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Complexity Theory

Exercise 4: Time Complexity

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Exercise 4.1. Consider the problem **CLIQUE**:

Input:	An undirected graph G and some $k \in \mathbb{N}$
Question:	Does there exists a clique in G of size at least k?

For an undirected graph $G = (V, E)$ (i.e., with symmetric $E \subseteq V \times V$), a *clique* in G of size $k \in \mathbb{N}$ is a subset of nodes $C \subseteq V$ with $|C| = k$ and $C \times C \subseteq E$.

Suppose **CLIQUE** can be solved in time $T(n)$ for some $T\colon \mathbb{N} \to \mathbb{N}$ with $T(n) \geq n$ for all $n \in \mathbb{N}$. Furthermore, show that then the optimisation problem

Input:	An undirected graph G
Compute:	A clique in G of maximal size

can be computed in time $\mathcal{O}(n \cdot T(n))$. You can assume that T is monotone.

Exercise 4.2. Show that if a language L is NP-complete, then $\overline{L}$ is coNP-complete.

Exercise 4.3. Show that if $\mathrm{P} = \mathrm{NP}$, then a polynomial-time algorithm exists that produces a satisfying assignment of a given satisfiable propositional formula.

Exercise 4.4. Show that finding paths of a given length in undirected graphs, i.e.,

$$\textsc{Path} = \{ \langle G, s, t, k \rangle \mid G \text{ contains a simple path from } s \text{ to } t \text{ of length } k \}$$

is NP-complete.

$*$ **Exercise 4.5.** Let $A \subseteq 1^*$. Show that if A is NP-complete, then $\mathsf{P} = \mathsf{NP}$.

Proceed as follows: Consider a polynomial-time reduction f from SAT to A. For a formula φ, let φ_{0100} be the reduced formula where variables x_1, x_2, x_3, x_4 in φ are set to the values 0, 1, 0, 0, respectively. (The particular choice of 4 variables as well as of 0100 is arbitrary here) What happens when one applies f to all of these exponentially many reduced formulas?