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The Expressive Power of Description Logics with Numerical Constraints over Restricted Classes of Models (Extended Version)

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The Expressive Power of Description Logics with Numerical Constraints over Restricted Classes of Models

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Abstract. For Description Logics (DLs), different approaches for extending the expressive power using numerical constraints have been introduced. Here, we consider the logic $\mathcal{ALCSCC}$, which can state powerful numerical constraints on the number of role successors satisfying certain properties, and logics of the form $\mathcal{ALC}(\mathfrak{D})$, in which individuals can be assigned numerical or other concrete values, which can be compared using predefined predicates of $\mathfrak{D}$. Instead of investigating the complexity of reasoning in these logics, we are interested in characterizing their expressive power. We improve on our previous work in this direction in several respects. For $\mathcal{ALCSCC}$, we develop a method that can deal with the finitely branching interpretations considered in the original paper on this logic, rather than moving to the variant $\mathcal{ALCSCC}^\infty$, where arbitrary interpretations are allowed. The main idea is to employ, in the proof of the characterization, locality properties of first-order logic over certain restricted classes of models (such as finite and finitely branching models) rather than compactness, which does not hold in the finitely branching case. For logics of the form $\mathcal{ALC}(\mathfrak{D})$, we consider a notion of expressive power that takes the concrete values assigned to individuals into account, rather than the abstract expressive power investigated in our previous work. The characterization of the expressive power of $\mathcal{ALC}(\mathfrak{D})$ obtained this way works not only for arbitrary interpretations, but also for finite and finitely branching ones.

1 Introduction

Description logics (DLs) [6,12] are a prominent family of logic-based knowledge representation languages, which can be used to formalize the terminological knowledge of an application domain in a machine-processable way. For instance, the standard Web Ontology Language OWL[3] is based on an expressive DL and the large medical ontology SNOMED CT[4] has been developed using a rather inexpressive DL. The *expressive power* of a DL is determined by the constructors

[3] https://www.w3.org/TR/owl2-overview/
[4] https://www.snomed.org/

that are available for building complex concept descriptions out of concept names (unary predicates) and role names (binary predicates). For example, the concept description $\mathsf{Person} \sqcap \exists \mathsf{pet.Dog}$, describing persons that have a dog as a pet, uses conjunction ($\sqcap$) and existential restriction ($\exists r.C$) as constructors, where Person and Dog are concept names and pet is a role name. To show that a given DL $\mathcal{L}_1$ can be expressed by another DL $\mathcal{L}_2$ using the same concept and role names, we can provide a semantic-preserving translation of $\mathcal{L}_1$ concept descriptions into $\mathcal{L}_2$ concept descriptions. Proving inexpressivity is more challenging. The first formal investigation of the expressive power of DLs was performed in [1,2], but in a rather ad hoc manner. More fundamental characterizations of the expressive power of various concept description languages up to the DL $\mathcal{ALC}$ based on the model-theoretic notion of *bisimulation* are given in [19]. Basically, this approach (pioneered by van Benthem [27] for the modal logic K, which is a syntactic variant of $\mathcal{ALC}$) characterizes a given DL as the fragment of first-order logic (FOL) that is invariant under an appropriate notion of bisimulation.

The expressive power of $\mathcal{ALC}$ can, for instance, be extended by enabling the use of numerical constraints within concept descriptions. In the extension $\mathcal{ALCQ}$ of $\mathcal{ALC}$, qualified number restrictions [17] can be employed to constrain the number of role successors belonging to a certain concept; e.g., $\mathsf{Person} \sqcap (\geq 3\,\mathsf{child.Female}) \sqcap (\leq 2\,\mathsf{pet.Dog})$ describes persons that have at least 3 daughters and at most 2 dogs as pets. The DL $\mathcal{ALCSCC}$ [3] extends $\mathcal{ALCQ}$ with very expressive counting constraints on role successors expressed in the logic QF-BAPA [18]. Since QFBAPA only considers finite sets and their cardinalities, the semantics of $\mathcal{ALCSCC}$ is restricted to finitely branching interpretations, where each element can have only finitely many role successors. In $\mathcal{ALCSCC}$ one can, e.g., describe persons that have more daughters than they have dogs as pets, without using specific numbers as upper/lower bounds for the numbers of pet dogs and daughters. Bisimulation-based characterizations of $\mathcal{ALCQ}$ (or its modal logic variant of K extended with graded modalities) can be found in [25,21,24]. In [7,8], we have investigated the expressivity of DLs with expressive counting constraints. However, to dispense with the requirement that interpretations be finitely branching, we used an infinite variant QFBAPA$^\infty$ of QFBAPA to formulate these constraints, which yields the variant $\mathcal{ALCSCC}^\infty$ of $\mathcal{ALCSCC}$. We were able to show that $\mathcal{ALCSCC}^\infty$ is not a fragment of FOL and characterized the first-order fragment of this logic ($\mathcal{ALCCQU}$ or equivalently $\mathcal{ALCQ}t$) using a form of counting bisimulation [21]. The *first major contribution* of the present paper is to prove the same results for $\mathcal{ALCSCC}$, where only finitely branching interpretations are available. The proof techniques used in [7,8], which were inspired by the ones in [21], cannot be employed in this setting since they depend on compactness of FOL, which does not hold for the restriction of FOL to finitely branching interpretations. Instead, we employ a proof technique inspired by [26,24], which utilizes locality properties of FOL rather than compactness. Interestingly, this approach can deal with arbitrary interpretations, finitely branching interpretations, and finite interpretations in a uniform way.

An orthogonal approach for employing numerical constraints within concept descriptions is the use of numerical concrete domains [20,13]. In a DL with a concrete domain, concrete objects such as numbers or strings can be assigned to individuals using partial functions called *features*. For example, the concept description $\mathsf{Person} \sqcap \exists \mathsf{child}\ \mathsf{age}, \mathsf{pet}\ \mathsf{age}.<$ describes persons that have a child that is younger than one of their pets. Here, age is a feature that assigns a rational number, their age, to some of the elements of the interpretation domain, and $<$ is the usual smaller relation between rational numbers. In [9,10], we have investigated the abstract expressive power of DLs with concrete domains, which only considers the abstract part of interpretations, i.e., ignores the values assigned to features. We have shown that the abstract expressive power of $\mathcal{ALC}(\mathfrak{D})$, i.e., $\mathcal{ALC}$ extended with the concrete domains $\mathfrak{D}$, is contained in FOL for certain concrete domains, but have also exhibited a large class of concrete domains for which this is not the case. The *second major contribution* of the present paper is to introduce a notion of concrete expressive power for DLs with concrete domains that also takes the feature values into account. For example, if we take two concrete domains over the rational numbers, where one has as only predicate $+_1$ (relating $q \in \mathbb{Q}$ with $q + 1$) and the other $+_2$ (relating $q \in \mathbb{Q}$ with $q + 2$), then the extensions of $\mathcal{ALC}$ with these concrete domains have the same abstract expressive power, but their concrete expressive power is incomparable. Using proof techniques similar to the ones employed for $\mathcal{ALCSCC}$ we can characterize $\mathcal{ALC}(\mathfrak{D})$ as the fragment of $\mathsf{FOL}(\mathfrak{D})$ (i.e., FOL extended with the concrete domain $\mathfrak{D}$) that is invariant under an appropriate notion of bisimulation.

2 Preliminaries

We start by introducing the base logic $\mathcal{ALC}$ before defining its two orthogonal extensions with numerical constraints. Since here we focus on the expressivity of concept description languages, we do not introduce TBoxes, ABoxes, or reasoning problems (see [12] for more details on $\mathcal{ALC}$ and other classical DLs).

The classical DL $\mathcal{ALC}$ Given disjoint, at most countable sets $\mathsf{N_C}$ and $\mathsf{N_R}$ of *concept* and *role names*, $\mathcal{ALC}$ *concept descriptions* (*concepts* for short) are built from concept names using negation ($\neg C$), conjunction ($C \sqcap D$), and *existential restrictions* ($\exists r.C$), where $r \in \mathsf{N_R}$ and C, D are $\mathcal{ALC}$ concept descriptions. As usual, we define $C \sqcup D := \neg(\neg C \sqcap \neg D)$ (disjunction), $\forall r.C := \neg\exists r.\neg C$ (value restriction) and $\top := A \sqcup \neg A$ (top concept). An *interpretation* $\mathcal{I}$ consists of a non-empty *domain* $\Delta^{\mathcal{I}}$ and a mapping $\cdot^{\mathcal{I}}$ assigning a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ to $A \in \mathsf{N_C}$ and a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ to $r \in \mathsf{N_R}$. For $d \in \Delta^{\mathcal{I}}$, we define $r^{\mathcal{I}}(d) := \{e \in \Delta^{\mathcal{I}} \mid (d, e) \in r^{\mathcal{I}}\}$. We extend $\cdot^{\mathcal{I}}$ to concepts by $(\neg C)^{\mathcal{I}} := \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$, $(C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}}$ and $(\exists r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(d) \cap C^{\mathcal{I}} \neq \emptyset\}$. In this DL, the concept of a person not having a dog as a pet can be written as $\mathsf{Person} \sqcap \forall \mathsf{pet}.\neg\mathsf{Dog}$.

The DL $\mathcal{ALCSCC}$ This DL employs the logic *QFBAPA* [18] to state cardinality constraints on role successors that are more expressive than existential and value restrictions. In QFBAPA, *set terms* are built from *set variables* and the constants $\emptyset$ and $\mathcal{U}$ using intersection $\cap$, union $\cup$ and complement ^{c}. A *QFBAPA formula* is a Boolean combination of *atomic formulae* of the form

$$m_0 + m_1|s_1| + \cdots + m_k|s_k| \leqslant n_0 + n_1|t_1| + \cdots + n_\ell|t_\ell| \tag{1}$$

where each s_i, t_j is a set term and each m_i, n_j is a natural number.[5] A *solution* σ of a QFBAPA formula ϕ assigns a *finite* set $\sigma(\mathcal{U})$ to $\mathcal{U}$, the empty set to $\emptyset$ and subsets of $\sigma(\mathcal{U})$ to set variables such that ϕ is satisfied by σ, in the standard way. Checking if a QFBAPA formula has a solution is an NP-complete problem [18]. The logic QFBAPA$^\infty$ [7] has the same syntax as QFBAPA, but solutions may assign infinite sets to $\mathcal{U}$. Its satisfiability problem is also NP-complete [7].

$\mathcal{ALCSCC}$ extends the syntax of $\mathcal{ALC}$ with the new constructor *role successor restriction* (or succ-*restriction*) $\mathsf{succ}(\mathfrak{con})$, where $\mathfrak{con}$ is an atomic QFBAPA formula with role names and $\mathcal{ALCSCC}$ concept descriptions as set variables [3]. For instance, the concept of all persons that have more daughters than they have dogs as pets can be expressed in $\mathcal{ALCSCC}$ as $\mathsf{succ}(|\mathsf{pet} \cap \mathsf{Dog}| < |\mathsf{child} \cap \mathsf{Female}|)$. Note that existential restrictions $\exists r.C$ are not needed as explicit constructors in this DL since they can be expressed as $\mathsf{succ}(|r \cap C| \geqslant 1)$.

When defining the semantics of $\mathcal{ALCSCC}$, interpretations $\mathcal{I}$ are required in [3] to be *finitely branching*, i.e. such that the set of all role successors $\mathsf{ars}^\mathcal{I}(d) := \bigcup_{r \in \mathsf{N_R}} r^\mathcal{I}(d)$ is finite, for all $d \in \Delta^\mathcal{I}$. Then, each $d \in \Delta^\mathcal{I}$ induces a QFBAPA assignment σ_d, where $\sigma_d(\mathcal{U}) := \mathsf{ars}^\mathcal{I}(d)$, $\sigma_d(r) := r^\mathcal{I}(d)$ for $r \in \mathsf{N_R}$ and $\sigma_d(C) := C^\mathcal{I} \cap \mathsf{ars}^\mathcal{I}(d)$ for concepts C. The mapping $\cdot^\mathcal{I}$ is extended to succ-restrictions by defining $d \in \mathsf{succ}(\mathfrak{con})^\mathcal{I}$ iff σ_d is a solution of $\mathfrak{con}$.

The DL $\mathcal{ALCSCC}^\infty$ is defined in [7] with the same syntax as $\mathcal{ALCSCC}$, but in the semantics arbitrary interpretations are allowed. Consequently, the assignment σ_d may be such that $\sigma_d(\mathcal{U})$ is infinite, and thus satisfaction of the constraint $\mathfrak{con}$ by σ_d is evaluated in QFBAPA$^\infty$ rather than QFBAPA.

In the definitions of $\mathcal{ALCSCC}^\infty$ and $\mathcal{ALCSCC}$, we considered two classes of first-order interpretations: the class $\mathbb{C}_{\mathsf{all}}$ of all interpretations and the class $\mathbb{C}_{\mathsf{fb}}$ of finitely branching interpretations. Later on, we will also consider the class $\mathbb{C}_{\mathsf{fin}}$ of all finite interpretations, which is also of interest in DL research [15,22]. Our results on the expressive power will be parameterized with a class $\mathbb{C}$ of interpretations satisfying certain restrictions. Since the syntax of $\mathcal{ALCSCC}^\infty$ and $\mathcal{ALCSCC}$ coincide, we will in the following always talk about $\mathcal{ALCSCC}$ concepts. However, if $\mathbb{C}$ contains interpretations that are not finitely branching, then the semantics uses QFBAPA$^\infty$ rather than QFBAPA.

DLs with concrete domains Following [11,20,13], we use the term *concrete domain* to refer to a relational structure $\mathfrak{D} = (D, \dots, P^D, \dots)$ over a non-empty, at most countable relational signature, where D is a non-empty set, and each predicate P has an associated arity $k_P \in \mathbb{N}$ and is interpreted by

[5] Following [8], we use a streamlined definition of QFBAPA that does not explicitly introduce set constraints and divisibility constraints.

a relation $P^D \subseteq D^{k_P}$. An example is the structure $\mathfrak{Q} := (\mathbb{Q}, <, =, >)$ over the rational numbers $\mathbb{Q}$ with standard binary ordering and equality relations. Given a countable set V of variables, a *constraint system* over V is a set $\mathfrak{C}$ of *constraints* $P(v_1, \ldots, v_k)$, where $v_1, \ldots, v_k \in V$ and P is a k-ary predicate of $\mathfrak{D}$. We denote by $V(\mathfrak{C})$ the set of variables that occur in $\mathfrak{C}$. The constraint system $\mathfrak{C}$ is *satisfiable* if there is a mapping $h\colon V(\mathfrak{C}) \to D$ such that $P(v_1, \ldots, v_k) \in \mathfrak{C}$ implies $(h(v_1), \ldots, h(v_k)) \in P^D$. The *constraint satisfaction problem* for $\mathfrak{D}$, denoted CSP($\mathfrak{D}$), asks if a given finite constraint system $\mathfrak{C}$ over $\mathfrak{D}$ is satisfiable. The CSP of $\mathfrak{Q}$ is decidable in polynomial time, by reduction to $<$-cycle detection: for example, the system $\{x_1 < x_2,\ x_2 < x_3,\ x_3 < x_1\}$ is unsatisfiable over $\mathfrak{Q}$.

When integrating such a concrete domain into the DL $\mathcal{ALC}$, it needs to satisfy certain restrictions to obtain a decidable DL. Without a TBox, admissibility is required in [11] whereas in the presence of a TBox the stronger ω-admissibility is required in [20,13]. In the context of our investigation of the expressive power of DLs with concrete domains, it is sufficient to assume that negated constraints can be expressed using one or more non-negated ones.

Definition 1. *A structure $\mathfrak{D}$ is* weakly closed under negation (WCUN) *if for all $k \geqslant 1$ and all k-ary relations P of $\mathfrak{D}$ there are k-ary relations $P_1, \ldots, P_{n_P}$ such that $(d_1, \ldots, d_k) \notin P^D$ iff $(d_1, \ldots, d_k) \in \bigcup_{i=1}^{n_P} P_i^D$ for all $d_1, \ldots, d_k \in D^k$.*

It is easy to see that both admissible and ω-admissible concrete domains satisfy this property. Examples of ω-admissible, and thus WCUN, concrete domains are Allen's interval algebra, RCC8 and $\mathfrak{Q}$ [20,13]. For example the negated predicate $\neq$ in $\mathfrak{Q}$ is obtained as the union of $<$ and $>$.

To integrate a given concrete domain $\mathfrak{D}$ into $\mathcal{ALC}$, we complement $\mathsf{N_C}$ and $\mathsf{N_R}$ with a finite set $\mathsf{N_F}$ of *feature names* that connect individuals with values in D [11]. A *feature path* p is of the form f or rf with $r \in \mathsf{N_R}$ and $f \in \mathsf{N_F}$. For instance, age is a feature name as well as a feature path, while child age is a feature path including the role name child. The DL $\mathcal{ALC}(\mathfrak{D})$ extends $\mathcal{ALC}$ with *concrete domain restrictions* (or *CD-restrictions*) of the form $\exists p_1, \ldots, p_k.P$ and $\forall p_1, \ldots, p_k.P$, where p_i are feature paths and P is a k-ary predicate of $\mathfrak{D}$. An interpretation $\mathcal{I}$ assigns to $f \in \mathsf{N_F}$ a *partial* function $f^\mathcal{I}\colon \Delta^\mathcal{I} \rightharpoonup D$. A feature path p is mapped to $p^\mathcal{I} \subseteq \Delta^\mathcal{I} \times D$ by defining[6] $p^\mathcal{I}(d) := \{f^\mathcal{I}(d)\}$ if $p = f$ and $p^\mathcal{I}(d) := \{f^\mathcal{I}(e) \mid e \in r^\mathcal{I}(d)\}$ if $p = rf$. Then we can define

$$(\exists p_1, \ldots, p_k.P)^\mathcal{I} := \{d \in \Delta^\mathcal{I} \mid \textit{some} \text{ tuple in } p_1^\mathcal{I}(d) \times \cdots \times p_k^\mathcal{I}(d) \text{ is in } P^D\}$$
$$(\forall p_1, \ldots, p_k.P)^\mathcal{I} := \{d \in \Delta^\mathcal{I} \mid \textit{every} \text{ tuple in } p_1^\mathcal{I}(d) \times \cdots \times p_k^\mathcal{I}(d) \text{ is in } P^D\}.$$

For example, one can describe individuals having a child that is younger than one of their pets using $\exists$child age, pet age.$<$.

3 The Expressive Power of $\mathcal{ALCSCC}$

In this section, we first introduce a notion of bisimulation, called Presburger bisimulation, such that $\mathcal{ALCSCC}$ concept descriptions are invariant under such

[6] In a slight abuse of notation, we view $f^\mathcal{I}(d)$ both as a value and as a singleton set.

bisimulations, i.e., bisimilar elements belong to the same $\mathcal{ALCSCC}$ concept descriptions. Next, we consider an approximate variant of Presburger bisimulation and show that, while not all $\mathcal{ALCSCC}$ concept descriptions are invariant under this notion, the ones that are expressible in first-order logic are. This shows that there are $\mathcal{ALCSCC}$ concept descriptions that are not expressible in FOL. Finally, we characterize the fragment of $\mathcal{ALCSCC}$ that is first-order definable as the logic $\mathcal{ALCQ}t$, for which successor constraints have a restricted form.

Presburger bisimulation Assume that $\mathsf{N_C}$ and $\mathsf{N_R}$ are finite. We base our definition of Presburger bisimulations on the notion of *safe role types*, which are non-empty subsets of $\mathsf{N_R}$. Intuitively, such a role type stands for the intersection of its elements intersected with the complements of the non-elements. For example, if $\mathsf{N_R} = \{r, s, t\}$, then the safe role type $\{r, s\}$ corresponds to the set term $r \cap s \cap t^c$. More formally, safe role types τ are interpreted in an interpretation $\mathcal{I}$ as the binary relation

$$\tau^{\mathcal{I}} := (\bigcap_{r\in\tau} r^{\mathcal{I}} \setminus (\bigcup_{r\in\mathsf{N_R}\setminus\tau} r^{\mathcal{I}})) \subseteq \bigcup_{r\in\mathsf{N_R}} r^{\mathcal{I}}.$$

The fact that safe role types are non-empty sets of role names ensures the inclusion stated above, i.e., any $\tau^{\mathcal{I}}$ is an $r^{\mathcal{I}}$ successor for at least one role name r, which justifies the name *safe*. Consequently, for all $d \in \Delta^{\mathcal{I}}$, the set $\tau^{\mathcal{I}}(d) := \{e \in \Delta^{\mathcal{I}} \mid (d, e) \in \tau^{\mathcal{I}}\}$ is a subset of $\mathsf{ars}^{\mathcal{I}}(d)$, and every $e \in \mathsf{ars}^{\mathcal{I}}(d)$ belongs to $\tau^{\mathcal{I}}(d)$ for exactly one safe role type τ. The set $\mathsf{N_R}$ must be finite, in order to encode safe role types as well-defined set terms. For $\mathcal{ALCSCC}^{\infty}$ it was shown in [7] that each set term s occurring within a succ-restriction can be rewritten as the disjoint union of terms of the form $\tau \cap C$ where τ is a safe role type and C an $\mathcal{ALCSCC}^{\infty}$ concept [7]. The same also holds for $\mathcal{ALCSCC}$. Following [7], we modify the notion of *counting bisimulation* from [21] by using safe role types in place of role names to obtain Presburger bisimulations (called $\mathcal{ALCQ}t$ bisimulations in [7]).

Definition 2. *Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be finite and $\mathbb{C}$ a class of interpretations. The binary relation $\rho \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is a* Presburger (Pr) bisimulation *between the interpretations $\mathcal{I}$ and $\mathcal{J}$ if for all $A \in \mathsf{N_C}$ and all safe role types τ over $\mathsf{N_R}$ the following properties are satisfied:*

Atomic *$(d, e) \in \rho$ implies $d \in A^{\mathcal{I}}$ iff $e \in A^{\mathcal{J}}$;*
Forth *if $(d, e) \in \rho$ and $D \subseteq \tau^{\mathcal{I}}(d)$ is finite, then there is a set $E \subseteq \tau^{\mathcal{J}}(e)$ such that ρ contains a bijection between D and E;*
Back *if $(d, e) \in \rho$ and $E \subseteq \tau^{\mathcal{J}}(e)$ is finite, then there is a set $D \subseteq \tau^{\mathcal{I}}(d)$ such that ρ contains a bijection between D and E.*

We call $d \in \Delta^{\mathcal{I}}$ and $e \in \Delta^{\mathcal{J}}$ Pr bisimilar *if $(d, e) \in \rho$ for some Pr bisimulation ρ between $\mathcal{I}$ and $\mathcal{J}$. A concept C is* $\mathbb{C}$-invariant *under Pr bisimulation if $d \in C^{\mathcal{I}}$ iff $e \in C^{\mathcal{J}}$ holds for all Pr bisimilar individuals $d \in \Delta^{\mathcal{I}}$, $e \in \Delta^{\mathcal{J}}$ with $\mathcal{I}, \mathcal{J} \in \mathbb{C}$.*

In [7] we proved that $\mathcal{ALCSCC}^{\infty}$ concepts are $\mathbb{C}_{\mathsf{all}}$-invariant under Pr bisimulation. A very similar proof (by induction on the structure of concept descriptions) can be used to show the corresponding result for $\mathcal{ALCSCC}$, where only finitely branching interpretations are considered.

Theorem 1. *Every $\mathcal{ALCSCC}$ concept is $\mathbb{C}_{\mathsf{fb}}$-invariant under Pr bisimulation.*

Proof. Let $\mathcal{I}, \mathcal{J} \in \mathbb{C}_{\mathsf{fb}}$ and ρ a Pr bisimulation relating $d \in \Delta^{\mathcal{I}}$ and $e \in \Delta^{\mathcal{J}}$. We show by induction on the structure of an $\mathcal{ALCSCC}$ concept C that $d \in C^{\mathcal{I}}$ iff $e \in C^{\mathcal{J}}$ holds. The cases where C is a concept name, a conjunction of concepts or the negation of a concept are similar to the analogous cases in the proof of a corresponding result for $\mathcal{ALC}$ [12], and are omitted.

Thus, we focus on the case $C = \mathsf{succ}(\mathfrak{con})$, where we inductively assume that every subconcept of C is $\mathbb{C}_{\mathsf{fb}}$-invariant under Pr bisimulation. Recall that $\mathfrak{con}$ is of the form (1). By applying distributivity of set intersection over set union, it is easy to show that any set term occurring in $\mathfrak{con}$ can be written as the disjoint union of set terms of the form $\tau \cap F$ where τ is a safe role type and F is a Boolean combination of concepts to which the induction assumption applies. The reason we can restrict the attention to safe role types here lies in the semantics of $\mathcal{ALCSCC}$, which considers only role successors when evaluating set terms. We provide for every $\mathcal{ALCSCC}$ concept F and safe role type τ over $\mathsf{N_R}$ an injective mapping from $D := \tau^{\mathcal{I}}(d) \cap F^{\mathcal{I}}$ to $E := \tau^{\mathcal{J}}(e) \cap F^{\mathcal{J}}$ and vice versa. This proves that these sets have the same size, and thus that $\mathfrak{con}$ is evaluated equally w.r.t. d and e. Note that, since $\mathcal{I}$ and $\mathcal{J}$ are finitely branching, the sets D and E are both finite. Overall, this implies that $d \in C^{\mathcal{I}}$ iff $e \in C^{\mathcal{J}}$.

The required injections are obtained as follows. Thanks to the forth property, we find a set $E' \subseteq \tau^{\mathcal{J}}(e)$ such that ρ contains a bijection between D and E'. By our induction hypothesis, the concept F is $\mathbb{C}_{\mathsf{fb}}$-invariant under Pr bisimulation, so we obtain that $E' \subseteq C^{\mathcal{J}}$. Then, $E' \subseteq E$ holds, and the bijection between D and E' is the sought injective mapping from D to E. Using the back property, we similarly prove that there is an injective mapping from E to D.

Together with the other cases, this concludes our proof, and thus we conclude that every $\mathcal{ALCSCC}$ concept is $\mathbb{C}_{\mathsf{fb}}$-invariant under Pr bisimulation. □

Since finite interpretations are finitely branching, this also implies $\mathbb{C}_{\mathsf{fin}}$-invariance of $\mathcal{ALCSCC}$ concepts under Pr bisimulation.

Comparing DLs with bisimulations. To compare the expressive power of DLs with and without concrete domains, we introduced the notion of *abstract expressive power* [10] based on *abstract models*, obtained from models $\mathcal{I}$ that interpret $\mathsf{N_F}$ by "forgetting" the interpretation of $\mathsf{N_F}$. For DLs without concrete domains, models and abstract models coincide. Then, two concepts C and D are *abstractly $\mathbb{C}$-equivalent* if the abstract models of C in $\mathbb{C}$ coincide with those of D. Using Theorem 1 we show that for some instances of $\mathfrak{D}$ we can find $\mathcal{ALC}(\mathfrak{D})$ concepts whose abstract expressive power cannot be captured in $\mathcal{ALCSCC}$.

Theorem 2. *There is an $\mathcal{ALC}(\mathfrak{Q})$ concept that is not abstractly $\mathbb{C}_{\mathsf{fb}}$-equivalent to any $\mathcal{ALCSCC}$ concept.*

Proof. We show that $D := \exists f, rf. <$ is the sought $\mathcal{ALC}(\mathfrak{D})$ concept. Assume, by contradiction, that there exists an $\mathcal{ALCSCC}$ concept C such that the finitely branching abstract models of D coincide with the models of C. Let $\mathcal{I}$ be the interpretation of $\mathsf{N_C}$ and $\mathsf{N_R}$ with $\Delta^{\mathcal{I}} := \{a\}$ and $r^{\mathcal{I}} := \{(a, a)\}$ with $r \in \mathsf{N_R}$. Let $\mathcal{J}$ be the interpretation of $\mathsf{N_C}$ and $\mathsf{N_R}$ whose domain is $\mathbb{N}$ and where $n + 1$ is an r-successor of n for $n \in \mathbb{N}$. The relation $\rho := \{a\} \times \mathbb{N}$ is then a Pr bisimulation, and by Theorem 1 it follows that $a \in C^{\mathcal{I}}$ iff $n \in C^{\mathcal{J}}$ for $n \in \mathbb{N}$.

Clearly, $\mathcal{J}$ is an abstract model of D: by using $f^{\mathcal{J}}(n) := n$ for $n \in \mathbb{N}$ as interpretation of $f \in \mathsf{N_F}$, we obtain that $n \in D^{\mathcal{J}}$ for $n \in \mathbb{N}$. By abstract $\mathbb{C}_{\mathsf{fb}}$-equivalence of C and D, then, $n \in C^{\mathcal{J}}$ and thus $a \in C^{\mathcal{I}}$ must hold. Using abstract $\mathbb{C}_{\mathsf{fb}}$-equivalence again, we deduce that there exists an interpretation of feature names $f^{\mathcal{I}}(a)$ such that $a \in D^{\mathcal{I}}$. This leads to a contradiction, because $a \in D^{\mathcal{I}}$ can happen iff $f^{\mathcal{I}}(a) < f^{\mathcal{I}}(a)$. Therefore, we conclude that C and D cannot be abstractly $\mathbb{C}_{\mathsf{fb}}$-equivalent. □

We can also use Pr bisimulations to compare $\mathcal{ALCSCC}$ with other DLs with expressive counting constraints. In [4], we introduced the logic $\mathcal{ALCSCC}^{++}$ where we replace the restrictions $\mathsf{succ}(\mathfrak{con})$ of $\mathcal{ALCSCC}$ with extended ones of the form $\mathsf{sat}(\mathfrak{con})$. The semantics of this DL is defined w.r.t. finite interpretations $\mathcal{I}$ and restrictions $\mathsf{sat}(\mathfrak{con})$ are interpreted using a QFBAPA assignment σ_d as in $\mathcal{ALCSCC}$, with the difference that here $\mathcal{U}$ is mapped to $\sigma_d(\mathcal{U}) := \Delta^{\mathcal{I}}$. We show that the newly introduced restrictions cannot be expressed in $\mathcal{ALCSCC}$. To compare two concepts w.r.t. a class of interpretations $\mathbb{C}$ we define C and D to be $\mathbb{C}$-equivalent if $C^{\mathcal{I}} = D^{\mathcal{I}}$ holds for all $\mathcal{I} \in \mathbb{C}$.

Theorem 3. *There are $\mathcal{ALCSCC}^{++}$ concepts that are not $\mathbb{C}_{\mathsf{fin}}$-equivalent to any $\mathcal{ALCSCC}$ concept.*

Proof. Assume, by contradiction, that there is an $\mathcal{ALCSCC}$ concept D that is $\mathbb{C}_{\mathsf{fin}}$-equivalent to $C := \mathsf{sat}(|A| \leqslant 1)$. Let $\mathcal{I}$ be the interpretation consisting of a single individual d with $d \in A^{\mathcal{I}}$, and let $\mathcal{J}$ consist of two individuals e, e' with both $e, e' \in A^{\mathcal{J}}$. Clearly, $d \in C^{\mathcal{I}}$ holds while $e, e' \notin C^{\mathcal{J}}$. By the assumption of $\mathbb{C}_{\mathsf{fin}}$-equivalence, we obtain that $d \in D^{\mathcal{I}}$ and $e, e' \notin D^{\mathcal{J}}$. However, the relation $\rho := \{(d, e), (d, e')\}$ is a Pr bisimulation. This leads to a contradiction, since by Theorem 1 it must hold that $d \in D^{\mathcal{I}}$ iff $e, e' \in D^{\mathcal{J}}$. Therefore, we conclude that C and D cannot be $\mathbb{C}_{\mathsf{fin}}$-equivalent. □

$\mathcal{ALCSCC}$ goes beyond FOL $\mathcal{ALC}$ and many other DLs are fragments of first-order logic (FOL) [14], in the sense that for every concept description C of the given DL there is a FOL formula $\phi(x)$ such that $\phi^{\mathcal{I}} = C^{\mathcal{I}}$ for all interpretations $\mathcal{I}$, where $\phi^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \mathcal{I} \models \phi(d)\}$. This notion of definability of a concept description by an FOL formula in one free variable can be relativized to a class of models $\mathbb{C}$ in an obvious way. $\mathbb{C}$-invariance of an FOL formula in one free variable under a given notion of bisimulation is also defined in an obvious way.

In [7], we have shown that there are $\mathcal{ALCSCC}^{\infty}$ concepts that are not FOL-definable in this sense w.r.t. $\mathbb{C}_{\mathsf{all}}$. However, since the semantics of $\mathcal{ALCSCC}$ is defined w.r.t. a restricted class of interpretations, this result does not directly transfer to $\mathcal{ALCSCC}$. Our tool for showing non-FOL-definability for $\mathcal{ALCSCC}$ (and incidentally also for $\mathcal{ALCSCC}^\infty$ w.r.t. other classes of interpretations) is a bounded version of Pr bisimulation where one makes only a bounded number ℓ of steps into the interpretation and bounds the cardinalities of the sets considered in the back and forth conditions by a number q. This notion of bisimulation is obtained by adapting the bisimulation-based characterization of modal logic with graded modalities w.r.t. finite models in [24] to our more expressive logic.

Definition 3. *Let* $\mathsf{N_C}$ *and* $\mathsf{N_R}$ *be finite and* $q, \ell \in \mathbb{N}$. *The relation* $\rho \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{J}$ *is a* Pr $(q,0)$-bisimulation *between the interpretations* $\mathcal{I}$ *and* $\mathcal{J}$ *if it satisfies the* (atomic) *condition of Definition 2, and it is a* Pr $(q,\ell+1)$-bisimulation *if it is a Pr* (q,ℓ)*-bisimulation that satisfies the following for all safe role types* τ*:*

(q,ℓ)-forth *if* $(d,e) \in \rho$ *and* $D \subseteq \tau^\mathcal{I}(d)$ *with* $|D| \leqslant q$, *then there are* $E \subseteq \tau^\mathcal{J}(e)$ *and a Pr* (q,ℓ)*-bisimulation* ρ' *that contains a bijection between* D *and* E*;*
(q,ℓ)-back *if* $(d,e) \in \rho$ *and* $E \subseteq \tau^\mathcal{J}(e)$ *with* $|E| \leqslant q$, *then there are* $D \subseteq \tau^\mathcal{I}(d)$ *and a Pr* (q,ℓ)*-bisimulation* ρ' *that contains a bijection between* D *and* E*.*

The notions of Pr (q,ℓ)*-bisimilarity and* $\mathbb{C}$*-invariance w.r.t. Pr* (q,ℓ)*-bisimulation are defined similarly to how it was done in Definition 2.*

Theorem 1 states that all $\mathcal{ALCSCC}$ concepts are invariant under Pr bisimulation. For Pr (q,ℓ)-bisimulation, this need not hold, as stated in the next theorem.

Theorem 4. *There is an* $\mathcal{ALCSCC}$ *concept* C *such that, for all values of* q *and* ℓ*, the concept* C *is not* $\mathbb{C}_\mathsf{fb}$*-invariant under Pr* (q,ℓ)*-bisimulation.*

Proof. Consider the $\mathcal{ALCSCC}$ concept $C := \mathsf{succ}(|r \cap A| = |r \cap \neg A|)$, which has been used in [7] to show that $\mathcal{ALCSCC}^\infty$ is not a fragment of FOL. For $n, m \in \mathbb{N}$, let $\mathcal{I}_{m,n}$ be the finitely branching interpretation containing individuals d and d_i for $i = 1, \ldots, m+n$, where r is interpreted as the set of tuples (d, d_i) for $i = 1, \ldots, m+n$, every d_i with $i = 1, \ldots, m$ is in A and every other individual is not in A. Given $q \in \mathbb{N}$ we consider $\mathcal{I}_{q,q}$ and $\mathcal{I}_{q,q+1}$, and notice that $d \in \Delta^{\mathcal{I}_{q,q}}$ and $d \in \Delta^{\mathcal{I}_{q,q+1}}$ are Pr (q,ℓ)-bisimilar: the relation mapping $d \in \Delta^{\mathcal{I}_{q,q}}$ to $d \in \Delta^{\mathcal{I}_{q,q+1}}$ and $d_i \in \Delta^{\mathcal{I}_{q,q}}$ to $d_i \in \Delta^{\mathcal{I}_{q,q+1}}$ is a Pr (q,ℓ)-bisimulation for all $\ell \in \mathbb{N}$. However, $d \in C^{\mathcal{I}_{q,q}}$ holds, whereas $d \notin C^{\mathcal{I}_{q,q+1}}$. □

Our goal is now to show that this cannot happen for $\mathcal{ALCSCC}$ concepts that are FOL-definable w.r.t. $\mathbb{C}_\mathsf{fb}$ or $\mathbb{C}_\mathsf{fin}$ (or more generally a class $\mathbb{C}$ of interpretations satisfying certain closure properties). The proof of this result uses certain locality properties of FOL formulae that are invariant under Pr bisimulation.

Definition 4. *Let* $\mathcal{I}$ *be an interpretation. The* distance *of* d *and* d' *in* $\mathcal{I}$ *is the smallest value* $\ell \in \mathbb{N}$ *for which there is a sequence of elements* $d_1, \ldots, d_{\ell+1} \in \Delta^\mathcal{I}$ *where* $d_1 = d$, $d_{\ell+1} = d'$ *and* d_i *is a role successor or predecessor of* d_{i+1} *for* $i = 1, \ldots, \ell$*, or* ∞ *if such a number does not exist. The* ℓ-neighborhood $\mathcal{N}_\ell^\mathcal{I}[\![d]\!]$ *of* d *is derived from* $\mathcal{I}$ *by taking the substructure consisting of all individuals with distance at most* ℓ *from* d*.*

The class $\mathbb{C}$ *of interpretations is* closed under neighborhoods *if* $\mathcal{N}_\ell^\mathcal{I}[\![d]\!] \in \mathbb{C}$ *for all* $\mathcal{I} \in \mathbb{C}$*,* $d \in \Delta^\mathcal{I}$ *and* $\ell \in \mathbb{N}$*. The* FOL *formula* $\phi(x)$ *is* ℓ-local *w.r.t.* $\mathbb{C}$ *if for all* $\mathcal{I} \in \mathbb{C}$ *and all* $d \in \Delta^\mathcal{I}$ *we have that* $\mathcal{I} \models \phi(d)$ *iff* $\mathcal{N}_\ell^\mathcal{I}[\![d]\!] \models \phi(d)$*.*

We observed that every $\mathcal{ALCSCC}$ concept of depth ℓ is ℓ-local. Clearly, we cannot argue the same for first-order formulae $\phi(x)$ of quantifier depth ℓ. As an example, $\phi(x) = \exists y_1.\exists y_2.r(x, y_1) \wedge A(y_2)$ is not 2-local, as any potential individual replacing y_2 need not be in the 2-neighborhood of the individual replacing x. This formula is in particular not ℓ-local for all values of ℓ.

Interestingly, there is a close relationship between ℓ-locality of FOL formulae and invariance under finite disjoint union.

Definition 5 (Disjoint union). *Given a finite index set* $\mathbb{I}$ *and a family of interpretations* $(\mathcal{I}_\nu)_{\nu\in\mathbb{I}} \subseteq \mathbb{C}$*, their* finite disjoint union $\mathcal{I}$ *is defined by:*

$$\Delta^\mathcal{I} := \{(d, \nu) \mid \nu \in \mathbb{I} \text{ and } d \in \Delta^{\mathcal{I}_\nu}\},$$
$$A^\mathcal{I} := \{(d, \nu) \mid \nu \in \mathbb{I} \text{ and } d \in A^{\mathcal{I}_\nu}\} \quad \text{for all } A \in \mathsf{N_C},$$
$$r^\mathcal{I} := \{((d, \nu), (e, \nu)) \mid \nu \in \mathbb{I} \text{ and } (d, e) \in r^{\mathcal{I}_\nu}\} \quad \text{for all } r \in \mathsf{N_R}.$$

The FOL *formula* $\phi(x)$ *is* $\mathbb{C}$-invariant under finite disjoint unions *if, for any finite disjoint union constructed as above,* $\mathcal{I}_\nu \models \phi(d)$ *iff* $\mathcal{I} \models \phi((d, \nu))$ *holds for every* $\nu \in \mathbb{I}$ *and* $d \in \Delta^{\mathcal{I}_\nu}$*. We say that* $\mathbb{C}$ *is* closed under finite disjoint unions *if* $\mathcal{I}_\nu \in \mathbb{C}$ *for all* $\nu \in \mathbb{I}$ *implies that the disjoint union of* $(\mathcal{I}_\nu)_{\nu\in\mathbb{I}}$ *also belongs to* $\mathbb{C}$ *whenever the index set* $\mathbb{I}$ *is finite.*

By proving that $\rho := \{(d, (d, \nu)) \mid d \in \Delta^{\mathcal{I}_\nu}, \nu \in \mathbb{I}\}$ is a Pr bisimulation, we obtain the following property for formulae that are $\mathbb{C}$-invariant under Pr bisimulation.

Proposition 1. *If the* FOL *formula* $\phi(x)$ *is* $\mathbb{C}$*-invariant under Pr bisimulation, then it is* $\mathbb{C}$*-invariant under finite disjoint unions.*

By Theorem 1, this implies that FOL formulae that are equivalent to $\mathcal{ALCSCC}$ concepts are $\mathbb{C}_{\mathsf{fb}}$- and $\mathbb{C}_{\mathsf{fin}}$-invariant under disjoint union. Before we can state the crucial lemma from [23], we must introduce one more notation. We call the class $\mathbb{C}$ of interpretations *localizable* if it is closed under both neighborhoods and finite disjoint unions.[7] Note that our classes $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$ and $\mathbb{C}_{\mathsf{fin}}$ are localizable.

Lemma 1 ([23]). *If* $\mathbb{C}$ *is localizable, then any* FOL *formula* $\phi(x)$ *of quantifier depth* q *that is* $\mathbb{C}$*-invariant under finite disjoint unions is* $(2^q - 1)$*-local w.r.t.* $\mathbb{C}$*.*

Combining this lemma with Proposition 1, we can now link ℓ-locality with invariance under Pr bisimulation.

Corollary 1. *If* $\mathbb{C}$ *is localizable, then any* FOL *formula* $\phi(x)$ *of quantifier depth* q *that is* $\mathbb{C}$*-invariant under Pr bisimulation is* ℓ*-local w.r.t.* $\mathbb{C}$ *for* $\ell := 2^q - 1$*.*

[7] These conditions on $\mathbb{C}$ are not stated explicitly in [23], but are implicitly assumed.

Our next goal is now to show that, for FOL formulae, invariance under Pr bisimulation is equivalent to invariance under Pr (q,ℓ)-bisimulation for some $q, \ell \in \mathbb{N}$. In the proof of Theorem 4 we exhibit, for all values of q, two tree-shaped interpretations whose roots are Pr (q,ℓ)-bisimilar for all values of ℓ, and that are distinguished by the fact that one root satisfies a certain $\mathcal{ALCSCC}$ concept C and the other does not. We prove that this cannot happen for FOL formulae $\phi(x)$ that are ℓ-local and have quantifier depth q.

The notion of tree-shaped interpretation is based on *paths* of length ℓ in $\mathcal{I}$ seen as sequences $p := \langle d_0 \cdots d_{\ell+1}\rangle$ such that $d_{i+1} \in \mathsf{ars}^{\mathcal{I}}(d_i)$ for $i = 0, \ldots, \ell$ and whose *endpoint* is $\mathsf{end}(p) := d_{\ell+1}$. Then, $\mathcal{I}$ is a *tree* of depth ℓ if there exists $d \in \Delta^{\mathcal{I}}$, called the *root* of $\mathcal{I}$, such that every other element in $\mathcal{I}$ is connected to d by exactly one path of length at most ℓ and d is not the endpoint of a path.

To prove that two trees of depth ℓ that have Pr (q,ℓ)-bisimilar roots d, e are such that these roots satisfy the same formulae $\phi(x)$ of quantifier depth at most q, we make use of the Ehrenfeucht-Fraïsse method, which is based on the notion of *q-isomorphism* between d and e (see [16], Definition 1.2.1 of *partial isomorphism* and Definition 1.3.1 of *q-isomorphism*).

Definition 6. *A* partial isomorphism *between interpretations $\mathcal{I}$, $\mathcal{J}$ of* $\mathsf{N_C}$, $\mathsf{N_R}$ *is an injective partial function $p\colon \Delta^{\mathcal{I}} \rightharpoonup \Delta^{\mathcal{J}}$ s.t. $d \in A^{\mathcal{I}}$ iff $p(d) \in A^{\mathcal{J}}$ and $(d, d') \in r^{\mathcal{I}}$ iff $(p(d), p(d')) \in r^{\mathcal{J}}$ holds for all $d, d' \in \Delta^{\mathcal{I}}$ for which p is defined, $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$. A* q-isomorphism *between $d \in \Delta^{\mathcal{I}}$ and $e \in \Delta^{\mathcal{J}}$ is a sequence $I_0, \ldots, I_q$ of non-empty sets of partial isomorphisms between $\mathcal{I}$ and $\mathcal{J}$ with $\{d \mapsto e\} \in I_q$ satisfying the following properties for all $0 \leqslant i < q$:*

i-**forth** *if $p \in I_{i+1}$ and $d' \in \Delta^{\mathcal{I}}$ then there exists $p' \in I_i$ that extends p and such that $p'(d') = e'$ for some $e' \in \Delta^{\mathcal{J}}$;*

i-**back** *if $p \in I_{i+1}$ and $e' \in \Delta^{\mathcal{J}}$ then there exists $p' \in I_i$ that extends p and such that $e' = p'(d')$ for some $d' \in \Delta^{\mathcal{I}}$.*

We say that d, e are q-isomorphic *if there is a q-isomorphism between d and e.*

The following, fundamental result from finite model theory relates the existence of a q-isomorphism between two individuals and the satisfiability of FOL formulae of quantifier depth q w.r.t. these individuals (see [16], Theorem 1.3.2).

Theorem 5 (Ehrenfeucht-Fraïsse). *The individuals d and e are q-isomorphic iff they satisfy the same* FOL *formulae $\phi(x)$ of quantifier depth at most q.*

We show that a Pr (q,ℓ)-bisimulation between the roots of two trees of depth ℓ induces a q-isomorphism between these roots, leading to the following result.

Theorem 6. *If $\mathcal{I}$, $\mathcal{J}$ are trees of depth at most ℓ with roots d, e that are Pr (q,ℓ)-bisimilar, then these roots satisfy the same* FOL *formulae $\phi(x)$ of quantifier depth at most q.*

Proof. If d and e are Pr (q,ℓ)-bisimilar we can define a q-isomorphism $I_0, \ldots, I_q$ between d and e such that for all $p \in I_{q-i}$ and $i = 0, \ldots, q$ the following hold:

i-left if $\langle d_0 \cdots d_m \rangle$ with $d_0 = d$ is a path in $\mathcal{I}$ and $p(d_m)$ is defined, then for $j = 0, \ldots, m$ there is $e_j \in \Delta^{\mathcal{J}}$ such that d_j, e_j are Pr $(q, \ell - j)$-bisimilar and $p(d_j) = e_j$, and $\langle e_0 \cdots e_m \rangle$ with $e_0 = e$ is a path in $\mathcal{J}$;

i-right if $\langle e_0 \cdots e_m \rangle$ with $e_0 = e$ is a path in $\mathcal{J}$ and $e_m = p(d_m)$, then for $j = 0, \ldots, m$ there is $d_j \in \Delta^{\mathcal{I}}$ such that d_j, e_j are Pr $(q, \ell - j)$-bisimilar and $p(d_j) = e_j$, and $\langle d_0 \cdots d_m \rangle$ with $d_0 = d$ is a path in $\mathcal{I}$;

i-branches p is defined on individuals belonging at most i different branches of $\mathcal{I}$ and maps to individuals belonging to at most i diverging paths of $\mathcal{J}$.

Two paths are *diverging* if their length is greater than 1 and neither of the two is a prefix of the other. Clearly, $p := \{d \mapsto e\}$ satisfies all three properties and is a partial isomorphism: d and e satisfy the same concept names by (q,ℓ)-bisimilarity, and they vacuously agree on $\mathsf{N_R}$ since trees do not contain role loops. Let $I_q := \{\{d \mapsto e\}\}$. For $0 \leqslant i < q$, we assume that I_{q-i} is defined and show how to define $I_{q-(i+1)}$ so that i-forth and i-back in Definition 6 are satisfied.

Let $p \in I_{q-i}$ and $d' \in \Delta^{\mathcal{I}}$, consider the unique path $\langle d_0, \ldots, d_{m'} \rangle$ with $m' \leqslant \ell$ between $d_0 := d$ and $d_{m'} := d'$ in $\mathcal{I}$ and let m with $0 \leqslant m < m'$ be the greatest value for which $p(d_{m'})$ is defined. If $m = m'$, we simply add p to $I_{q-(i+1)}$. Otherwise, for $m \leqslant j < m'$ we assume that the partial isomorphism p_j extending p with values for $d_m, \ldots, d_j$ is defined and satisfies $(i+1)$-left, $(i+1)$-right and $(i+1)$-$\mathsf{branches}$, and show how to extend p_j to p_{j+1} by adding a value e_{j+1} for d_{j+1} so that p_{j+1} also satisfies these conditions.

Let τ be the unique safe role type s.t. $(d_j, d_{j+1}) \in \tau^{\mathcal{I}}$. Then, the set D' of τ-successors of d_j for which p_j is defined must contain at most $i < q$ individuals: for $j = m$, this is a clear consequence of i-$\mathsf{branches}$, and for $m < j < m'$ the set D' must be empty as otherwise $p_j(d_j)$ would already have been defined, and we would contradict our definition of m. Since d_j, e_j must be Pr $(q, \ell - j)$-bisimilar due to i-left, i-right and their $(i+1)$-versions, and $D := D' \cup \{d_{j+1}\} \subseteq \tau^{\mathcal{I}}(d_j)$ has size $i + 1 \leqslant q$, there exists a set $E \subseteq \tau^{\mathcal{J}}(e_j)$ of $i + 1$ elements and a bijection $f \colon D \mapsto E$ such that d_x, $f(d_x)$ are Pr $(q,\ \ell - (j + 1))$-bisimilar for $d_x \in D$. We notice that if $d_x \in D'$ then $p_j(d_x) \in \tau^{\mathcal{J}}(e_j)$ must hold because p_j is a partial isomorphism, and that d_x, $p_j(d_x)$ are $(q,\ \ell - (j + 1))$-bisimilar by our assumptions on p_j, hence we can assume that $f(d_x) = p_j(d_x)$. Moreover, $e_{j+1} := f(d_{j+1}) \in E$ cannot be in the image of p_j: this is a direct consequence of i-$\mathsf{branches}$ for $j = m$ as p_j would otherwise map to values over $i + 1$ different branches, and for $m < j < m'$ this would contradict the definition of m.

We define p_{j+1} by extending p_j with $p_{j+1}(d_{j+1}) := e_{j+1}$ and verify that it is a partial isomorphism. First, notice that p_j is injective by assumption and that $p_{j+1}(d_{j+1}) \neq p_{j+1}(d_x)$ if $d_x \neq d_{j+1}$ by definition, hence p_{j+1} is injective. Next, $d_x \in A^{\mathcal{I}}$ iff $p_{j+1}(d_x) \in A^{\mathcal{J}}$ holds if $p_j(d_x)$ is defined, so it is sufficient to notice that $d_{j+1} \in A^{\mathcal{I}}$ iff $p_{j+1}(d_{j+1}) \in A^{\mathcal{J}}$ follows from the fact that d_{j+1} and e_{j+1} are Pr $(q,\ell - (j + 1))$-bisimilar thanks to the atomic condition to conclude that p_{j+1} is a partial isomorphism w.r.t. $\mathsf{N_C}$. To check that $(d_x, d_y) \in r^{\mathcal{I}}$ iff $(p_{j+1}(d_x), p_{j+1}(d_y)) \in r^{\mathcal{J}}$ for all d_x, d_y for which p_{j+1} is defined, we consider the cases not covered by p_j. In the first case, $d_y = d_{j+1}$ with $m \leqslant j < m'$ and so $(d_x, d_{j+1}) \in r^{\mathcal{I}}$ may occur iff $d_x = d_j$, and since we chose $p_{j+1}(d_{j+1})$ to be a τ-successor of $p_{j+1}(d_j)$ iff $d_{j+1} \in \tau^{\mathcal{I}}(d_j)$ we conclude that $(d_x, d_{j+1}) \in r^{\mathcal{I}}$ iff $(p_{j+1}(d_x), p_{j+1}(d_{j+1})) \in r^{\mathcal{I}}$. In the second case, $d_x = d_{j+1}$ and so $(d_{j+1}, d_y) \in r^{\mathcal{I}}$ may occur iff $d_y = d_{(j+1)+1}$ with $m < j' := j+1 < m'$, and so we fall in the first case applied to $d_y = d_{j'+1}$. We thus showed that p_{j+1} is a partial isomorphism w.r.t. $\mathsf{N_R}$ and we conclude that it is a partial isomorphism and add it to $I_{q-(i+1)}$.

The process above shows that $I_0, \ldots, I_q$ satisfies the i-forth condition for $i = 0, \ldots, q$. Using a similar strategy, we show, for a given $e' \in \Delta^{\mathcal{J}}$, how to add $p' \in I_{q-(i+1)}$ that extends $p \in I_{q-i}$ and such that $p(d') = e'$ for some $d' \in \Delta^{\mathcal{I}}$, thus showing that $I_0, \ldots, I_q$ satisfies the i-back condition for $0 \leqslant i < q$. We obtain a q-isomorphism between d and e and conclude by Theorem 5 that they satisfy the same FOL formulae $\phi(x)$ of quantifier depth at most q. □

While not all interpretations in a class $\mathbb{C}$ need to be tree-shaped, we show that, for every interpretation in $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$ or $\mathbb{C}_{\mathsf{fin}}$, it is possible to find a Pr bisimilar interpretation in this class where the ℓ-neighborhood of a specific individual d is a tree with root d. Normally, this is achieved by unravelling [12], but this may yield an infinite interpretation, and is thus not suitable for our setting, where we are also interested in the class $\mathbb{C}_{\mathsf{fin}}$. Instead, we introduce *partial unravelling* of $\mathcal{I}$, which preserves finiteness and (like unraveling) finite branching. Intuitively, the *ℓ-unravelling* of an interpretation $\mathcal{I}$ at an element $d \in \Delta^{\mathcal{I}}$ applies unraveling up to length ℓ, and then adds a copy of $\mathcal{I}$ at the end. The exact definition of this operation, which is an adaptation of the unravelling operation described in [12], is as follows.

Definition 7. *Given an interpretation $\mathcal{I}$ with $d \in \Delta^{\mathcal{I}}$ and $\ell \in \mathbb{N}$, let $\mathcal{I}^u_\ell$ be the interpretation whose domain $\Delta^{\mathcal{I}^u_\ell}$ is the set of all paths of $\mathcal{I}$ of length at most ℓ starting in d, with the following interpretation of $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$:*

$$A^{\mathcal{I}^u_\ell} := \{p \in \Delta^{\mathcal{I}^u_\ell} \mid \mathsf{end}(p) \in A^{\mathcal{I}}\},$$
$$r^{\mathcal{I}^u_\ell} := \{(\langle d_0, \ldots, d_k\rangle, \langle d_0, \ldots, d_k, d_{k+1}\rangle) \in \Delta^{\mathcal{I}^u_\ell} \times \Delta^{\mathcal{I}^u_\ell} \mid (d_k, d_{k+1}) \in r^{\mathcal{I}}\}.$$

The ℓ-unravelling *$\mathcal{I}^d_\ell$ of $\mathcal{I}$ at d is obtained as the union of $\mathcal{I}$ and $\mathcal{I}^u_\ell$ where we additionally add to $r^{\mathcal{I}^d_\ell}$ all $(p, e) \in \Delta^{\mathcal{I}^u_\ell} \times \Delta^{\mathcal{I}}$ such that p has length ℓ and $(\mathsf{end}(p), e) \in r^{\mathcal{I}}$. Then, $\mathbb{C}$ is* closed under partial unravelling *if $\mathcal{I}^d_\ell \in \mathbb{C}$ for all $\mathcal{I} \in \mathbb{C}$, $d \in \Delta^{\mathcal{I}}$ and $\ell \in \mathbb{N}$.*

As mentioned above, the ℓ-unravelling of $\mathcal{I}$ at d provides an element $\langle d\rangle$ that is Pr bisimilar to $d \in \mathcal{I}$ and whose ℓ-neighborhood is tree-shaped.

Proposition 2. *Let $\mathcal{I}^d_\ell$ be the ℓ-unravelling of the interpretation $\mathcal{I}$ at $d \in \Delta^{\mathcal{I}}$, $\langle d\rangle$ the element corresponding to d in $\mathcal{I}^d_\ell$. Then,*

1. *The elements $d \in \Delta^{\mathcal{I}}$ and $\langle d\rangle \in \Delta^{\mathcal{I}^d_\ell}$ are Pr bisimilar.*
2. *The ℓ-neighborhood $\mathcal{N}^{\mathcal{I}^d_\ell}_\ell[\![\langle d\rangle]\!]$ of $\langle d\rangle$ in $\mathcal{I}^d_\ell$ is a tree of depth at most ℓ with root $\langle d\rangle$.*

Proof. Using the notation of Definition 7, we prove that

$$\rho := \{(d, d) \mid d \in \Delta^{\mathcal{I}}\} \cup \{(d, p) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}^u_\ell} \mid d = \mathsf{end}(p)\}$$

is the sought relation. Given that Pr bisimulations are closed under union, it is enough to show that all $(e,p) \in \rho$ satisfy the conditions of Definition 2 to conclude that ρ is an Pr bisimulation, as the first relation in the union trivially is an Pr bisimulation. For each $(e,p) \in \rho$, the (atomic) condition is implied by definition of $A^{\mathcal{I}_\ell}$ for $A \in \mathsf{N_C}$.

Next, we show that ρ satisfies (forth). Let $(e,p) \in \rho$ and $D \subseteq \tau^{\mathcal{I}}(e)$ a finite set for some safe role type τ over $\mathsf{N_R}$. If p is a directed path of length ℓ with starting point d, then every τ-successor of p is an element of $\Delta^{\mathcal{I}}$ and in particular it is a τ-successor of e. We define $D' := D$ and obtain a finite subset of $\tau^{\mathcal{I}_\ell}(p)$ such that ρ contains a bijection between D and D'. If, on the other hand, p has length less than ℓ, all of its τ-successors in $\mathcal{I}_\ell$ are directed paths of the form $p' = p\langle e'\rangle$ for which $(\mathtt{end}(p), e') \in \tau^{\mathcal{I}}$. Since $\mathtt{end}(p) = e$, we deduce that $p' := p\langle e'\rangle \in \tau^{\mathcal{I}_\ell}(p)$ holds for all $e' \in \tau^{\mathcal{I}}(e)$. Since ρ contains all tuples (e', p') of the form above, we conclude that it contains a bijection between D and the finite subset $D' := \{p\langle e'\rangle \mid e' \in D\}$ of $\tau^{\mathcal{I}_\ell}(p)$.

Finally, we show that the (back) direction holds. Let $D' \subseteq \tau^{\mathcal{I}_\ell}(p)$ be a finite set. If p has length ℓ, reusing our previous observations, we derive that $D' \subseteq \Delta^{\mathcal{I}}$. In particular, from $e = \mathtt{end}(p)$ and the definition of $r^{\mathcal{I}_\ell}$ we derive that $e' \in \tau^{\mathcal{I}}(e)$ for all $e' \in D'$. Thus, $D := D'$ is a finite subset of $\tau^{\mathcal{I}}(e)$ and, since $(e', e') \in \rho$ for all $e' \in \Delta^{\mathcal{I}}$, it follows that ρ contains a bijection between D and D'. If p has length less than ℓ, then each element of D' is a d-dipath p' such that $p' = p\langle e'\rangle$ and $(\mathtt{end}(p), e') \in \tau^{\mathcal{I}}$. Since $e = \mathtt{end}(p)$, it follows that $e' \in \tau^{\mathcal{I}}(e)$. Moreover, $(e', p') \in \rho$ by definition of ρ. We deduce that $D := \{\mathtt{end}(p') \mid p' \in D'\} \subseteq \tau^{\mathcal{I}}(e)$ and ρ contains a bijection between D and D'. We proved that ρ satisfies all relevant conditions, hence it is a Pr bisimulation between $\mathcal{I}$ and $\mathcal{I}_\ell$. □

The following result links invariance under Pr bisimulation with invariance under Pr (q,ℓ)-bisimulation for FOL formulae.

Theorem 7. *Let $\mathbb{C}$ be localizable and closed under partial unravelling. For all* FOL *formulae $\phi(x)$, the following are equivalent:*

1. *$\phi(x)$ is $\mathbb{C}$-invariant under Pr bisimulation.*
2. *$\phi(x)$ is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation for some $q, \ell \in \mathbb{N}$.*

Proof. The implication "2. ⇒ 1." is an immediate consequence of the fact that every Pr bisimulation is also a Pr (q,ℓ)-bisimulation for all $q, \ell \in \mathbb{N}$.

To prove the other direction, we assume 1. and that $\phi(x)$ has quantifier depth q. By Corollary 1 we deduce that $\phi(x)$ is ℓ-local w.r.t. $\mathbb{C}$ for $\ell := 2^q - 1$. Given $\mathcal{I}, \mathcal{J} \in \mathbb{C}$ and $d \in \Delta^{\mathcal{I}}$, $e \in \Delta^{\mathcal{J}}$, we know that the ℓ-unravellings $\mathcal{I}_\ell^d$ and $\mathcal{J}_\ell^e$ and the ℓ-neighborhoods $\mathcal{N}_d := \mathcal{N}_\ell^{\mathcal{I}_\ell^d}[\![\langle d\rangle]\!]$ and $\mathcal{N}_e := \mathcal{N}_\ell^{\mathcal{J}_\ell^e}[\![\langle e\rangle]\!]$ also belong to $\mathbb{C}$. Since $\phi(x)$ is $\mathbb{C}$-invariant under Pr bisimulation and ℓ-local w.r.t. $\mathbb{C}$ we obtain

$$\begin{aligned} &\mathcal{I} \models \phi(d) \text{ iff } \mathcal{I}_\ell^d \models \phi(\langle d\rangle) \text{ iff } \mathcal{N}_d \models \phi(\langle d\rangle) \text{ and} \\ &\mathcal{J} \models \phi(e) \text{ iff } \mathcal{J}_\ell^e \models \phi(\langle e\rangle) \text{ iff } \mathcal{N}_e \models \phi(\langle e\rangle). \end{aligned} \quad \text{(by Proposition 2)}$$

If ρ is a Pr (q,ℓ)-bisimulation with $(d,e) \in \rho$, then combining this relation with the Pr bisimulations linking d and $\langle d\rangle$ and e and $\langle e\rangle$ shows that there is a Pr (q,ℓ)-bisimulation ρ' between $\mathcal{I}_\ell^d$ and $\mathcal{I}_\ell^e$ with $(\langle d\rangle, \langle e\rangle) \in \rho'$. Since such a bisimulation looks only ℓ steps into the interpretation, the restriction of ρ' to the respective ℓ-neighborhoods $\mathcal{N}_d$ and $\mathcal{N}_e$ is also a Pr (q,ℓ)-bisimulation. Proposition 2 says that these neighborhoods are trees of depth at most ℓ, and thus we can apply Theorem 6 to obtain $\mathcal{N}_d \models \phi(\langle d\rangle)$ iff $\mathcal{N}_e \models \phi(\langle e\rangle)$. □

Together with Theorem 4, this yields the desired non-definability results since the classes $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$, and $\mathbb{C}_{\mathsf{fin}}$ are localizable and closed under partial unravelling.

Corollary 2. *Let $\mathbb{C}$ be localizable and closed under partial unravelling. Then there are $\mathcal{ALCSCC}$ concepts that are not FOL-definable w.r.t. $\mathbb{C}$.*

The first-order fragment of $\mathcal{ALCSCC}$. In [7], we have established that the FOL-definable subset of $\mathcal{ALCSCC}^\infty$ corresponds to the DL $\mathcal{ALCQ}t$. This DL can be seen both as the extension of $\mathcal{ALCQ}$ where safe role types instead of just role names can be used in qualified number restrictions, and as the restriction of $\mathcal{ALCSCC}$ where only successor restrictions of the form $\mathsf{succ}(|\tau \cap C| \geqslant q)$ are available, where τ is a safe role type, $q \in \mathbb{N}$, and C is an $\mathcal{ALCQ}t$ concept. To make the relationship to qualified number restrictions clear, we write such successor restrictions as $(\geqslant q\, \tau.C)$, and call them qualified number restrictions. Saying that this result was proved in [7] for $\mathcal{ALCSCC}^\infty$ means that it was shown w.r.t. the class $\mathbb{C}_{\mathsf{all}}$. In the following we prove that it also holds for the classes $\mathbb{C}_{\mathsf{fb}}$ and $\mathbb{C}_{\mathsf{fin}}$.

It is easy to see that every $\mathcal{ALCQ}t$ concept can be translated into an equivalent FOL formula with one free variable, and thus $\mathcal{ALCQ}t$ is a FOL-definable fragment of $\mathcal{ALCSCC}$. We will show that all FOL-definable concepts of $\mathcal{ALCSCC}$ are equivalent to one in $\mathcal{ALCQ}t$. We define the *depth* of an $\mathcal{ALCQ}t$ concept to be the maximal nesting of qualified number restrictions and the *breadth* to be the maximal number occurring in a qualified number restriction. With $\mathcal{ALCQ}t_{q,\ell}$ we denote the set of $\mathcal{ALCQ}t$ concepts of depth at most ℓ and breadth at most q. The following results for $\mathcal{ALCQ}t_{q,\ell}$ are established here.

Proposition 3. *Let $\mathbb{C}$ be a class of interpretations, $q, \ell \in \mathbb{N}$, and assume that $\mathsf{N_C}$ and $\mathsf{N_R}$ are finite. Then the following holds:*

1. *Every $\mathcal{ALCQ}t_{q,\ell}$ concept is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation.*
2. *Up to $\mathbb{C}$-equivalence, there are only finitely many $\mathcal{ALCQ}t_{q,\ell}$ concepts.*
3. *For every $\mathcal{I} \in \mathbb{C}$ and $d \in \Delta^\mathcal{I}$ there is an $\mathcal{ALCQ}t_{q,\ell}$ concept $\mathsf{Bisim}^q_\ell[d]$ such that $d \in \mathsf{Bisim}^q_\ell[d]^\mathcal{I}$ and $e \in \mathsf{Bisim}^q_\ell[d]^\mathcal{J}$ for an interpretation $\mathcal{J} \in \mathbb{C}$ and $d \in \Delta^\mathcal{J}$ implies that d and e are (q,ℓ)-bisimilar.*

First, we prove that the first point of Proposition 3 holds.

Theorem 8. *For all classes $\mathbb{C}$ of interpretations and all $q, \ell \in \mathbb{N}$, every $\mathcal{ALCQ}t_{q,\ell}$ concept is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation.*

Proof. We fix $q \geqslant 0$ and prove that every $\mathcal{ALCQ}t_{q,\ell}$ concept is invariant under Pr (q,ℓ)-bisimulation by induction over ℓ. Let $\mathcal{I}, \mathcal{J} \in \mathbb{C}$ be interpretations related by a Pr (q,ℓ)-bisimulation ρ with $(d, e) \in \rho$.

For $\ell = 0$, the fact that all $\mathcal{ALCQ}t_{q,0}$ concepts are Boolean combinations of concept names and that d and e satisfy the same concept names thanks to the atomic condition satisfied by ρ implies that they satisfy the same $\mathcal{ALCQ}t_{q,0}$ concepts, hence that $\mathcal{ALCQ}t_{q,0}$ concepts are $\mathbb{C}$-invariant under Pr $(q,0)$-bisimulation.

We assume inductively that every $\mathcal{ALCQ}t_{q,\ell}$ concept is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation and show that this implies that all $\mathcal{ALCQ}t_{q,\ell+1}$ concepts are $\mathbb{C}$-invariant under Pr $(q,\ell+1)$-bisimulation. Let ρ be a Pr $(q,\ell+1)$-bisimulation with $(d,e) \in \rho$. We show by structural induction over C an $\mathcal{ALCQ}t_{q,\ell+1}$ concept that d and e satisfy the same $\mathcal{ALCQ}t_{q,\ell+1}$ concepts. If $C = A$ is a concept name, this trivially follows from the fact that ρ satisfies the atomic condition. We inductively assume that if a $\mathcal{ALCQ}t_{q,\ell}$ concept D is a proper subconcept of C, then $d \in D^{\mathcal{I}}$ iff $e \in D^{\mathcal{J}}$. Let $C = (\geqslant q'\, \tau.D)$ with D an $\mathcal{ALCQ}t_{q,\ell}$ concept and $q' \leqslant q$. If $d \in C^{\mathcal{I}}$, then there is a set D_C of size $q' \leqslant q$ of τ-successors of d such that $d' \in D^{\mathcal{I}}$ for $d' \in D_C$. Thanks to the (q,ℓ)-forth condition, we find a set E_C of size $q' \leqslant q$ of τ-successors of e and a bijection h from D_C to E_C such that d' and $h(d')$ are Pr (q,ℓ)-bisimilar for $d' \in D_C$. Using our inductive hypothesis on ℓ, we deduce that $e' \in D^{\mathcal{J}}$ for $e' \in E_C$ and conclude that $e \in C^{\mathcal{J}}$. Similarly, we show that $e \in C^{\mathcal{J}}$ implies $d \in C^{\mathcal{I}}$, this time using the (q,ℓ)-back condition.

If $C = \neg D$, then the semantics of negation and our inductive hypothesis on D imply that $d \in (\neg D)^{\mathcal{I}}$ iff $d \notin D^{\mathcal{I}}$ iff $e \notin D^{\mathcal{J}}$ iff $e \in (\neg D)^{\mathcal{J}}$. The treatment is similar for $C = D_0 \sqcap D_1$. We conclude that d and e satisfy the same $\mathcal{ALCQ}t_{q,\ell+1}$ concepts, hence that $\mathcal{ALCQ}t_{q,\ell+1}$ concepts are $\mathbb{C}$-invariant under Pr $(q,\ell+1)$-bisimulation. This concludes our proof by induction over ℓ. □

We prove that $\mathcal{ALCQ}t_{q,\ell}$, unlike $\mathcal{ALCQ}t$, only contains finitely many concepts (up to $\mathbb{C}$-equivalence) if we assume that $\mathsf{N_C}$ and $\mathsf{N_R}$ are finite. This is well-known for $\mathcal{ALC}$, i.e. the modal logic K [26] and for $\mathcal{ALCQ}$, i.e. modal logic with graded modalities [24] and the proof of these facts can be easily extended to $\mathcal{ALCQ}t$. We observe that if we only restricted w.r.t. q, then for $q \geqslant 1$ we could define concepts of arbitrary depth, and similarly if we only restricted w.r.t. ℓ we could write concepts of arbitrary breadth for $\ell \geqslant 1$, while restricting only w.r.t. ℓ is sufficient in logics such as $\mathcal{ALC}$ and $\mathcal{ALC}(\mathfrak{D})$ (as shown in the next section).

Proposition 4. *If $\mathsf{N_C}$ and $\mathsf{N_R}$ are finite sets, then for all values of q and ℓ and all classes of interpretations $\mathbb{C}$ the logic $\mathcal{ALCQ}t_{q,\ell}$, is finite (up to $\mathbb{C}$-equivalence).*

Proof. We fix $q \geqslant 0$ and proceed by induction over ℓ. For $\ell = 0$, we notice that every $\mathcal{ALCQ}t_{q,0}$ concept is a Boolean combination of concept names. Since $\mathsf{N_C}$ is assumed to be finite, we conclude that $\mathcal{ALCQ}t_{q,0}$ is finite up to $\mathbb{C}$-equivalence. Next, we inductively assume that $\mathcal{ALCQ}t_{q,\ell}$ is finite. Then, there exist finitely many qualified number restrictions $(\geqslant k\, \tau.C)$ with $k \leqslant q$, C an $\mathcal{ALCQ}t_{q,\ell}$ concept description and τ a safe role type over $\mathsf{N_R}$ (up to $\mathbb{C}$-equivalence). This holds by finiteness of $\mathsf{N_C}$ and $\mathsf{N_R}$. Every $\mathcal{ALCQ}t_{q,\ell+1}$ concept description is equivalent to a Boolean combination of $\mathcal{ALCQ}t_{q,\ell}$ concepts and qualified number restrictions of the form above. Since there are finitely many such combinations up to $\mathbb{C}$-equivalence, we conclude that $\mathcal{ALCQ}t_{q,\ell+1}$ is finite. □

We want to show that the "converse" of the first statement of this proposition also holds, i.e., individuals that behave the same w.r.t. all $\mathcal{ALCQ}t_{q,\ell}$ concepts are Pr (q,ℓ)-bisimilar.

Definition 8. *Given an interpretation $\mathcal{I}$ with $d \in \Delta^{\mathcal{I}}$, a safe role type τ over $\mathsf{N_R}$ and $q, \ell \in \mathbb{N}$ we consider the mutually recursive $\mathcal{ALCQ}t_{q,\ell}$ concepts*

$$\mathsf{Atomic}[d] := \textstyle\bigsqcap\{A \in \mathsf{N_C} \mid d \in A^{\mathcal{I}}\} \sqcap \bigsqcap\{\neg A \mid A \in \mathsf{N_C}, d \notin A^{\mathcal{I}}\} \qquad (\mathsf{atomic})$$

$$\mathsf{Forth}^{q,\ell}_{\tau}[d] := \textstyle\bigsqcap_{d' \in \tau^{\mathcal{I}}(d)} \mathsf{Forth}^{q,\ell}_{\tau,d'}[d] \qquad ((q,\ell)\text{-}\mathsf{forth})$$

$$\mathsf{Back}^{q,\ell}_{\tau}[d] := \begin{cases} \neg(\geqslant 1\, \tau.(\bigsqcap_{d' \in \tau^{\mathcal{I}}(d)} \neg \mathsf{Bisim}^{q}_{\ell}[d'])) & \text{if } q \geqslant 1 \\ \top & \text{otherwise} \end{cases} \qquad ((q,\ell)\text{-}\mathsf{back})$$

where, assuming that $k \geqslant 1$ is the number of τ-successors of d in $(\mathsf{Bisim}^{q}_{\ell}[d'])^{\mathcal{I}}$,

$$\mathsf{Forth}^{q,\ell}_{\tau,d'}[d] := \begin{cases} (\geqslant k\, \tau.\mathsf{Bisim}^{q}_{\ell}[d']) \sqcap \neg(\geqslant k+1\, \tau.\mathsf{Bisim}^{q}_{\ell}[d']) & \text{if } k < q, \\ (\geqslant q\, \tau.\mathsf{Bisim}^{q}_{\ell}[d']) & \text{otherwise;} \end{cases}$$

and finally

$$\mathsf{Bisim}^{q}_{0}[d] := \mathsf{Atomic}[d]$$

$$\mathsf{Bisim}^{q}_{\ell+1}[d] := \mathsf{Bisim}^{q}_{\ell}[d] \sqcap \textstyle\bigsqcap\{\mathsf{Back}^{q,\ell}_{\tau}[d] \sqcap \mathsf{Forth}^{q,\ell}_{\tau}[d] \mid \tau \text{ safe role type over } \mathsf{N_R}\}.$$

We call $\mathsf{Bisim}^{q}_{\ell}[d]$ the (q,ℓ)-characteristic $\mathcal{ALCQ}t$ concept *of d.*

If $\mathsf{N_C}$ and $\mathsf{N_R}$ are finite then Proposition 4 ensures that characteristic concepts are well-defined, even if $\mathcal{I}$ is not finitely branching, since the conjunctions in $\mathsf{Forth}^{q,\ell}_{\tau}[d]$ and $\mathsf{Back}^{q,\ell}_{\tau}[d]$ contain only finitely many non-equivalent conjuncts. Using the fact that the concepts (atomic), ((q,ℓ)-forth) and ((q,ℓ)-back) in Definition 8 encode the corresponding properties in Definition 3 we show that the relation $\rho_\ell := \{(d,e) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}} \mid e \in (\mathsf{Bisim}^{q}_{\ell}[d])^{\mathcal{J}}\}$ is a Pr (q,ℓ)-bisimulation and obtain the following correspondence.

Theorem 9. *Let $\mathsf{N_C}$, $\mathsf{N_R}$ be finite and $q, \ell \in \mathbb{N}$. Then, $d \in \Delta^{\mathcal{I}}$ and $e \in \Delta^{\mathcal{J}}$ are Pr (q,ℓ)-bisimilar iff they satisfy the same $\mathcal{ALCQ}t_{q,\ell}$ concepts.*

Proof. If d and e are Pr (q,ℓ)-bisimilar then they satisfy the same $\mathcal{ALCQ}t_{q,\ell}$ concepts by Theorem 8. We show by induction over ℓ that the relation ρ_ℓ defined above satisfies all the conditions stated in Definition 3, which implies that ρ_ℓ is a Pr (q,ℓ)-bisimulation. If d and e satisfy the same $\mathcal{ALCQ}t_{q,\ell}$ concepts, then $d \in (\mathsf{Bisim}^{q}_{\ell}[d])^{\mathcal{I}}$ implies that $e \in (\mathsf{Bisim}^{q}_{\ell}[d])^{\mathcal{J}}$ and so we conclude that $(d,e) \in \rho_\ell$.

For $\ell \in \mathbb{N}$, we observe that $e \in (\mathsf{Atomic}[d])^{\mathcal{I}}$ iff for all $A \in \mathsf{N_C}$ it holds that $d \in A^{\mathcal{I}}$ iff $e \in A^{\mathcal{J}}$; since this concept occurs as a conjunct in $\mathsf{Bisim}^{q}_{\ell}[d]$, we conclude that ρ_ℓ fulfills the atomic condition. In particular, this implies that ρ_0 is a Pr $(q,0)$-bisimulation.

Next, we inductively assume that ρ_ℓ is a Pr (q,ℓ)-bisimulation and show that $\rho_{\ell+1}$ is a Pr $(q,\ell+1)$-bisimulation. We start by showing that $\rho_{\ell+1}$ satisfies the

(q,ℓ)-forth condition. Assume that $(d, e) \in \rho_{\ell+1}$ and let $D \subseteq \tau^{\mathcal{I}}(d)$ be a set of size $q' \leqslant q$ for some safe role type τ over $\mathsf{N_R}$. We partition D into sets $D_{d'}$ for $d' \in D$ with $D_{d'} := D \cap (\mathsf{Bisim}^q_\ell[d'])^{\mathcal{I}}$; then, it holds that $q_{d'} := |D_{d'}| \leqslant q$. In particular, $(\geqslant q'\, \tau.(\mathsf{Bisim}^q_\ell[d']))$ with $q' \geqslant q_{d'}$ is a conjunct of $\mathsf{Forth}^{q,\ell}_\tau[d]$, so we conclude that $e \in (\geqslant q'\, \tau.(\mathsf{Bisim}^q_\ell[d']))^{\mathcal{J}}$, hence $e \in (\geqslant q_{d'}\, \tau.(\mathsf{Bisim}^q_\ell[d']))^{\mathcal{J}}$. Thus, there exists a set $E_{d'} \subseteq \mathsf{Bisim}^q_\ell[d']^{\mathcal{J}}$ of τ-successors of e of size $q_{d'}$. Together with the definition of ρ_ℓ, we obtain that $D_{d'} \times E_{d'} \subseteq \rho_\ell$, and since the two sets are of the same size we can find a bijection $f_{d'} \subseteq \rho_\ell$ between them. By combining all mappings $f_{d'}$ with $d' \in D$ we are able to find a bijection $f \subseteq \rho_\ell$ between D and $E := \bigcup_{d' \in D} E_{d'}$. Since ρ_ℓ is inductively assumed to be a Pr (q,ℓ)-bisimulation, we conclude that $\rho_{\ell+1}$ satisfies (q, ℓ)-forth.

Finally, we show that $\rho_{\ell+1}$ satisfies the (q,ℓ)-back condition. Assume that $(d, e) \in \rho_{\ell+1}$ and let E be a subset of $\tau^{\mathcal{J}}(e)$ of cardinality $q' \leqslant q$. Since $e \in (\mathsf{Bisim}^q_{\ell+1}[d])^{\mathcal{J}}$ and $\mathsf{Back}^{q,\ell}_\tau[d]$ is a conjunct of $\mathsf{Bisim}^q_{\ell+1}[d]$, we deduce that for every $e' \in E$ there is some τ-successor d' of d such that $e' \in \mathsf{Bisim}^q_\ell[d']^{\mathcal{J}}$. As done in the previous paragraph, then, we define sets $E_{d'} := E \cap (\mathsf{Bisim}^q_\ell[d'])^{\mathcal{J}}$ with $q_{d'} := |E_{d'}| \leqslant q$ and use them to find a set $D \subseteq \tau^{\mathcal{I}}(d)$ of size q' and a bijection $f \colon E \to D$ included in ρ_ℓ, concluding that $\rho_{\ell+1}$ satisfies (q,ℓ)-back.

Since $\rho_{\ell+1}$ satisfies all the conditions of Definition 3, we conclude that it is a Pr $(q,\ell + 1)$-bisimulation with $(d, e) \in \rho_{\ell+1}$. □

Combining these results with Theorems 1 and 7, we obtain the following characterization of the FOL fragment on $\mathcal{ALCSCC}$.

Theorem 10. *Let $\mathbb{C}$ be localizable and closed under partial unravelling and $\mathsf{N_C}$, $\mathsf{N_R}$ be finite. For all FOL formulae $\phi(x)$, the following are equivalent:*

1. *$\phi(x)$ is $\mathbb{C}$-equivalent to some $\mathcal{ALCSCC}$ concept.*
2. *$\phi(x)$ is $\mathbb{C}$-invariant under Pr bisimulation.*
3. *$\phi(x)$ is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation for some $q, \ell \in \mathbb{N}$.*
4. *$\phi(x)$ is $\mathbb{C}$-equivalent to some $\mathcal{ALCQ}t$ concept.*

Proof. That 1. implies 2. follows from Theorem 1 and the equivalence between 2. and 3. is stated in Theorem 7. In addition, 4. trivially implies 1.

Thus, it is sufficient to show that 3. implies 4. To this purpose, we define $C_\phi := \bigsqcup\{\mathsf{Bisim}^q_\ell[d] \mid \mathcal{I} \in \mathbb{C}, d \in \Delta^{\mathcal{I}} \text{ and } \mathcal{I} \models \phi(d)\}$. By 2. of Proposition 3, this disjunction is finite (up to equivalence), and thus C_ϕ is a well-formed $\mathcal{ALCQ}t_{q,\ell}$ concept. First, assume that $\mathcal{I} \models \phi(d)$ with $\mathcal{I} \in \mathbb{C}$ and $d \in \Delta^{\mathcal{I}}$. Then, $d \in C^{\mathcal{I}}_\phi$ trivially follows from the fact that $\mathsf{Bisim}^q_\ell[d]$ occurs as a disjunct in C_ϕ.

Conversely, if $d \in C^{\mathcal{I}}_\phi$, then $d \in (\mathsf{Bisim}^q_\ell[e])^{\mathcal{I}}$ for some $\mathcal{J} \in \mathbb{C}$ and $e \in \Delta^{\mathcal{J}}$ such that $\mathcal{J} \models \phi(e)$. By 3. of Proposition 3, this implies that d and e are Pr (q,ℓ)-bisimilar. Hence, 3. of the present proposition implies that $\mathcal{I} \models \phi(d)$. Thus, we have shown that $\phi(x)$ and C_ϕ are $\mathbb{C}$-equivalent. □

Recall that the classes $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$ or $\mathbb{C}_{\mathsf{fin}}$ satisfy the assumptions of Theorem 10.

4 The Expressive Power of DLs with Concrete Domains

In [9,10] we have investigated the *abstract expressive power* of DLs with concrete domains, which only considers the abstract part of interpretations, i.e., ignores the values assigned to features. This allowed us to compare classical logics like $\mathcal{ALC}$ and FOL with DLs with concrete domains. Here, we want to compare extensions of $\mathcal{ALC}$ with different concrete domains using an appropriate notion of bisimulation, called $\mathfrak{D}$ bisimulation if $\mathfrak{D}$ is the concrete domain under consideration, and characterize $\mathcal{ALC}(\mathfrak{D})$ as the fragment of $\mathsf{FOL}(\mathfrak{D})$ that is invariant under $\mathfrak{D}$ bisimulation. The employed notion of bisimulation is the one for $\mathcal{ALC}$ (see, e.g., [12]) extended with an additional clause that deals with feature values. As in the previous section, we show our results not only for the class of all interpretations, but also for the restrictions to finitely branching and finite ones.

Definition 9. *Let $\mathfrak{D}$ be a concrete domain and $\mathcal{I}$, $\mathcal{J}$ interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ that assign elements of $\mathfrak{D}$ to features from $\mathsf{N_F}$. The relation $\rho \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is a $\mathfrak{D}$* bisimulation *between $\mathcal{I}$ and $\mathcal{J}$ if for all $A \in \mathsf{N_C}$, all $r \in \mathsf{N_R}$, all k-ary relations P of $\mathfrak{D}$, and all feature paths $p_1, \dots, p_k$ over $\mathsf{N_R}$ and $\mathsf{N_F}$:*

atomic *if $(d, e) \in \rho$ then $d \in A^{\mathcal{I}}$ iff $e \in A^{\mathcal{J}}$;*
forth *if $(d, e) \in \rho$ and $d' \in r^{\mathcal{I}}(d)$, then there is $e' \in r^{\mathcal{J}}(e)$ such that $(d', e') \in \rho$;*
back *if $(d, e) \in \rho$ and $e' \in r^{\mathcal{J}}(e)$, then there is $d' \in r^{\mathcal{I}}(d)$ such that $(d', e') \in \rho$.*
features *if $(d, e) \in \rho$, then there is $(v_1, \dots, v_k) \in P^D$ with $v_1 \in p_1^{\mathcal{I}}(d)$, …, $v_k \in p_k^{\mathcal{I}}(d)$ iff there is $(w_1, \dots, w_k) \in P^D$ with $w_1 \in p_1^{\mathcal{J}}(e)$, …, $w_k \in p_k^{\mathcal{J}}(e)$.*

Bisimilarity between individuals and $\mathbb{C}$-invariance w.r.t. $\mathfrak{D}$ bisimulation are defined similarly to how it was done in Definition 2 w.r.t. Pr bisimulation.

A result analogous to Theorem 1 holds for $\mathcal{ALC}(\mathfrak{D})$ concepts if the concrete domain $\mathfrak{D}$ is weakly closed under negation.

Theorem 11. *If $\mathfrak{D}$ is WCUN and $\mathbb{C}$ is a class of interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ that assign elements of $\mathfrak{D}$ to features from $\mathsf{N_F}$, then every $\mathcal{ALC}(\mathfrak{D})$ concept is $\mathbb{C}$-invariant under $\mathfrak{D}$ bisimulation.*

Proof. The proof by structural induction on the concept C proceeds like the one for $\mathcal{ALC}$ in [12], except for the cases where C is a CD-restriction. We only consider these cases explicitly here. Thus, let ρ be a $\mathfrak{D}$ bisimulation between $\mathcal{I}$ and $\mathcal{J}$ with $(d, e) \in \rho$. We show that d and e satisfy the same CD-restrictions.

If $C := \exists p_1, \dots, p_k.P$ then $d \in C^{\mathcal{I}}$ implies the existence of $v_1 \in p_1^{\mathcal{I}}(d)$,…,$v_k \in p_k^{\mathcal{I}}(d)$ such that $(v_1, \dots, v_k) \in P^D$. Since ρ satisfies features, there must be $w_1 \in p_1^{\mathcal{J}}(e)$, …, $w_k \in p_k^{\mathcal{J}}(e)$ such that $(w_1, \dots, w_k) \in P^D$, hence $e \in C^{\mathcal{J}}$. Similarly, we can show that $e \in C^{\mathcal{J}}$ implies $d \in C^{\mathcal{I}}$.

If $C := \forall p_1, \dots, p_k.P$, then $d \in C^{\mathcal{I}}$ implies that $(v_1, \dots, v_k) \in P^D$ for all values $v_1 \in p_1^{\mathcal{I}}(d)$, …, $v_k \in p_k^{\mathcal{I}}(d)$. Since $\mathfrak{D}$ is WCUN, this is the case iff there are relations P_1, …, P_{n_P} of $\mathfrak{D}$ such that $(v_1, \dots, v_k) \notin P_i^D$ for $i = 1, \dots, n_P$. Using the features condition of ρ, we deduce that $(w_1, \dots, w_k) \notin P_i^D$ for all $w_1 \in p_1^{\mathcal{J}}(e)$, …, $w_k \in p_k^{\mathcal{J}}(e)$ and $i = 1, \dots, n_P$. By WCUN it follows that $(w_1, \dots, w_k) \in P^D$, and we conclude that $e \in C^{\mathcal{J}}$. The proof of the other direction is symmetric. □

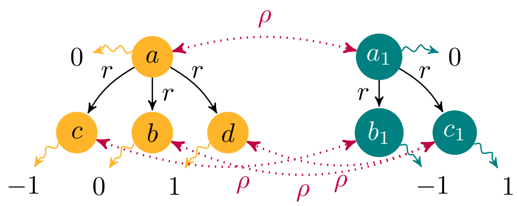

Fig. 1. A $\mathfrak{Q}_{+_2}$ bisimulation ρ between $\mathcal{I}$ (left) and $\mathcal{J}$ (right).

A non-expressivity result. We can use the notion of $\mathfrak{D}$ bisimulation to show that $\mathcal{ALC}(\mathfrak{D})$ cannot express certain concepts of the DL $\mathcal{ALC}(\mathfrak{D}')$, where $\mathfrak{D}'$ has the same domain set as $\mathfrak{D}$, but different relations. Coming back to the example in the introduction, we compare the expressive power of $\mathfrak{Q}_{+1}$ and $\mathfrak{Q}_{+2}$, both having domain set $\mathbb{Q}$, where the former has a binary relation $+_1$ relating $q \in \mathbb{Q}$ and $q+1$ (and the complementary relation $\neq_{+_1}$) and the latter has a binary relation $+_2$ relating q and $q+2$ (and the complementary relation $\neq_{+_2}$).

These two DLs have the same *abstract expressive power*. In fact, we can interchange CD-restrictions using relations $+_1$ and $\neq_{+_1}$ with restrictions of the same kind (existential or universal) using relations $+_2$ and $\neq_{+_2}$. Abstract models of a concept in one of these DLs are then the same as of the corresponding concept in the other DL: in one direction, we just double the feature values, and in the other we halve them. Nevertheless, we can show that their *concrete expressive power*, which takes the feature values into account, is incomparable.

Proposition 5. *Let $\mathbb{C}$ be $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$, or $\mathbb{C}_{\mathsf{fin}}$. There are $\mathcal{ALC}(\mathfrak{Q}_{+_1})$ concepts that are not $\mathbb{C}$-equivalent to any $\mathcal{ALC}(\mathfrak{Q}_{+_2})$ concept (and vice versa).*

Proof. First, consider the $\mathcal{ALC}(\mathfrak{Q}_{+_1})$ concept $C := \exists rf, rf.+_1$ and assume by contradiction that it is $\mathbb{C}_{\mathsf{all}}$-equivalent to some $\mathcal{ALC}(\mathfrak{Q}_{+_2})$ concept D. Let us consider the interpretations $\mathcal{I}$ and $\mathcal{J}$ depicted in Figure 1. Then, $a \in C^{\mathcal{I}}$ and by equivalence $a \in D^{\mathcal{I}}$, while $a_1 \notin C^{\mathcal{J}}$ and so $a_1 \notin D^{\mathcal{J}}$ by equivalence. This leads to a contradiction, since the relation ρ between $\mathcal{I}$ and $\mathcal{J}$ is a $\mathfrak{Q}_{+_2}$ bisimulation relating a and a_1, and by Theorem 11 this means that $a \in D^{\mathcal{I}}$ iff $a_1 \in D^{\mathcal{J}}$. Therefore, we conclude that C and D cannot be equivalent w.r.t. any class of interpretations that contains the two interpretations of Figure 1. Vice versa, we can show with a similar argument that $\exists rf, rf.+_2$ cannot be expressed in $\mathcal{ALC}(\mathfrak{Q}_{+_1})$, but this requires slightly different interpretations. □

We can also use $\mathfrak{D}$ bisimulations to show that some extended CD-restrictions cannot be simulated by normal CD-restrictions. Here, we show that $\mathcal{ALC}(\mathfrak{Q})$ is less expressive than its extension $\mathcal{ALC}_{pp}(\mathfrak{Q})$ where we allow CD-restrictions of the form $\exists p_1, \ldots, p_k.\phi(x_1, \ldots, x_k)$ with $\phi(x_1, \ldots, x_k)$ a conjunction of atomic formulae $P(y_1, \ldots, y_n)$ where $y_1, \ldots, y_n \in \{x_1, \ldots, x_k\}$ and P is a relation of $\mathfrak{Q}$.

Proposition 6. *Let $\mathbb{C}$ be the class of all interpretations. There are $\mathcal{ALC}_{pp}(\mathfrak{Q})$ concepts that are not equivalent to any $\mathcal{ALC}(\mathfrak{Q})$ concept.*

Proof. Consider the $\mathcal{ALC}_{pp}(\mathfrak{Q})$ concept $C := \exists rf, rf, rf.(x < y \wedge y < z)$ and assume by contradiction that it is $\mathbb{C}$-equivalent to some $\mathcal{ALC}(\mathfrak{Q})$ concept D. Let us consider the interpretations $\mathcal{I}$ and $\mathcal{J}$ depicted in Figure 1. Then, $a \in C^{\mathcal{I}}$ and by equivalence $a \in D^{\mathcal{I}}$, while $a_1 \notin C^{\mathcal{J}}$ and so $a_1 \notin D^{\mathcal{J}}$ by equivalence. This leads to a contradiction, since the relation ρ between $\mathcal{I}$ and $\mathcal{J}$ is a $\mathfrak{D}$ bisimulation relating a and a_1 and by Theorem 11 this means that $a \in D^{\mathcal{I}}$ iff $a_1 \in D^{\mathcal{J}}$. Therefore, we conclude that C and D are not $\mathbb{C}$-equivalent. □

FOL with concrete domains and $\mathcal{ALC}(\mathfrak{D})$. Since we are interested in characterizing the concrete expressive power of $\mathcal{ALC}(\mathfrak{D})$, which takes the feature values into account, we cannot compare $\mathcal{ALC}(\mathfrak{D})$ with FOL, where no such values are available. Instead, we consider the extension FOL($\mathfrak{D}$) of FOL with the concrete domain $\mathfrak{D}$ as introduced in [9,10]. The logic FOL($\mathfrak{D}$) is obtained from FOL by adding *definedness predicates* $\mathrm{Def}(f)(t)$ with $f \in \mathsf{N_F}$ and t a first-order term, and *concrete domain predicates* $P(f_1, \ldots, f_k)(t_1, \ldots, t_k)$ where P is a k-ary relation of $\mathfrak{D}$, each t_i is a first-order term and $f_i \in \mathsf{N_F}$ for $i = 1, \ldots, k$.

The semantics of FOL($\mathfrak{D}$) formulae is defined in terms of first-order interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ that additionally assign partial functions $f^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \rightharpoonup D$ to $f \in \mathsf{N_F}$. The semantics of terms, Boolean connectives and first-order quantifiers is defined as usual. Denoting the interpretation of a first-order term t w.r.t $\mathcal{I}$ and a variable assignment w as $t^{\mathcal{I},w}$, the new predicates are interpreted as follows:

- $\mathcal{I} \models \mathrm{Def}(f)(t^{\mathcal{I},w})$ if $f^{\mathcal{I}}(t^{\mathcal{I},w})$ is defined, and
- $\mathcal{I} \models P(f_1, \ldots, f_k)(t_1^{\mathcal{I},w}, \ldots, t_k^{\mathcal{I},w})$ if $(f_1^{\mathcal{I}}(t_1^{\mathcal{I},w}), \ldots, f_k^{\mathcal{I}}(t_k^{\mathcal{I},w})) \in P^D$.

Note that if $(f_1^{\mathcal{I}}(t_1^{\mathcal{I},w}), \ldots, f_k^{\mathcal{I}}(t_k^{\mathcal{I},w})) \in P^D$ then each $f_i^{\mathcal{I}}(t_i^{\mathcal{I},w})$ must be defined.

It is easy to see (and explicitly shown in [9,10]) that $\mathcal{ALC}(\mathfrak{D})$ is a fragment of FOL($\mathfrak{D}$). Our goal is to prove that it is the fragment of FOL($\mathfrak{D}$) that is invariant under $\mathfrak{D}$ bisimulation, not just for the class of all interpretations, but also for finite and finitely branching interpretations. For this, we use an approach that is very similar to the one employed in Section 3. Recall that Lemma 1 turned out to be an important model-theoretic tool in that approach since it provided us with locality results for FOL formulae expressing $\mathcal{ALCSCC}$ concepts. The corresponding result also holds for FOL($\mathfrak{D}$). Note that the notions of *finite disjoint union* and the corresponding $\mathbb{C}$-invariance w.r.t. classes $\mathbb{C}$ of interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ are obtained by extending Definition 5 to account for feature names in the obvious way. For interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ we define ℓ-neighborhoods by using the same notion of distance employed in Definition 4. This means that the distance of two individuals is not determined by concrete domain predicates, but only by role names. The notions of ℓ-locality of a FOL($\mathfrak{D}$) formula and of $\mathbb{C}$-invariance w.r.t. classes $\mathbb{C}$ of interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ are obtained by extending Definition 4 using this notion of neighborhood. In particular, the extension of the classes $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$, and $\mathbb{C}_{\mathsf{fin}}$ to interpretations taking feature names into account are defined in the obvious way, and these classes are localizable.

Lemma 2. *If $\mathbb{C}$ is localizable, then a* FOL($\mathfrak{D}$) *formula $\phi(x)$ of quantifier depth q that is $\mathbb{C}$-invariant under disjoint unions is ℓ-local w.r.t. $\mathbb{C}$ for $\ell := 2^q - 1$.*

Proof. We adopt the same transformation used in [9,10] to map $\phi(x)$ to a FOL formula $\phi^{\mathsf{FOL}}(x)$ of the same quantifier depth and $\mathcal{I} \in \mathbb{C}$ to a FOL interpretation $\mathcal{I}^{\mathsf{FOL}}$. Formally, we replace every atom $P(f_1, \dots, f_k)(t_1, \dots, t_k)$ in $\phi(x)$ with $P^{f_1,\dots,f_k}(t_1, \dots, t_k)$, where $P^{f_1,\dots,f_k}$ is a fresh k-ary predicate symbol for all k-ary relations P of $\mathfrak{D}$ and all $f_1, \dots, f_k \in \mathsf{N_F}$, and every atom of the form $\mathrm{Def}(f)(t)$ in $\phi(x)$ with $\mathrm{Def}_f(t)$ where Def_f is a new predicate symbol for $f \in \mathsf{N_F}$. No newly quantified variable is introduced in this transformation, and so $\phi^{\mathsf{FOL}}(x)$ has quantifier depth q, like $\phi(x)$. We associate to $\mathcal{I} \in \mathbb{C}$ an expansion $\mathcal{I}^{\mathsf{FOL}}$ by the following interpretation of the newly introduced predicates:

- $d \in (\mathrm{Def}_f)^{\mathcal{I}^{\mathsf{FOL}}}$ iff $f^{\mathcal{I}}(d)$ is defined
- $(d_1, \dots, d_k) \in (P^{f_1,\dots,f_k})^{\mathcal{I}^{\mathsf{FOL}}}$ iff $(f_1^{\mathcal{I}}(d_1), \dots, f_k^{\mathcal{I}}(d_k)) \in P^D$.

We denote with $\mathbb{C}^{\mathsf{FOL}}$ the resulting class of interpretations. By the semantics of $\mathsf{FOL}(\mathfrak{D})$, we obtain that for all $\mathsf{FOL}(\mathfrak{D})$ formula $\phi(x)$, all $\mathcal{I} \in \mathbb{C}$ and all $d \in \Delta^{\mathcal{I}}$

$$\mathcal{I} \models \phi(d) \text{ iff } \mathcal{I}^{\mathsf{FOL}} \models \phi^{\mathsf{FOL}}(d). \tag{$\star$}$$

We fix $d \in \Delta^{\mathcal{I}}$ and consider the ℓ-neighborhood $\mathcal{N}$ of d. Let $\mathcal{M}$ be the disjoint union of q copies $\mathcal{I}_1, \dots, \mathcal{I}_q$ of $\mathcal{I}$ and q copies $\mathcal{N}_1, \dots, \mathcal{N}_q$ of $\mathcal{N}$. We define $\mathcal{I}_\star$ as the disjoint union of $\mathcal{I}_0 := \mathcal{I}$ and $\mathcal{M}$, and $\mathcal{N}_\diamond$ as the disjoint union of $\mathcal{N}_0 := \mathcal{N}$ and $\mathcal{M}$. For each $e \in \Delta^{\mathcal{I}}$, $i = 0, \dots, q$ and $j = 1, \dots, q$ we denote with $(e, \mathcal{I}_i)_\star$ the individual in $\mathcal{I}_\star$ corresponding to $e \in \Delta^{\mathcal{I}_i}$ and with $(e, \mathcal{I}_j)_\diamond$ the individual in $\mathcal{N}_\diamond$ corresponding to $e \in \Delta^{\mathcal{I}_j}$. Similarly, if $e \in \Delta^{\mathcal{N}}$ then we introduce the notation $(e, \mathcal{N}_i)_\diamond$ and $(e, \mathcal{N}_j)_\star$.

Since $\mathbb{C}$ is localizable, we deduce that $\mathcal{I}_\star, \mathcal{N}_\diamond \in \mathbb{C}$. By $\mathbb{C}$-invariance under disjoint union of $\phi(x)$ we obtain that

$$\mathcal{I} \models \phi(d) \text{ iff } \mathcal{I}_\star \models \phi((d, \mathcal{I}_0)_\star) \text{ and } \mathcal{N} \models \phi(d) \text{ iff } \mathcal{N}_\diamond \models \phi((d, \mathcal{N}_0)_\diamond),$$

and using ($\star$) we observe that

$$\mathcal{I}_\star \models \phi((d, \mathcal{I}_0)_\star) \text{ iff } \mathcal{I}_\star^{\mathsf{FOL}} \models \phi^{\mathsf{FOL}}((d, \mathcal{I}_0)_\star),$$
$$\mathcal{N}_\diamond \models \phi((d, \mathcal{N}_0)_\diamond) \text{ iff } \mathcal{N}_\diamond^{\mathsf{FOL}} \models \phi^{\mathsf{FOL}}((d, \mathcal{N}_0)_\diamond).$$

We show that $(d, \mathcal{I}_0)_\star$ and $(d, \mathcal{N}_0)_\diamond$ are q-isomorphic. By Theorem 5, this implies that they satisfy the same FOL formulae of quantifier depth at most q and in particular that

$$\mathcal{I}_\star^{\mathsf{FOL}} \models \phi^{\mathsf{FOL}}((d, \mathcal{I}_0)_\star) \text{ iff } \mathcal{N}_\diamond^{\mathsf{FOL}} \models \phi^{\mathsf{FOL}}((d, \mathcal{N}_0)_\diamond),$$

which together with all our previous observations implies that $\mathcal{I} \models \phi(d)$ iff $\mathcal{N} \models \phi(d)$, hence that $\phi(x)$ is ℓ-local.

Note that in this case, a partial isomorphism p between $\mathcal{I}_\star^{\mathsf{FOL}}$ and $\mathcal{N}_\diamond^{\mathsf{FOL}}$ must be defined not only in terms of $\mathsf{N_C}$ and $\mathsf{N_R}$ but also according to the newly introduced definedness and concrete predicates:

$$\mathcal{I}_\star^{\mathsf{FOL}} \models \mathrm{Def}_f(e) \text{ iff } \mathcal{N}_\diamond^{\mathsf{FOL}} \models \mathrm{Def}_f(p(e)) \text{ and}$$
$$\mathcal{I}_\star^{\mathsf{FOL}} \models P^{f_1,\dots,f_k}(e_1, \dots, e_k) \text{ iff } \mathcal{N}_\diamond^{\mathsf{FOL}} \models P^{f_1,\dots,f_k}(p(e_1), \dots, p(e_k))$$

must hold for all feature names $f, f_1, \ldots, f_k$ and $e, e_1, \ldots, e_k$ in $\mathcal{I}_\star^{\mathsf{FOL}}$. Nevertheless, we consider the *distance* of two individuals in $\mathcal{I}_\star^{\mathsf{FOL}}$ and $\mathcal{N}_\diamond^{\mathsf{FOL}}$ as introduced in Definition 4, i.e. in terms of elements connected by roles in $\mathsf{N_R}$ and thus consider neighborhoods in $\mathcal{I}_\star^{\mathsf{FOL}}$ and $\mathcal{N}_\diamond^{\mathsf{FOL}}$ according to this notion of distance.

Following Otto's construction in [23], we build a q-isomorphism $I_0, \ldots, I_q$ such that for $i = 0, \ldots, q$, $p \in I_{q-i}$ and all elements $e := (e', \mathcal{K})_\star$ of $\mathcal{I}_\star^{\mathsf{FOL}}$ for which p is defined we have that, having defined $\ell_i := (2^{q-i} - 1)$, the ℓ_i-neighborhoods of e and $p(e)$ are equal (up to renaming of the elements) and in particular that $p(e) = (e', \mathcal{K}')_\diamond$, where $\mathcal{K}$ and $\mathcal{K}'$ are any of the interpretations considered in the construction of $\mathcal{I}_\star$ and $\mathcal{N}_\diamond$. First, we set $I_q := \{\{(d, \mathcal{I}_0)_\star \mapsto (d, \mathcal{N}_0)_\diamond\}\}$. Since $\mathcal{N}$ is assumed to be the ℓ-neighborhood of d in $\mathcal{I}$ and $\ell = \ell_0$, it is clear that the mapping in I_q satisfies our requirement of equality of the neighborhoods. It is also trivial to see that this is a partial isomorphism w.r.t. $\mathsf{N_C}$, $\mathsf{N_R}$ and the newly introduced predicates, as a consequence of the fact that $f^{\mathcal{I}}(d) = f^{\mathcal{N}}(d)$ holds for all $f \in \mathsf{N_F}$. Assuming that we have defined I_{q-i} with $0 \leqslant i < q$, we show how to define $I_{q-(i+1)}$ so that i-forth Definition 6 is satisfied.

Let $p \in I_{q-i}$ and e an individual in $\mathcal{I}_\star^{\mathsf{FOL}}$. We show how to define a mapping p' that extends p by adding a value $p'(e)$ for e. First, we consider the case where every element e' for which $p(e')$ is defined has distance greater than $\ell_{i+1} + 1$ from e. If e is of the form $(d', \mathcal{N}_j)_\star$, we choose $p'(e)$ to be of the form $(d', \mathcal{N}_k)_\diamond$ for a value $1 \leqslant k \leqslant q$ such that no other element of the form $(d'', \mathcal{N}_k)_\diamond$ is in the image of p. This is always possible, since $\mathcal{N}_\diamond$ contains q copies of $\mathcal{N}$. Similarly, we treat the case where e is of the form $(d', \mathcal{I}_j)_\star$. Next, we consider the case where e has distance at most $\ell_{i+1} + 1$ from some element e' for which $p(e')$ is defined. By construction of I_{q-i} and the fact that $p \in I_{q-i}$, we deduce that e' and $p(e')$ have the same ℓ_i-neighborhoods up to renaming. Assuming that e' is of the form $(d', \mathcal{K})_\star$ with $\mathcal{K}$ of the form $\mathcal{I}_j$ or $\mathcal{N}_j$, we know that e is of the form $(d'', \mathcal{K})_\star$. Moreover, we know that $p(e')$ is of the form $(d'', \mathcal{K}')_\diamond$ with $\mathcal{K}'$ of the form $\mathcal{I}_j$ or $\mathcal{N}_j$, and we thus choose $p'(e)$ to be $(d'', \mathcal{K}')_\diamond$.

We verify that the ℓ_{i+1}-neighborhoods of e and $p'(e)$ are equal. Since for all other elements for which p' is defined this is a trivial consequence of $p \in I_{q-i}$, this is sufficient to conclude that p' satisfies this property for all the individuals on which is it defined. We distinguish two cases. In the first case, this is trivially a consequence of choosing the same individual w.r.t. the same original interpretation (either $\mathcal{N}$ or $\mathcal{I}$). In the second case, we have chosen the same individual (up to renaming) w.r.t. the identical ℓ_i-neighborhoods of two elements e' and $p(e')$ and both individuals have distance at most $\ell_{i+1} + 1$ from e' and $p(e')$ (respectively), which means that the ℓ_{i+1}-neighborhoods of e and $p'(e)$ are fully enclosed in the larger ℓ_i-neighborhoods of e' and $p(e')$ and thus are identical.

What is left is to prove that p' is a partial isomorphism w.r.t. $\mathsf{N_C}$, $\mathsf{N_R}$ and the newly introduced predicates. It is clear that p' is injective, because of the way we choose $p'(e)$ and by inductive hypothesis on p. It is also clear that, by this choice, $e \in A^{\mathcal{I}_\star^{\mathsf{FOL}}}$ iff $p'(e) \in A^{\mathcal{N}_\star^{\mathsf{FOL}}}$ holds for all $A \in \mathsf{N_C}$. Since p is a partial isomorphism w.r.t. $\mathsf{N_C}$ by inductive hypothesis this is sufficient to conclude that p' is a partial isomorphism w.r.t. $\mathsf{N_C}$. Next, we show that p' is a partial isomorphism w.r.t. $\mathsf{N_R}$.

We notice that for all $e', e'' \in \mathcal{I}_\star^{\mathsf{FOL}}$ the fact that $(e, e') \in r^{\mathcal{I}_\star^{\mathsf{FOL}}}$ iff $(p'(e), p'(e')) \in r^{\mathcal{N}_\star^{\mathsf{FOL}}}$ holds follows from the fact that in this case, e' and e'' must have distance $1 \leqslant \ell_{i+1}$ in $\mathcal{I}_\star^{\mathsf{FOL}}$, which means that the corresponding values of $p'(e')$ and $p'(e'')$ also have distance 1 and moreover are respectively equal to e' and e'' up to renaming. Finally, we show that p' is a partial isomorphism w.r.t. the newly introduced predicates. By construction of $\mathcal{N}$, $\mathcal{N}_\diamond$ and $\mathcal{I}_\star$ we know that $f^{\mathcal{N}}(e') = f^{\mathcal{I}}(e')$ for all $e' \in \Delta^{\mathcal{N}}$ and all $f \in \mathsf{N_F}$. Using the definition of disjoint union, we then obtain that $f^{\mathcal{I}_\star}((e', \mathcal{N}_j)_\star) = f^{\mathcal{I}}(e')$ and $f^{\mathcal{N}_\diamond}((e', \mathcal{N}_k)_\diamond) = f^{\mathcal{I}}(e')$ for all $j = 1, \ldots, q$ and $k = 0, \ldots, q$. Clearly, for all $e' \in \Delta^{\mathcal{I}}$ and all $f \in \mathsf{N_F}$ it also holds that $f^{\mathcal{I}_\star}((e', \mathcal{I}_k)_\star) = f^{\mathcal{I}}(e')$ and $f^{\mathcal{N}_\diamond}((e', \mathcal{N}_j)_\diamond) = f^{\mathcal{I}}(e')$ for $j = 1, \ldots, q$ and $k = 0, \ldots, q$. In other words, the feature values of each individual in $\mathcal{I}$ and $\mathcal{N}$ are duplicated over all copies. This clearly implies that $\mathcal{I}_\star^{\mathsf{FOL}} \models \mathrm{Def}_f(e)$ iff $\mathcal{N}_\diamond^{\mathsf{FOL}} \models \mathrm{Def}_f(p'(e))$, and this property is already satisfied for all other elements for which p' is defined by inductive hypothesis on p. Finally, we show that

$$\mathcal{I}_\star^{\mathsf{FOL}} \models P^{f_1,\ldots,f_k}(e_1, \ldots, e_k) \text{ iff } \mathcal{N}_\diamond^{\mathsf{FOL}} \models P^{f_1,\ldots,f_k}(p'(e_1), \ldots, p'(e_k)) \qquad (\dagger)$$

Assuming that $e_j = (e'_j, \mathcal{K}_j)$ for $j = 1, \ldots, k$, we have that $p'(e_j) = (e'_j, \mathcal{K}'_j)$ by construction of p' and the inductive hypothesis on p. Combined with the above, we obtain that

$$\mathcal{I}_\star^{\mathsf{FOL}} \models P^{f_1,\ldots,f_k}(e_1, \ldots, e_k) \text{ iff } \mathcal{I} \models P(f_1, \ldots, f_k)(e'_1, \ldots, e'_k)$$

and

$$\mathcal{N}_\diamond^{\mathsf{FOL}} \models P^{f_1,\ldots,f_k}(p'(e_1), \ldots, p'(e_k)) \text{ iff } \mathcal{I} \models P(f_1, \ldots, f_k)(e'_1, \ldots, e'_k)$$

so we conclude that (†) holds.

We thus showed that p' is a partial isomorphism, and that $I_{q-(i+1)}$ satisfies the i-forth condition. Similarly, we show how to use $p \in I_{q-i}$ and $e' \in \mathcal{N}_\diamond^{\mathsf{FOL}}$ to add a partial isomorphism p' to $I_{q-(i+1)}$ such that $p'(e) = e'$ for some $e \in \mathcal{I}_\star^{\mathsf{FOL}}$, and therefore prove that I_{q-i} satisfies the i-back condition. Overall, we conclude that $I_0, \ldots, I_q$ is a q-isomorphism.

As mentioned in the first part of the proof, this implies that $\mathcal{I} \models \phi(d)$ iff $\mathcal{N} \models \phi(c)$ and thus that $\phi(x)$ is ℓ-local w.r.t. $\mathbb{C}$. □

In the following, we assume that the concrete domain $\mathfrak{D}$ is WCUN and has finitely many relations; both conditions are always satisfied by ω-admissible concrete domains [20,13]. Following the approach employed in the previous section, we introduce a bounded version of $\mathfrak{D}$ bisimulation, where now only the depth is bounded since there are no cardinality constraints.

Definition 10. *Let $\mathcal{I}$, $\mathcal{J}$ be interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ and $\ell \in \mathbb{N}$. The relation $\rho \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is a $\mathfrak{D}$* 0-bisimulation *if ρ satisfies the* atomic *condition of Definition 9 and for all k-ary relations P of $\mathfrak{D}$ and $f_1, \ldots, f_k \in \mathsf{N_F}$:*

values *if $(d, e) \in \rho$ then $(f_1^{\mathcal{I}}(d), \ldots, f_k^{\mathcal{I}}(d)) \in P^D$ iff $(f_1^{\mathcal{J}}(e), \ldots, f_k^{\mathcal{J}}(e)) \in P^D$.*

The relation ρ is a $\mathfrak{D}$ $(\ell+1)$-bisimulation *if it is a $\mathfrak{D}$ ℓ-bisimulation that additionally satisfies the* features *conditions of Definition 9, and for all $r \in \mathsf{N_R}$ the following are satisfied:*

ℓ-**forth** *if $(d,e) \in \rho$ and d' is an r-successor of d, then there exist an r-successor e' of e and a $\mathfrak{D}$ ℓ-bisimulation ρ' such that $(d',e') \in \rho'$;*

ℓ-**back** *if $(d,e) \in \rho$ and e' is an r-successor of e, then there exist an r-successor d' of d and a $\mathfrak{D}$ ℓ-bisimulation ρ' such that $(d',e') \in \rho'$.*

The notions of bisimilarity and $\mathbb{C}$-invariance w.r.t. $\mathfrak{D}$ ℓ-bisimulation are defined similarly to how it was done in Definition 2.

We show that, under the assumption that the concrete domain $\mathfrak{D}$ is WCUN and has finitely many relations, results analogous to Proposition 1, Corollary 1, Theorem 6, Proposition 2, Theorem 7, and Proposition 3 also hold for $\mathsf{FOL}(\mathfrak{D})$ and $\mathcal{ALC}(\mathfrak{D})$, where $\mathcal{ALC}(\mathfrak{D})$ plays both the role of $\mathcal{ALCSCC}$ and of $\mathcal{ALCQ}t$. Since we can prove that every $\mathsf{FOL}(\mathfrak{D})$ formula that is $\mathbb{C}$-invariant under $\mathfrak{D}$ bisimulation is $\mathbb{C}$-invariant under finite disjoint unions similarly to what is done in Proposition 1 for FOL formulae w.r.t. Pr bisimulation, we obtain the following corollary which is analogous to Corollary 1.

Corollary 3. *If $\mathbb{C}$ is localizable, a $\mathsf{FOL}(\mathfrak{D})$ formula $\phi(x)$ of quantifier depth q that is $\mathbb{C}$-invariant under $\mathfrak{D}$ bisimulation is ℓ-local w.r.t. $\mathbb{C}$ for $\ell := 2^q - 1$.*

We obtain the notions of *tree* and *partial unravelling* to interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ by extending Definition 7 to feature names in the obvious way. For trees of depth ℓ, the corresponding version of Theorem 6 for $\mathfrak{D}$ ℓ-bisimulation is simplified to the following, where q-isomorphism is replaced by $\mathfrak{D}$ bisimilarity.

Lemma 3. *If $\mathcal{I}$, $\mathcal{J}$ are trees of depth ℓ with roots d, e that are $\mathfrak{D}$ ℓ-bisimilar, then these roots are $\mathfrak{D}$ bisimilar.*

Proof. We show that a $\mathfrak{D}$ ℓ-bisimulation ρ between d and e induces a $\mathfrak{D}$ bisimulation ρ' such that if $(d',e') \in \rho'$ then d' and e' have the same distance $0 \leqslant \ell' \leqslant \ell$ from d and e and are $\mathfrak{D}$ $(\ell-\ell')$-bisimilar.

We begin by setting $\rho' := \{(d,e)\}$. Clearly, the tuple (d,e) satisfies the property above with $\ell' = 0$. Assuming that $(d',e') \in \rho'$ are $\mathfrak{D}$ $(\ell-\ell')$-bisimilar, for every r-successor d'' of d' we add to ρ' a tuple (d'',e'') where e'' is an r-successor of e' that is $\mathfrak{D}$ $(\ell-(\ell'+1))$-bisimilar to d''. This is always possible: if $\ell' < \ell$ then this is guaranteed by the $(\ell-\ell')$-forth condition, and if $\ell' = \ell$ then d' has no r-successors in $\mathcal{I}$ and so the above is vacuously true. In the first, both d'' and e'' have distance $\ell'+1 \leqslant \ell$ from d and e. Similarly, for every r-successor e'' of e' we add to ρ' a tuple (d'',e'') where d'' is an r-successor of d' that is $\mathfrak{D}$ $(\ell-(\ell'+1))$-bisimilar to e''.

We show that the relation ρ' obtained by exhaustively repeating the process above for $\ell' = 0,\dots,\ell$ is a $\mathfrak{D}$ bisimulation. Since $(d',e') \in \rho'$ implies that d' and e' are $\mathfrak{D}$ ℓ'-bisimilar with $\ell' \geqslant 0$, it clearly holds that ρ' satisfies the atomic condition. By construction of ρ', the forth and back conditions are also clearly satisfied. To see that $\mathsf{features}$ is satisfied by ρ', let $p_1, \ldots, p_k$ be feature paths over $\mathsf{N_R}$ and $\mathsf{N_F}$. If $p_i = f_i$ holds for $i = 1, \ldots, k$, then the values condition of $\mathfrak{D}$ ℓ-bisimulations applied to d' and e' implies that $\mathsf{features}$ is satisfied for $p_1, \ldots, p_k$. Otherwise, $p_i = r_i f_i$ holds for some $1 \leqslant i \leqslant k$. If $(v_1, \ldots, v_k) \in P^D$ with $v_1 \in p_1^{\mathcal{I}}(d'), \ldots, v_k \in p_k^{\mathcal{I}}(d')$ then d' has some role successors, which means d' and e' are $\mathfrak{D}$ ℓ'-bisimilar with $\ell' > 0$ and so we use the $\mathsf{features}$ property of $\mathfrak{D}$ ℓ'-bisimulation to derive that there are $w_1 \in p_1^{\mathcal{J}}(e'), \ldots, w_k \in p_k^{\mathcal{J}}(e')$ such that $(w_1, \ldots, w_k) \in P^D$. The other implication is proved similarly, and we conclude that ρ' satisfies the $\mathsf{features}$ property. Therefore, ρ' is a $\mathfrak{D}$ bisimulation. □

Similar to the case of Pr (q,ℓ)-bisimulation and q-isomorphism in **??**, we obtain the following for $\mathfrak{D}$ ℓ-bisimulation on trees of depth ℓ.

Corollary 4. *If $\mathbb{C}$ is closed under partial unravelling and $\mathcal{I}, \mathcal{J} \in \mathbb{C}$ contain $d \in \Delta^{\mathcal{I}}$, $e \in \Delta^{\mathcal{J}}$ that are $\mathfrak{D}$ ℓ-bisimilar, then $\langle d \rangle \in \Delta^{\mathcal{I}_\ell}$ and $\langle e \rangle \in \Delta^{\mathcal{J}_\ell}$ satisfy the same ℓ-local $\mathsf{FOL}(\mathfrak{D})$ formulae $\phi(x)$ that are $\mathbb{C}$-invariant under $\mathfrak{D}$ bisimulation.*

By adapting the proof of Theorem 7 to use Corollary 4 instead of **??** and Corollary 3 instead of Corollary 1, we obtain the following analogous of Theorem 7 for $\mathfrak{D}$ bisimulation.

Theorem 12. *Let $\mathbb{C}$ be localizable and closed under partial unravelling. Then, a $\mathsf{FOL}(\mathfrak{D})$ formula $\phi(x)$ is $\mathbb{C}$-invariant under $\mathfrak{D}$ bisimulation iff it is $\mathbb{C}$-invariant under $\mathfrak{D}$ ℓ-bisimulation for some value of ℓ.*

To conclude, we will show that for $\mathsf{FOL}(\mathfrak{D})$ formulae $\phi(x)$ $\mathbb{C}$-invariance under $\mathfrak{D}$ ℓ-bisimulation implies $\mathbb{C}$-equivalence to some $\mathcal{ALC}(\mathfrak{D})$ concept C, where C is in particular a concept of depth ℓ. Similarly to what was done earlier for $\mathcal{ALCQ}t$, we define $\mathcal{ALC}(\mathfrak{D})_\ell$ as the subset of $\mathcal{ALC}(\mathfrak{D})$ whose concepts have nesting level at most ℓ, where the depth of a CD-restriction $\exists p_1, \ldots, p_k.P$ is 1 if $p_i = r_i f_i$ for some $i = 1, \ldots, k$ and 0 otherwise. As in the case of $\mathcal{ALCQ}t_{q,\ell}$ we observe that $\mathcal{ALC}(\mathfrak{D})_\ell$ is finite, up to $\mathbb{C}$-equivalence.

Proposition 7. *If $\mathfrak{D}$ has finitely many relations and $\mathsf{N_C}$, $\mathsf{N_R}$, $\mathsf{N_F}$ are finite, then $\mathcal{ALC}(\mathfrak{D})_\ell$ has finitely many concepts (up to $\mathbb{C}$-equivalence) for all $\ell \in \mathbb{N}$.*

Proof. If $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ are finite then there are only finitely many k-tuples of feature paths over $\mathsf{N_R}$ and $\mathsf{N_F}$ for all values of k; since $\mathfrak{D}$ has finitely many relations, this means that there are only finitely many CD-restrictions in $\mathcal{ALC}(\mathfrak{D})_\ell$.

We prove that our claim holds by induction over ℓ. For $\ell = 0$, this trivially holds because $\mathsf{N_C}$ is finite and by our observation regarding CD-restrictions. For the inductive step, we assume that the claim holds for ℓ and show that the same applies for $\ell + 1$. Every concept in $\mathcal{ALC}(\mathfrak{D})_{\ell+1}$ is a Boolean combination of CD-restrictions, $\mathcal{ALC}(\mathfrak{D})_\ell$ concepts and role restrictions $\exists r.C$ with $r \in \mathsf{N_R}$ and C a $\mathcal{ALC}(\mathfrak{D})_\ell$ concept. By inductive hypothesis, there can be only finitely many role restrictions of this form and $\mathcal{ALC}(\mathfrak{D})_\ell$ concepts (up to $\mathbb{C}$-equivalence). Together with our observation above on the number of CD-restrictions, we deduce that there can only be finitely many non-equivalent Boolean combinations of the described form. Therefore, we conclude that the claim holds for $\mathcal{ALC}(\mathfrak{D})_{\ell+1}$. □

Moreover, under our assumptions on $\mathfrak{D}$, we obtain the invariance of $\mathcal{ALC}(\mathfrak{D})_\ell$ under $\mathfrak{D}$ ℓ-bisimulation.

Proposition 8. *If $\mathfrak{D}$ is WCUN and has finitely many relations, then $\mathcal{ALC}(\mathfrak{D})_\ell$ concepts are invariant under $\mathfrak{D}$ ℓ-bisimulation.*

It is again possible, as in the case of Pr (q,ℓ)-bisimulation and $\mathcal{ALCQ}t_{q,\ell}$, to define a *characteristic ℓ-concept* in $\mathcal{ALC}(\mathfrak{D})_\ell$ that describes all individuals that are $\mathfrak{D}$ ℓ-bisimilar to $d \in \Delta^\mathcal{I}$. Assuming that $\hat{f}^\mathcal{I}(d) := (f_1^\mathcal{I}(d), \dots, f_k^\mathcal{I}(d))$ with $f_1, \dots, f_k \in \mathsf{N_F}$ and that $p_1, \dots, p_k$ are feature paths over $\mathsf{N_R}$, $\mathsf{N_F}$ we define

$$\begin{aligned}
\mathsf{Values}_\exists[d] &:= \textstyle\bigsqcap\{\exists f_1,\dots,f_k.P \mid \hat{f}^\mathcal{I}(d) \in P^D\}\\
\mathsf{Values}_\forall[d] &:= \textstyle\bigsqcap\{\forall f_1,\dots,f_k.P \mid \text{if } f_1^\mathcal{I}(d),\dots,f_k^\mathcal{I}(d) \text{ are defined then } \hat{f}^\mathcal{I}(d) \in P^D\}\\
\mathsf{Features}_\exists[d] &:= \textstyle\bigsqcap\{\exists p_1,\dots,p_k.P \mid \textit{some} \text{ tuple in } p_1^\mathcal{I}(d) \times \dots \times p_k^\mathcal{I}(d) \text{ is in } P^D\}\\
\mathsf{Features}_\forall[d] &:= \textstyle\bigsqcap\{\forall p_1,\dots,p_k.P \mid \textit{every} \text{ tuple in } p_1^\mathcal{I}(d) \times \dots \times p_k^\mathcal{I}(d) \text{ is in } P^D\}
\end{aligned}$$

By Proposition 7, these concepts are well-defined. Next, we define

$$\begin{aligned}
\mathsf{Forth}_\ell[d] &:= \textstyle\bigsqcap_{r\in\mathsf{N_R}}\bigsqcap_{e\in r^\mathcal{I}(d)}\exists r.\mathsf{Bisim}_\ell[e]\\
\mathsf{Back}_\ell[d] &:= \textstyle\bigsqcap_{r\in\mathsf{N_R}}\forall r.(\bigsqcup_{e\in r^\mathcal{I}(d)}\mathsf{Bisim}_\ell[e])
\end{aligned}$$

and finally introduce the concepts

$$\begin{aligned}
\mathsf{Bisim}_0[d] &:= \mathsf{Values}_\exists[d] \sqcap \mathsf{Values}_\forall[d] \sqcap \mathsf{Atomic}[d]\\
\mathsf{Bisim}_{\ell+1}[d] &:= \mathsf{Bisim}_\ell[d] \sqcap \mathsf{Features}_\exists[d] \sqcap \mathsf{Features}_\forall[d] \sqcap \mathsf{Forth}_\ell[d] \sqcap \mathsf{Back}_\ell[d]
\end{aligned}$$

where the concept $\mathsf{Atomic}[d]$ is defined as in Definition 8.

Theorem 13. *If $\mathfrak{D}$ is WCUN and has finitely many relations and $\mathsf{N_C}$, $\mathsf{N_R}$, $\mathsf{N_F}$ are finite then $d \in \Delta^\mathcal{I}$, $e \in \Delta^\mathcal{J}$ are $\mathfrak{D}$ ℓ-bisimilar iff they satisfy the same $\mathcal{ALC}(\mathfrak{D})_\ell$ concepts.*

Proof. The proof is similar to that of Theorem 9, where we additionally need to test that $\rho_\ell := \{(d,e) \in \Delta^\mathcal{I}\times\Delta^\mathcal{J} \mid e \in (\mathsf{Bisim}_\ell[d])^\mathcal{J}\}$ satisfies the values condition and additionally the features condition if $\ell > 0$. To prove that ρ_ℓ satisfies values, assume that for $f_1, \dots, f_k \in \mathsf{N_F}$ and a k-ary relation P of $\mathfrak{D}$ it holds that $(f_1^\mathcal{I}(d), \dots, f_k^\mathcal{I}(d)) \in P^D$. Then, $\exists f_1, \dots, f_k.P$ is a conjunct of $\mathsf{Values}_\exists[d]$, and since $e \in \mathsf{Bisim}_\ell[d]^\mathcal{J}$ we derive that $(f_1^\mathcal{J}(e), \dots, f_k^\mathcal{J}(e)) \in P^D$. Vice versa, if $(f_1^\mathcal{I}(d), \dots, f_k^\mathcal{I}(d)) \notin P^D$, then by WCUN we find k-ary predicates $P_1, \dots, P_{n_P}$ such that if $f_j^\mathcal{I}(d)$ is defined for $j = 1, \dots, k$ then $(f_1^\mathcal{I}(d), \dots, f_k^\mathcal{I}(d)) \in P_i^D$ holds for some $1 \leqslant i \leqslant n_P$. This means that $\forall f_1, \dots, f_k.P_i$ is a conjunct of $\mathsf{Values}_\forall[d]$. Then, either $f_j^\mathcal{J}(e)$ is undefined for some $1 \leqslant j \leqslant k$ or $(f_1^\mathcal{J}(e), \dots, f_k^\mathcal{J}(e)) \in P_i^D$ holds, and by WCUN this implies that $(f_1^\mathcal{J}(e), \dots, f_k^\mathcal{J}(e)) \notin P^D$. We conclude that ρ_ℓ satisfies values. Similarly, we verify that ρ_ℓ satisfies features if $\ell > 0$, taking care of replacing tuples $f_1, \dots, f_k \in \mathsf{N_F}$ with tuples $p_1, \dots, p_k$ of feature paths and replacing $\mathsf{Values}_\exists[d]$, $\mathsf{Values}_\forall[d]$ with $\mathsf{Features}_\exists[d]$, $\mathsf{Features}_\forall[d]$. □

Similarly to the proof of Theorem 7, these results can be combined to show the following characterization of $\mathcal{ALC}(\mathfrak{D})$ as the fragment of $\mathsf{FOL}(\mathfrak{D})$ that is invariant under $\mathfrak{D}$ bisimulation.

Theorem 14. *Let $\mathbb{C}$ be localizable and closed under partial unravelling, $\mathfrak{D}$ be WCUN and have finitely many relations, and $\mathsf{N_C}$, $\mathsf{N_R}$, $\mathsf{N_F}$ be finite. Then the following are equivalent for all $\mathsf{FOL}(\mathfrak{D})$ formulae $\phi(x)$:*

1. *$\phi(x)$ is $\mathbb{C}$-invariant under $\mathfrak{D}$ bisimulation.*
2. *$\phi(x)$ is $\mathbb{C}$-invariant under $\mathfrak{D}$ ℓ-bisimulation for some $\ell \in \mathbb{N}$.*
3. *$\phi(x)$ is equivalent to an $\mathcal{ALC}(\mathfrak{D})$ concept.*

Recall that the classes $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$ and $\mathbb{C}_{\mathsf{fin}}$ satisfy the assumptions of Theorem 14. We further remark that, in contrast to the case of $\mathcal{ALCSCC}$, where there are concepts that are not FOL-definable, every $\mathcal{ALC}(\mathfrak{D})$ concept is $\mathsf{FOL}(\mathfrak{D})$-definable.

5 Conclusion

We have investigated the expressive power of concept description languages that allow their users to employ numerical constraints when defining concepts in two orthogonal ways. In contrast to our previous results on the expressive power of such languages [7,8,9,10], the approach employed here also works for restricted classes of interpretations such as finitely branching or finite ones. In [8], we have characterized the expressive power of TBoxes and cardinality boxes of $\mathcal{ALCSCC}^\infty$ (where arbitrary interpretations are considered) using global Pr bisimulations. It is at the moment not clear to us whether the results obtained there can be extended to the restricted classes of interpretations considered in the present paper. Another interesting topic for future research is to study the expressive power of $\mathcal{ALCOSCC}(\mathfrak{D})$, a joint extension of both $\mathcal{ALCSCC}$ and $\mathcal{ALC}(\mathfrak{D})$, whose complexity has recently been analyzed in [5]. The DLs $\mathcal{ALCSCC}$ and $\mathcal{ALC}(\mathfrak{D})$ are closed under all Boolean operations, whereas Kurtonina and de Rijke [19] characterize the expressive power of sub-Boolean fragments of $\mathcal{ALC}$. It would be interesting to see whether their results can be extended to the corresponding fragments of $\mathcal{ALCSCC}$ and $\mathcal{ALC}(\mathfrak{D})$. Like most bisimulation-based characterizations of the expressive power of logics, we assume here that the concept D in the DL $\mathcal{L}_2$ expressing the concept C in the DL $\mathcal{L}_1$ must be built over the same signature as C, i.e., no auxiliary symbols may be used. It would again be interesting to see whether inexpressivity results such as the one in Proposition 5 still hold if the use of auxiliary symbols is allowed, as for instance in [1,2].

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