Finite and algorithmic model theory (Dresden, Winter 22/23): Exercises 3 (02.11.22 13:00)

1. An alternative definition of the random graph.[1] Consider a countable graph G whose universe is the set of primes congruent to 1 modulo 4. Put an edge between p and q if p is a quadratic residue modulo q.[2] Prove that G is isomorphic to the random graph.

2. Yet another alternative definition of the random graph.[3] We define the set HF of *hereditarily finite sets* as follows. The empty set $\emptyset$ is in HF and if $a_1, \dots, a_k$ are in HF (for any $k \in \mathbb{N}$) then $\{a_1, \dots, a_k\} \in$ HF. Consider a countable graph G whose domain is HF and we put the edge between two nodes u, v iff $u \in v$ or $v \in u$. Prove that G is isomorphic to the random graph.

3. And yet another alternative definition of the random graph.[4] Consider a countable graph G whose universe is $\mathbb{N}$ and there is an undirected edge between any (i, j) for which $j < i$ and $\mathsf{BIT}(i, j)$ is `true` (i.e. the j-th bit of the binary expansion of i is 1). Prove that G is isomorphic to the random graph.

4. Łoś-Tarski made algorithmic. Given a first-order formula φ that is preserved under substructures (over all structures), show that we can compute an equivalent (over all structures) universal formula that it is equivalent to φ.

5. The alternative statement of Łoś-Tarski theorem is: "A first-order formula φ is preserved under extensions[5] iff it is equivalent to a existential formula". Assuming Łoś-Tarski Preservation Theorem, show that its alternative version holds.

6. Repeat (with small modifications) our proof of Łoś-Tarski to show that the following Lyndon-Tarski "homomorphism preservation theorem" holds: a formula φ is preserved under inverse homomorphism images[6] then φ is equivalent to a universal negative formula[7].

[1] Difficult? Show that the constructed graph satisfies extension axioms.
[2] A number p is a quadratic residue modulo q if there is a number x so that $x^2 \equiv p \pmod{q}$.
[3] Difficult? Show that the constructed graph satisfies extension axioms.
[4] Difficult? Show that the constructed graph satisfies extension axioms.
[5] $\mathfrak{B}$ is an extension of $\mathfrak{A}$ iff $\mathfrak{A}$ is a substructure of $\mathfrak{B}$
[6] If φ holds in $\mathfrak{B}$ and there is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then φ is also true in $\mathfrak{A}$.
[7] i.e. the formula that can use negation of atoms, $\vee, \wedge$ and $\forall$

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