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The first seven exercises are *must-solve*. Do your best to solve them on your own.

### Exercise 1

Is it true that every satisfiable FO-theory that has an infinite model has also a finite model?

### Exercise 2

Is it true that every satisfiable FO-theory that has a finite model has also an infinite model?

### Exercise 3

Show that if an FO-theory has finite models of arbitrary large sizes[1] then it also has an infinite model.

### Exercise 4

Employ compactness theorem to show that the following property is not-$\mathsf{FO}[\{U\}]$-definable over finite models (with unary relational symbol $U$): "for a finite $\mathfrak{A}$ both $|U^{\mathfrak{A}}|$ and $|A \setminus U^{\mathfrak{A}}|$ are even".

### Exercise 5

Consider the following finitary analogous of the compactness theorem and show that it is not true.

**False Theorem 1.** *Let $\mathcal{T}$ be an* FO *theory. If every finite $\mathcal{T}_0 \subseteq \mathcal{T}$ has a finite model then also $\mathcal{T}$ has a finite model.*

### Exercise 6

Here we assume that vocabularies are finite. Show that over finite structures elementary equivalence collapse to isomorphism, i.e. for any two structures $\mathfrak{A}, \mathfrak{B}$ if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy exactly the same first-order formulae then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. Hint: It suffices to show that $\mathfrak{A}$ can be characterised up-to-isomorphism by an FO formula.

### Exercise 7

Does the Skolem theorem make sense over finite models?

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The following exercises are a bit more difficult than the previous ones, but are much more interesting.

### Exercise 8

Show with compactness theorem that in $\mathsf{FO}[<]$ it is not possible to axiomatize $(\omega, <)$. More precisely, there is no formula $\varphi_\omega$ such that if satisfied in $\mathfrak{A}$ iff $\mathfrak{A}$ is isomorphic to natural numbers with their standard ordering.
Hint: Use an extra symbol $c$ and write that it lies "far" from the beginning.

### Exercise 9

Prove that if an FO-theory $\mathcal{T}$ over $\Sigma$ has an infinite model than it has also has a model of any cardinality $\kappa$ with $\kappa \geq \max(\aleph_0, |\Sigma|)$. Conclude that, in stark contrast to Exercise 6, no infinite structure can be described up to isomorphism by a first-order formula. Hint: For the proof make $|\Sigma|$ as big as it suits you.

### Exercise 10

In the *model-checking problem*, we ask, for a given input $(\mathfrak{A}, \varphi)$ composed of a finite structure $\mathfrak{A}$ and an FO formula $\varphi$, whether $\mathfrak{A} \models \varphi$ holds. Show that the model-checking problem is decidable.

### Exercise 11

Let $\mathcal{L}$ be a sublogic of FO (i.e. the set of $\mathcal{L}$ sentences is a subset of the set of FO sentences). We say that $\mathcal{L}$ has the *finite model property* (FMP) if every satisfiable formula from $\mathcal{L}$ is also finitely satisfiable (i.e. has a finite model). Does FO has FMP? Show that if $\mathcal{L}$ has FMP then it is decidable, namely there exists an algorithm that answers "yes" iff an input $\varphi \in \mathcal{L}$ is satisfiable and "no" otherwise.
Hint: You must heavily rely on Gödel's completeness theorem and the fact that proofs are finite. Do not care about the running time of your program (actually it might work as long as it wants).

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[1] i.e. for each natural number $n$ it has a finite model with more than $n$ elements.