# 3. Petri Nets: The Basics

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# Petri Nets: Basic Notions

**Definition 3.1:** A triple $N = (P, T, F)$ is a **net structure** if $P$ and $T$ are disjoint finite sets, and $F \subseteq (P \times T) \cup (T \times P)$.

- elements $p \in P$ are called places, elements $t \in T$ transitions.
- $F$ is a flow relation; its elements are called arcs
- for a node $n \in P \cup T$,

  the preset of $n$ is ${}^{\bullet}n := \{m \mid (m, n) \in F\}$ and
  the postset of $n$ is $n^{\bullet} := \{o \mid (n, o) \in F\}$

# Petri Nets: Token Game

- states of a Petri net are distributions of so-called tokens on the places of a net
- we use **multisets**: for set $S$, $m : S \to \mathbb{N}$ is a multiset over $S$
- for multisets $m_1, m_2$ over $S$, define
  - union $m_1 + m_2$, such that for $s \in S$, $(m_1 + m_2)(s) = m_1(s) + m_2(s)$;
  - difference $m_1 - m_2$, such that for $s \in S$, $(m_1 - m_2)(s) = \max\{m_1(s) - m_2(s), 0\}$;
  - inclusion $m_1 \leq m_2$, which holds if, and only if, $\forall s \in S : m_1(s) \leq m_2(s)$
- note, $m_1 \not\leq m_2$ if $m_2 \leq m_1$ or $m_1$ and $m_2$ are incomparable

**Definition 3.2:** Let $N = (P, T, F)$ a net structure. We call a multiset $m$ over $P$ a **marking of** $N$. A transition $t \in T$ is **enabled under** $m$ if ${}^\bullet t \leq m$. An enabled transition $t \in T$ (under $m$) my **fire**, producing the successor marking

$$m'(p) := \begin{cases} m(p) - 1 & \text{if } p \in {}^\bullet t \setminus t^\bullet \\ m(p) + 1 & \text{if } p \in t^\bullet \setminus {}^\bullet t \\ m(p) & \text{otherwise.} \end{cases}$$

## Petri Nets

- Enabledness of $t$ (in net $N$) under $m$ is denoted by $m[t\rangle_N$
- If $m'$ is the successor marking of $m$ by firing $t$ (in net $N$), we write $m[t\rangle_N m'$
- The set of reachable markings from $m$ in $N = (P, T, F)$ is $[N, m\rangle$, defined inductively
  1. $m \in [N, m\rangle$ and
  2. if $m_1 \in [N, m\rangle$ and $m_1[t\rangle_N m_2$ for some $t \in T$, then $m_2 \in [N, m\rangle$.

**Definition 3.3:** A **Petri net** (elementary net system) is a quadruple $N = (P, T, F, m_0)$, where $(P, T, F)$ is a net structure and $m_0$ a marking for $(P, T, F)$. We call $m_0$ the **initial marking** of $N$. $[N\rangle := [(P, T, F), m_0\rangle$ is the set of all reachable markings of $N$.

**Definition 3.4:** The **reachability graph** of a Petri net $N = (P, T, F, m_0)$ is the directed graph $\mathcal{R}(N) = (V, E)$ with $V = [N\rangle$ and if $m, m' \in V$ and $m[t\rangle_N m'$, then $(m, m') \in E$. The reachability graph may also be labeled, having $(m, t, m') \in E$ if $m[t\rangle_N m'$.

# Some Extensions

- place capacities
- arc weights
- read arcs
- reset arcs
- inhibitor arcs
- colors (aka. data types)

# Labeled Petri Nets

Petri nets are **unlabeled**, meaning that different transitions model different actions.

**Definition 3.5:** Let $\Sigma$ be a labeling alphabet. For a Petri net $N = (P, T, F, m_0)$, $l : T \to \Sigma$ is a **transition labeling function**. Instead of using transition as labels for the labeled version of the reachability graph, we may also write $m \xrightarrow{l(t)} m'$ if $m[t\rangle_N m'$. If $l$ is injective, $N$ is effectively unlabeled under $l$.

Using the (labeled) reachability graph, we can define all equivalences from before for (labeled) Petri nets.

# Analysis Tasks for Petri Nets

Let $N = (P, T, F, m_0)$ be a Petri net.

1. Termination: $N$ is **terminating** if its reachability graph is finite and acyclic.
2. Deadlock-freedom: A marking $m$ for $N$ is **dead** if for all $t \in T$, $m[t\rangle_N$ does not hold. $N$ is **deadlock-free** if there is no dead marking $m \in [N\rangle$.
3. Liveness. Weak liveness. Quasi-liveness.
4. Boundedness: For $k \in \mathbb{N}$, $N$ is $k$-**bounded** if for all $m \in [N\rangle$ and all $p \in P$, $m(p) \leq k$. $N$ is **bounded** if there is a $k \in \mathbb{N}$, such that $N$ is $k$-bounded.
5. Reversibility: $N$ is **reversible** if for each $m \in [N\rangle$, $m_0 \in [N, m\rangle$.
6. Reachability: A marking $m$ is **reachable** in $N$ if $m \in [N\rangle$.
7. Equivalence (e. g., bisimilarity, trace equivalence, isomorphism): Two (labeled) Petri nets $N_1$ and $N_2$ are **bisimilar**, denoted $N_1 \leftrightarrow N_2$ if $\mathcal{R}(N_1) \leftrightarrow \mathcal{R}(N_2)$.

# Monotonicity Lemma

**Lemma 3.6:** Let $N = (P, T, F, m_0)$ be a Petri net and $l$ a marking for $N$. For each marking $m$ for $N$, if $m[t\rangle_N m'$, then $m + l[t\rangle_N m'$.

**Proof:** Since $m[t\rangle_N m'$, $t$ is enabled under $m$, meaning that $t \leq m$. That means, for each $p \in t$, $m(p) \geq 1$. For each $p \in P$, $m(p) \leq m + l(p)$. Hence, $m + l(p) \geq 1$ for each $p \in t$. Hence, $t$ is enabled under $m + l$.

If $p \in t \setminus t$, $m'(p) = m(p) - 1$ and for $m + l[t\rangle_N \hat{m}$, we have $\hat{m}(p) = m + l(p) - 1 = m(p) + l(p) - 1$. Since $m(p) \geq 1$, we can equivalently say that $\hat{m}(p) = (m(p) - 1) + l(p) = m'(p) + l(p) = m' + l(p)$.

If $p \in t \setminus t$, $m'(p) = m(p) + 1$. We get $\hat{m}(p) = m + l(p) + 1 = m(p) + l(p) + 1 = (m(p) + 1) + l(p) = m'(p) + l(p) = m' + l(p)$.

Otherwise, $m'(p) = m(p)$ and $\hat{m}(p) = m + l(p) = m(p) + l(p) = m'(p) + l(p) = m' + l(p)$. □

# Boundedness

**Theorem 3.7:** A Petri net $N = (P, T, F, m_0)$ is unbounded if, and only if, there are two markings $m_1$ and $m_2$ for $N$, such that $m_1 \in [N, m_0\rangle$, $m_2 \in [N, m_1\rangle$, $m_1 \leq m_2$, and $m_1(p) < m_2(p)$ for some $p \in P$.

**Proof:** "$\Leftarrow$": Since $m_2 \in [N, m_1\rangle$, there is a finite sequence $t_1, t_2, \ldots, t_n \in T$, such that $m_1[t_1\rangle[t_2\rangle \ldots [t_n\rangle m_2$. As $m_1(p) < m_2(p)$, there is a non-empty marking $l$, such that $m_2 = m_1 + l$. By reasoning with the monotonicity lemma over all of the $n$ firing transitions, we have that $m_2[t_1\rangle[t_2\rangle \ldots [t_n\rangle m_3$ and $m_3 = m_2 + l$. As $l$ is non-empty, there is again place $p$ with $m_3(p) > m_2(p)$. Suppose now, there was a $k \in \mathbb{N}$ such that $N$ is $k$-bounded. Then this bound applies to $p$, meaning $k \geq m_i(p)$ $(i = 0, 1, 2, 3)$. Repeating the firing sequence $t_1, t_2, \ldots, t_n$ from $m_3$ for $k - m_3(p) + 1$ times yields a marking $\hat{m}$ with $\hat{m}(p) > k$, contradicting the assumption that $N$ is $k$-bounded.

"$\Rightarrow$": Here, we need two more tools.

# König's Lemma (for Petri Nets)

**Lemma 3.8:** Let $N = (P, T, F, m_0)$ be a Petri net. If $[N\rangle$ is infinite, then $\mathcal{R}(N)$ has an infinite path $m_0 \xrightarrow{t_1} m_1 \xrightarrow{t_2} \ldots$, where each $m_i \neq m_j$ for $i \neq j$.

**Proof:** Since $T$ is finite, there are only finitely many successor marking of $m_0$. Thus, there is a marking $m_1 \in [N\rangle \setminus \{m_0\}$, such that $m_0[t_1\rangle m_1$ for some $t \in T$ and $[N, m_1\rangle$ is infinite because $[N\rangle$ is infinite. We proceed constructing the path from $m_1$ as initial marking with the same arguments as before. □

As unboundedness of $N$ implies $[N\rangle$ to be infinite, we have already derived an infinite path with distinct markings (by König's Lemma). But how do we get to $m_1 \leq m_2$?

# Dickson's Lemma

**Lemma 3.9:** For any infinite sequence $a_1, a_2, a_3, \ldots \in (\mathbb{N}^k)^\omega$, there is an infinite sequence of indizes $i_1 < i_2 < i_3 < \ldots$, such that $a_{i_1} \leq a_{i_2} \leq a_{i_3} \leq \ldots$.

**Proof:** By induction on $k \in \mathbb{N}$.

**Base:** We have $a_1, a_2, a_3, \ldots \mathbb{N}^\omega$. There is a unique minimal element of the set $A_1 = \{a_1, a_2, a_3, \ldots\}$, say $\min A_1$. Set $i_1$ to be the first occurrence of $\min A_1$ with $a_{i_1} = \min A_1$. Consider now the infinite sequence $a_{i_1+1}, a_{i_1+2}, \ldots$ and its set representation $A_2 = \{a_{i_1+1}, a_{i_1+2}, \ldots\}$, which also has a minimal element $\min A_2$. $\min A_1 \leq \min A_2$ since $A_2 \subseteq A_1$ and $\min A_2 < \min A_1$ contradicts the minimality of $\min A_1$ w. r. t. $A_1$. We set $i_2$ to be the first occurrence with $a_{i_2} = \min A_2$ in the remaining sequence. We proceed by repeating the procedure starting from $a_{i_2+1}$.

# Dickson's Lemma (cont'd)

For any infinite sequence $a_1, a_2, a_3, \ldots \in (\mathbb{N}^k)^\omega$, there is an infinite sequence of indizes $i_1 < i_2 < i_3 < \ldots$, such that $a_{i_1} \leq a_{i_2} \leq a_{i_3} \leq \ldots$.

**Proof:**

**Step:** We argue now from $k \to k+1$. Let $a_1, a_2, \ldots \in (\mathbb{N}^{k+1})^\omega$. Consider $\hat{a_1}, \hat{a_2}, \ldots \in (\mathbb{N}^k)^\omega$ where each $\hat{a_i}$ is $a_i$ up to the $k$-th component. By induction hypothesis, there is an infinite sequence $i_1 < i_2 < i_3 < \ldots$, such that $\hat{a_{i_1}} \leq \hat{a_{i_2}} \leq \hat{a_{i_3}} \leq \ldots$.
Consider now the infinite sequence $\bar{a_{i_1}}, \bar{a_{i_2}}, \bar{a_{i_3}}, \ldots \mathbb{N}^\omega$, where each $\bar{a_i}$ is the $k+1$-st component of $a_i$. By induction hypothesis, there is an infinite sequence $j_1 < j_2 < j_3 < \ldots$, such that $\bar{a_{j_1}} \leq \bar{a_{j_2}} \leq \ldots$. Hence, $a_{j_1} \leq a_{j_2} \leq \ldots$. □

# Boundedness (cont'd)

A Petri net $N = (P, T, F, m_0)$ is unbounded if, and only if, there are two markings $m_1$ and $m_2$ for $N$, such that $m_1 \in [N, m_0\rangle$, $m_2 \in [N, m_1\rangle$, $m_1 \leq m_2$, and $m_1(p) < m_2(p)$ for some $p \in P$.

**Proof:** "$\Rightarrow$": Since $N$ is unbounded, $[N\rangle$ is infinite. Hence, there is an infinite sequence of distinct markings $m_0, m_1, m_2, \ldots$ with $m_0[t_1\rangle m_1[t_2\rangle m_2[t_3\rangle \ldots$ (by König's Lemma). By Dickson's Lemma, there is an infinite sequence of indizes $i_1 < i_2 < i_3 < \ldots$, such that $m_{i_1} \leq m_{i_2} \leq m_{i_3} \leq \ldots$. We set $m_1 = m_{i_1}$ and $m_2 = m_{i_2}$. Since all the markings are distinct, there is a place $p \in P$, such that $m_1(p) < m_2(p)$. □

# Boundedness: Decidability

**Theorem 3.10:** Boundedness is decidable.

1. Compute the reachability graph by a BFS from $m_0$.
2. If we find a marking $m_2$, such that there is a marking $m_1 \leq m_2$ with a path to $m_2$ and for some $p$, $m_1(p) < m_2(p)$, **return unbounded**.
3. If the BFS terminates, **return bounded**.

**Proof:** Every step of the BFS triggers finitely many transitions (as $T$ is finite). By Theorem 3.7, there will eventually be marking $m_1$ and $m_2$ revealing unboundedness. If the net is bounded, the reachability graph is finite and will be computed by the BFS (terminating). □

# Making the Most Out Of Monotonicity

**Definition 3.11:** Let $N = (P, T, F, m_0)$ be a Petri net. A transition $t \in T$ is **quasi-live** if there is a marking $m \in [N\rangle$ such that $m[t\rangle$. $N$ is quasi-live if all transitions $t \in T$ are quasi-live.

By monotonicity, $m$ could be equal to $t$ or any bigger marking than $t$.

**Definition 3.12:** For markings $m_1$ and $m_2$, we say that $m_2$ **covers** $m_1$ if $m_1 \leq m_2$.d
For Petri net $N = (P, T, F, m_0)$, a marking $m$ is **coverable** by $N$ if there is a marking $m' \in [N\rangle$ covering $m$.

Hence, it is sufficient to check whether $t$ is coverable in a Petri net.

The **Coverability Problem** — that is given a Petri net $N$ and a marking $m$ for $N$, is $m$ coverable by $N$? — is decidable.