

Complexity Theory
Exercise 9: Circuit Complexity
14th January 2025

Exercise 9.1. Define the function $\text{maj}_n : \{0, 1\}^n \rightarrow \{0, 1\}$ by

$$\text{maj}_n(x_1, \dots, x_n) := \begin{cases} 0 & \text{if } \sum x_i < n/2 \\ 1 & \text{if } \sum x_i \geq n/2. \end{cases}$$

Devise a circuit to compute maj_3 and test it on the example input 101 and 010.

Exercise 9.2. Denote with $\text{add} : \{0, 1\}^{2n} \rightarrow \{0, 1\}^{n+1}$ the function that takes two binary n -bit numbers x and y and returns their $n + 1$ -bit sum. Show that add can be computed with size $\mathcal{O}(n)$ circuits.

Exercise 9.3. Show $\text{NC}^1 \subseteq \text{L}$.

Exercise 9.4. Show that every Boolean function with n variables can be computed with a circuit of size $\mathcal{O}(n \cdot 2^n)$.

Exercise 9.5. Show that every language $L \subseteq \{1^n \mid n \in \mathbb{N}\}$ is contained in P/poly. Conclude that P/poly contains undecidable languages.

Exercise 9.6. Find a decidable language in P/poly that is not contained in P.

Hint:

Take a language over $\{0, 1\}$ that is EXP-TIME-hard and consider its unary encoding.

Exercise 9.7. Show how to compute maj_n with circuits of size $\mathcal{O}(n \log n)$.

Exercise 9.8. Show that $\text{NC} \neq \text{PSPACE}$.