and (16) and Rule (14), derive goal and stop.

$$Child(John)$$

# Forward Chaining and Resolution

$\mathcal{S}_{fw}$: select all negative literals in clauses, and the (unique) positive literal if the clause does not have negative literals.

$$\neg JuvArthritis(x) \lor JuvDisease(x)$$
$$JuvArthritis(JRA)$$

We obtain the following by resolution:

$$JuvDisease(JRA)$$

# Forward Chaining and Resolution

$\mathcal{S}_{fw}$: select all negative literals in clauses, and the (unique) positive literal if the clause does not have negative literals.

Deriving a new fact by matching other facts to a rule may require several resolution steps (Hyperresolution).

$$\neg JuvDisease(x) \vee \neg Affects(x, y) \vee Child(y)$$
$$Affects(JRA, John)$$
$$JuvDisease(JRA)$$

We obtain the following by resolution:

$$\neg JuvDisease(JRA) \vee Child(John)$$
$$Child(John)$$

In forward chaining, we do both steps in one.

# Forward Chaining

**procedure** Forward($\langle \mathcal{R}, \mathcal{F} \rangle$, $P$)
  $\mathcal{F}' := \mathcal{F}$
  **repeat**
    **for** each rule $R = \neg B_1 \vee \neg B_2 \vee \ldots, \vee \neg B_n \vee H \in \mathcal{R}$ **do**
      **if** $\{D_1, \ldots, D_n\} \subseteq \mathcal{F}'$ such that $B_i$ unifies with $D_i$ **then**
        $\theta :=$ Unify($\{B_1 \doteq D_1, \ldots, B_n \doteq D_n\}$)
        $\mathcal{F}' := \mathcal{F}' \cup \{H\theta\}$
      **end if**
    **end for**
  **until** No new atom can be added to $\mathcal{F}'$ or $P \in \mathcal{F}'$ or $\Box \in \mathcal{F}'$
  **if** $P \in \mathcal{F}'$ or $\Box \in \mathcal{F}'$ **then**
    **return** true
  **else**
    **return** false
  **end if**
**end procedure**

# Backward Chaining (Example)

Check whether following rules and facts imply $Child(John)$:

$$\forall x.(JuvArthritis(x) \to JuvDisease(x)) \quad (18)$$

$$\forall x.(\forall y.(JuvDisease(x) \wedge Affects(x,y) \to Child(y))) \quad (19)$$

$$JuvArthritis(JRA) \quad (20)$$

$$Affects(JRA, John) \quad (21)$$

Match “goal” $Child(John)$ to rule heads and facts to derive new goals.

To prove $Child(John)$, by Rule (19) it is sufficient to show

$$JuvDisease(x) \quad \text{and} \quad Affects(x, John)$$

Then, by Fact (21), it would be sufficient to show $JuvDisease(JRA)$.
Another possibility is to use Rule (18) and get the following sub-goals

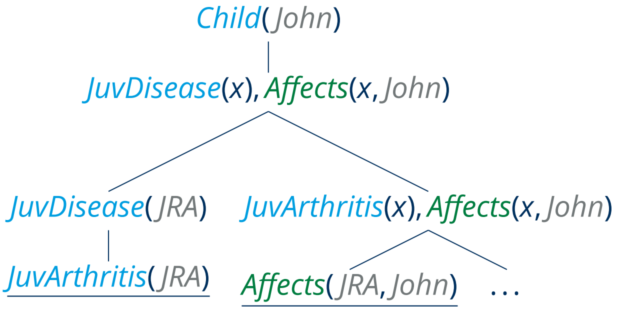
$$JuvArthritis(x) \quad \text{and} \quad Affects(x, John)$$

And so on . . .

# Backward Chaining (Example)

We can represent this kind of backwards reasoning in an AND-OR tree:

$$\forall x.(\mathit{JuvArthritis}(x) \rightarrow \mathit{JuvDisease}(x))$$
$$\forall x.(\forall y.(\mathit{JuvDisease}(x) \wedge \mathit{Affects}(x,y) \rightarrow \mathit{Child}(y)))$$
$$\mathit{JuvArthritis}(\mathit{JRA})$$
$$\mathit{Affects}(\mathit{JRA},\mathit{John})$$



# Backward Chaining and Resolution

$\mathcal{S}_{bw}$: select the unique positive literal in clauses, and
all negative literals if the clause does not have positive literals.

Matching the goal to a rule head or a fact corresponds to one resolution step.

$$\frac{\neg JuvDisease(x) \lor \neg Affects(x,y) \lor Child(y) \qquad \neg Child(John)}{\neg JuvDisease(x) \lor \neg Affects(x, John)}$$

# Termination Issues

Resolution with free selection may not terminate with $\mathcal{S}_{bw}$.

Example: Show that John is a Scientist.

$$\neg worksWith(x, y) \vee \neg Scientist(y) \vee Scientist(x) \tag{22}$$

$$worksWith(John, Mary) \tag{23}$$

$$\neg Scientist(John) \tag{24}$$

We start resolving on selected atoms:

$$\neg worksWith(John, y) \vee \neg Scientist(y) \qquad (22) + (24) \tag{25}$$

$$\neg worksWith(John, y_1) \vee \neg worksWith(y_1, y_2) \vee \neg Scientist(y_2) \qquad (22) + (25) \tag{26}$$

$$\ldots$$

Keep on generating clauses with chains of *worksWith* atoms of increasing length (variable proliferation).

Thus, the backward chaining tree can have infinite branches.

# Other Considerations

Implementing Forward and Backward chaining efficiently is non-trivial:

- Forward chaining: set of deduced facts might get huge
- Backward chaining: recursion may be too deep or search tree too wide.

There are many ways to optimise these algorithms

Semi-naive evaluation, Magic sets, . . .

But, this is beyond the scope of this course.

There are many optimised systems that implement forward/backward chaining.

The KR languages we have described are related to:

- Databases: Datalog query language, and deductive databases
- Logic programming: Prolog