Agenda

1. Introduction
2. Constraint Satisfaction (CSP)
3. Uninformed Search versus Informed Search (Best First Search, A* Search, Heuristics)
4. Local Search, Stochastic Hill Climbing, Simulated Annealing
5. Tabu Search
6. Answer-set Programming (ASP)
7. Structural Decomposition Techniques (Tree/Hypertree Decompositions)
8. Evolutionary Algorithms/ Genetic Algorithms
Traditional Methods

- There are many classic algorithms to search spaces for an optimal solution.
- Broadly, they fall into two disjoint classes:
  - Algorithms that only evaluate complete solutions (exhaustive search, local search, ...).
  - Algorithms that require the evaluation of partially constructed or approximate solutions.
- Algorithms that treat complete solutions can be stopped any time, and give at least one potential answer.
- If you interrupt an algorithm that works on partial solutions, the results might be useless.
Complete Solutions

- All decision variables are specified.
- For example, binary strings of length $n$ constitute complete solutions for any $n$-variable SAT.
- Permutations of $n$ cities constitute complete solutions for a TSP.
- We can compare two complete solutions using an evaluation function.
- Many algorithms rely on such comparisons, manipulating one single complete solution at a time.
- When a new solution has a better evaluation than the previous best solution, it replaces that prior solution.
- Exhaustive search, local search, hill climbing as well as modern heuristic methods such as simulated annealing, tabu search and evolutionary algorithms fall into this category.
Partial Solutions

There are two forms:

1. **incomplete solution** to the problem originally posed, and
2. **complete solution** to a reduced (i.e. simpler) problem.

- Incomplete solutions reside in a subset of the original problem’s search space.
  - In an SAT, consider all of the binary strings where the first two variables were assigned the value 1 (i.e. TRUE).
  - In a TSP, consider every permutation of cities that contains the sequence 7 – 11 – 2 – 16.
  - We fix the attention on a subset of the search space that has a partial property.
  - Hopefully, that property is also shared by the real solution!
Partial Solutions ctd.

- **Decompose** original problem into a set of **smaller** and **simpler** problems.
  - Hope: solving each of the easier problems and **combine the partial solutions**, results in an answer for the original problem.
  - In a TSP, consider only \( k \) out of \( n \) cities and try to establish the shortest path from city \( i \) to \( j \) that passes through all \( k \) of these cities.
  - **Reduce the size of the search space** significantly and search for a complete solution within the restricted domain.
  - Such partial solutions can serve as **building blocks** for the solution to the original problem.
Partial Solutions ctd.

- **Decompose** original problem into a set of **smaller** and **simpler** problems.
  - Hope: solving each of the easier problems and **combine the partial solutions**, results in an answer for the original problem.
  - In a TSP, consider only $k$ out of $n$ cities and try to establish the shortest path from city $i$ to $j$ that passes through all $k$ of these cities.
  - **Reduce the size of the search space** significantly and search for a complete solution within the restricted domain.
  - Such partial solutions can serve as **building blocks** for the solution to the original problem.

- But, algorithms that work on partial solutions pose **additional difficulties**. One needs to
  - devise a way to **organize the sub-spaces** so that they can be searched efficiently, and
  - create a **new evaluation function** that can assess the quality of partial solutions.
Exhaustive Search

- Checks every solution in the search space until the best global solution has been found.
- Can be used only for small instances of problems.
- Exhaustive (enumerative) algorithms are simple.
- Search space can be reduced by backtracking.
- Some optimization methods, e.g., branch and bound and A* are based on an exhaustive search.
Exhaustive Search

- Checks every solution in the search space until the best global solution has been found.
- Can be used only for small instances of problems.
- Exhaustive (enumerative) algorithms are simple.
- Search space can be reduced by backtracking.
- Some optimization methods, e.g., branch and bound and A* are based on an exhaustive search.
- How can we generate a sequence of every possible solution to the problem?
  - The order in which the solutions are generated and evaluated is irrelevant (because we evaluate all of them).
  - The answer for the question depends on the selected representation.
Enumerating the SAT

- We have to generate every possible binary string of length $n$.
- All solutions correspond to whole numbers in a one-to-one mapping.
- Generate all non-negative integers from 0 to $2^n - 1$ and convert each of these integers into the matching binary string of length $n$.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>0</td>
<td>0100</td>
<td>4</td>
</tr>
<tr>
<td>0001</td>
<td>1</td>
<td>0101</td>
<td>5</td>
</tr>
<tr>
<td>0010</td>
<td>2</td>
<td>0110</td>
<td>6</td>
</tr>
<tr>
<td>0011</td>
<td>3</td>
<td>0111</td>
<td>7</td>
</tr>
<tr>
<td>1000</td>
<td>8</td>
<td>1001</td>
<td>9</td>
</tr>
<tr>
<td>1010</td>
<td>10</td>
<td>1101</td>
<td>13</td>
</tr>
<tr>
<td>1011</td>
<td>11</td>
<td>1111</td>
<td>15</td>
</tr>
<tr>
<td>1100</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1110</td>
<td>14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1111</td>
<td>15</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Bits of the string are the truth assignments of the decision variables.
- Organize the search space, for example partition into two disjoint sub-spaces. First contains all the vectors where $x_1 = f$ (FALSE), and the second contains all vectors where $x_1 = t$ (TRUE).
Enumerating the SAT ctd.

Binary search tree for SAT
Search Strategies

A strategy is defined by picking the order of node expansion. Strategies are evaluated along the following dimensions:

- **Completeness** - does it always find a solution if one exists?
- **Time complexity** - number of nodes generated/expanded.
- **Space complexity** - maximum number of nodes in memory.
- **Optimality** - does it always find a least-cost solution?

Time and space complexity are measured in terms of:

- $b$ - maximum branching factor of the search tree;
- $d$ - depth of the least-cost solution;
- $m$ - maximum depth of the state space (may be $\infty$).
Group Work - Posters

- Uninformed Search Strategies
- Informed Search Strategies
Uninformed Search Strategies

Breath first and Depth first
- discover next node in structure
  - queue (FIFO)
    - new
      - optimal
      - time: \(O(b^d)\)
      - space: \(O(b^d)\)
  - stack (LIFO)
    - top
      - not optimal
      - time: \(O(b^m)\)
      - space: \(O(b \cdot m)\)

for non-monotonic rising edge cases

Completeness in graphs:
check for loops!

Iterative Deepening
- general strategy → finds best depth limit
  - depth increase → goal found
  + optimal
  + time: \(O(b^d)\)
  + space: \(O(bd)\)

Limit = 2
Informierte Suche

in Graphen

$F(n)$ Bewertung des Knotens $n$
   z.B. Distanz zum Ziel

$h(n)$ Heuristik ist (Unterschätzung)
   der Kosten des günstigsten Pfads vom aktuellen Knoten $n$ zum Ziel.

Greedy - Suche:
   - nutzt Heuristik als Bewertungsakt.
   - der nächst expandierte Knoten
     ist der billigste
   - nicht vollständig, nicht optimal (da lokale Opt.)

$A^*$ - Suche:
   - Kostenfkt $F(n) = g(n) + h(n)$, wobei
     $g(n)$ - bisherige Kosten
     $h(n)$ - geschätzte Kosten zum Ziel
   - Anforderungen an $h$:
     - Unterschätzung
     - $h(n) \geq 0$, $h(\tilde{e}) = 0$
     - $h(n) \leq c(n,a,n') + h(n')$
       (Konsistenzkriterium)
     - vollständig, optimal
Dynamic Programming

Principle of finding an overall solution by operating on an intermediate point that lies between where you are now and where you want to go.

- Procedure is recursive, each next intermediate point is a function of the points already visited.
- Prototypical problem suitable for dynamic programming has the following properties.
Dynamic Programming

Principle of finding an overall solution by operating on an intermediate point that lies between where you are now and where you want to go.

- Procedure is recursive, each next intermediate point is a function of the points already visited.
- Prototypical problem suitable for dynamic programming has the following properties.

- Can be decomposed into a sequence of decisions made at various stages.
- Each stage has a number of possible states.
- A decision takes you from a state at one stage to some state at the next stage.
- Best sequence of decisions (policy) at any stage is independent of the decisions made at prior stages.
- Well-defined cost for traversing from state to state across stages.
- There is a recursive relationship from choosing the best decisions to make.
Dynamic Programming ctd.

**Procedure**

- Starting at the goal and working backward to the current state.
- First, determine the best decision at last stage.
- From there, determine the best decision at the next to last stage, presuming we will make the best decision at the last stage.
- And so forth . . .
Dynamic Program for the TSP

\[
L = \begin{bmatrix}
0 & 7 & 12 & 8 & 11 \\
3 & 0 & 10 & 7 & 13 \\
4 & 8 & 0 & 9 & 12 \\
6 & 6 & 9 & 0 & 10 \\
7 & 7 & 11 & 10 & 0
\end{bmatrix}
\]

- Suppose, we start from city 1.
- We split the problem into smaller problems.
- \( g(i, S) \) length of the shortest path from city \( i \) to 1 that passes through each city in \( S \).
- \( g(4, \{5, 2, 3\}) \) is the shortest path from city 4 through cities 5, 2 and 3 (in some unspecified order) and then returns to 1.
- \( g(1, V - \{1\}) \) is the length of the shortest complete tour.
- In general, we claim that

\[
g(i, S) = \min_{j \in S} \{L(i, j) + g(j, S - \{j\})\}.
\]
Dynamic Program for the TSP ctd.

\[ L = \begin{bmatrix} 0 & 7 & 12 & 8 & 11 \\ 3 & 0 & 10 & 7 & 13 \\ 4 & 8 & 0 & 9 & 12 \\ 6 & 6 & 9 & 0 & 10 \\ 7 & 7 & 11 & 10 & 0 \end{bmatrix} \]

The problem is to find \( g(1, \{2, 3, 4, 5\}) \).
We start backwards with \( S = \emptyset \).

\[
\begin{align*}
g(2, \emptyset) &= L(2, 1) = 3, \\
g(3, \emptyset) &= L(3, 1) = 4, \\
g(4, \emptyset) &= L(4, 1) = 6, \text{ and} \\
g(5, \emptyset) &= L(5, 1) = 7.
\end{align*}
\]
Dynamic Program for the TSP ctd.

\[ L = \begin{bmatrix}
0 & 7 & 12 & 8 & 11 \\
3 & 0 & 10 & 7 & 13 \\
4 & 8 & 0 & 9 & 12 \\
6 & 6 & 9 & 0 & 10 \\
7 & 7 & 11 & 10 & 0
\end{bmatrix} \]

Next iteration, find the solutions to all problems where \(|S| = 1\) (12 sub-problems).

\[
g(2, \{3\}) = L(2, 3) + g(3, \emptyset) = 10 + 4 = 14, \\
g(2, \{4\}) = L(2, 4) + g(4, \emptyset) = 7 + 6 = 13, \text{ and} \\
g(2, \{5\}) = L(2, 5) + g(5, \emptyset) = 13 + 7 = 20.
\]
Dynamic Program for the TSP ctd.

For city 3:

\[ g(3, \{2\}) = L(3, 2) + g(2, \emptyset) = 8 + 3 = 11, \]
\[ g(3, \{4\}) = L(3, 4) + g(4, \emptyset) = 9 + 6 = 15, \]
\[ g(3, \{5\}) = L(3, 5) + g(5, \emptyset) = 12 + 7 = 19. \]

For city 4:

\[ g(4, \{2\}) = L(4, 2) + g(2, \emptyset) = 6 + 3 = 9, \]
\[ g(4, \{3\}) = L(4, 3) + g(3, \emptyset) = 9 + 4 = 13, \]
\[ g(4, \{5\}) = L(4, 5) + g(5, \emptyset) = 10 + 7 = 17. \]

For city 5:

\[ g(5, \{2\}) = L(5, 2) + g(2, \emptyset) = 7 + 3 = 10, \]
\[ g(5, \{3\}) = L(5, 3) + g(3, \emptyset) = 11 + 4 = 15, \]
\[ g(5, \{4\}) = L(5, 4) + g(4, \emptyset) = 10 + 6 = 16. \]
Dynamic Program for the TSP ctd.

\[ L = \begin{bmatrix}
0 & 7 & 12 & 8 & 11 \\
3 & 0 & 10 & 7 & 13 \\
4 & 8 & 0 & 9 & 12 \\
6 & 6 & 9 & 0 & 10 \\
7 & 7 & 11 & 10 & 0
\end{bmatrix} \]

Next iteration, \(|S| = 2\).

\[
g(2, \{3, 4\}) = \min\{L(2, 3) + g(3, \{4\}), L(2, 4) + g(4, \{3\})\}
\]
\[
= \min\{10 + 15, 7 + 13\} = \min\{25, 20\} = 20,
\]

\[
g(2, \{3, 5\}) = \min\{L(2, 3) + g(3, \{5\}), L(2, 5) + g(5, \{3\})\}
\]
\[
= \min\{10 + 19, 13 + 15\} = \min\{29, 28\} = 28,
\]

\[
g(2, \{4, 5\}) = \min\{L(2, 4) + g(4, \{5\}), L(2, 5) + g(5, \{4\})\}
\]
\[
= \min\{7 + 17, 13 + 16\} = \min\{24, 29\} = 24.
\]
Dynamic Program for the TSP ctd.

For city 3:

\[
g(3, \{2, 5\}) = \min\{L(3, 2) + g(2, \{5\}), L(3, 5) + g(5, \{2\})\} \\
= \min\{8 + 20, 12 + 10\} = \min\{28, 22\} = 22,
\]

\[
g(3, \{2, 4\}) = \min\{L(3, 2) + g(2, \{4\}), L(3, 4) + g(4, \{2\})\} \\
= \min\{8 + 13, 9 + 9\} = \min\{21, 18\} = 18,
\]

\[
g(3, \{4, 5\}) = \min\{L(3, 4) + g(4, \{5\}), L(3, 5) + g(5, \{4\})\} \\
= \min\{9 + 17, 12 + 16\} = \min\{26, 28\} = 26.
\]

For city 4:

\[
g(4, \{2, 3\}) = \min\{L(4, 2) + g(2, \{3\}), L(4, 3) + g(3, \{2\})\} \\
= \min\{6 + 14, 9 + 11\} = \min\{20, 20\} = 20,
\]

\[
g(4, \{2, 5\}) = \min\{L(4, 2) + g(2, \{5\}), L(4, 5) + g(5, \{2\})\} \\
= \min\{6 + 20, 10 + 10\} = \min\{26, 20\} = 20,
\]

\[
g(4, \{3, 5\}) = \min\{L(4, 3) + g(3, \{5\}), L(4, 5) + g(5, \{3\})\} \\
= \min\{9 + 19, 10 + 15\} = \min\{28, 25\} = 25.
\]
Dynamic Program for the TSP ctd.

For city 5:

\[ g(5, \{2, 3\}) = \min \{L(5, 2) + g(2, \{3\}), L(5, 3) + g(3, \{2\})\} \]
\[ = \min \{7 + 14, 11 + 11\} = \min \{21, 22\} = 21, \]

\[ g(5, \{2, 4\}) = \min \{L(5, 2) + g(2, \{4\}), L(5, 4) + g(4, \{2\})\} \]
\[ = \min \{7 + 13, 10 + 19\} = \min \{20, 29\} = 20, \]

\[ g(5, \{3, 4\}) = \min \{L(5, 3) + g(3, \{4\}), L(5, 4) + g(4, \{3\})\} \]
\[ = \min \{11 + 15, 10 + 13\} = \min \{26, 23\} = 23. \]
Dynamic Program for the TSP ctd.

\[
L = \begin{bmatrix}
0 & 7 & 12 & 8 & 11 \\
3 & 0 & 10 & 7 & 13 \\
4 & 8 & 0 & 9 & 12 \\
6 & 6 & 9 & 0 & 10 \\
7 & 7 & 11 & 10 & 0
\end{bmatrix}
\]

Next iteration, \(|S| = 3\).

\[
g(2, \{3, 4, 5\}) = \min\{L(2, 3) + g(3, \{4, 5\}), L(2, 4) + g(4, \{3, 5\}), L(2, 5) + g(5, \{3, 4\})\}
\]
\[
= \min\{10 + 26, 7 + 25, 13 + 23\} = \min\{36, 32, 34\} = 32,
\]
Dynamic Program for the TSP ctd.

Next iteration, $|S| = 3$.

\[
g(2, \{3, 4, 5\}) = \min\{L(2, 3) + g(3, \{4, 5\}), L(2, 4) + g(4, \{3, 5\}), L(2, 5) + g(5, \{3, 4\})\} = \min\{10 + 26, 7 + 25, 13 + 23\} = \min\{36, 32, 34\} = 32,
\]

\[
g(3, \{2, 4, 5\}) = \min\{L(3, 2) + g(2, \{4, 5\}), L(3, 4) + g(4, \{2, 5\}), L(3, 5) + g(5, \{2, 4\})\} = \min\{8 + 24, 9 + 20, 12 + 20\} = \min\{32, 29, 32\} = 29,
\]

\[
g(4, \{2, 3, 5\}) = \min\{L(4, 2) + g(2, \{3, 5\}), L(4, 3) + g(3, \{2, 5\}), L(4, 5) + g(5, \{2, 3\})\} = \min\{6 + 28, 9 + 22, 10 + 21\} = \min\{34, 31, 31\} = 31.
\]

\[
g(5, \{2, 3, 4\}) = \min\{L(5, 2) + g(2, \{3, 4\}), L(5, 3) + g(3, \{2, 4\}), L(5, 4) + g(4, \{2, 3\})\} = \min\{7 + 20, 11 + 18, 10 + 20\} = \min\{27, 29, 30\} = 27.
\]
Dynamic Program for the TSP ctd.

$$L = \begin{bmatrix} 0 & 7 & 12 & 8 & 11 \\ 3 & 0 & 10 & 7 & 13 \\ 4 & 8 & 0 & 9 & 12 \\ 6 & 6 & 9 & 0 & 10 \\ 7 & 7 & 11 & 10 & 0 \end{bmatrix}$$

Last iteration, $|S| = 4$, original problem:

$$g(1, \{2, 3, 4, 5\}) = \min\{L(1, 2) + g(2, \{3, 4, 5\}), L(1, 3) + g(3, \{2, 4, 5\}), L(1, 4) + g(4, \{2, 3, 5\}), L(1, 5) + g(5, \{2, 3, 4\})\}$$

$$= \min\{7 + 32, 12 + 29, 8 + 31, 11 + 27\} = \min\{39, 41, 39, 38\} = 38.$$ 

Shortest tour has length 38.
Which tour is that?
Dynamic Program for the TSP ctd.

Last iteration, $|S| = 4$, original problem:

$$g(1, \{2, 3, 4, 5\}) = \min\{L(1, 2) + g(2, \{3, 4, 5\}), L(1, 3) + g(3, \{2, 4, 5\}),$$

$$L(1, 4) + g(4, \{2, 3, 5\}), L(1, 5) + g(5, \{2, 3, 4\})\}$$

$$= \min\{7 + 32, 12 + 29, 8 + 31, 11 + 27\} = \min\{39, 41, 39, 38\} = 38.$$

Shortest tour has length 38. Which tour is that?

- Additional data structure $W$ with information on the next city with minimal path.
- $W(1, \{2, 3, 4, 5\}) = 5.$
- $W(5, \{2, 3, 4\}) = 2$, $W(2, \{3, 4\}) = 4$, $W(4, \{3\}) = 3$,
- last we arrive at city 1.
- Length of this tour is $11 + 7 + 7 + 9 + 4 = 38$. 

TU Dresden, 28th April 2017  
PSSAI  
slide 29 of 32
Properties of Dynamic Programming

- Computationally intensive: $O(n^2 2^n)$.
- DP algorithms tend to be complicated to understand, because the construction of the program depends on the problem.
- How to formulate sub-problems?
Summary

- Heuristic functions estimate costs of shortest paths
- Good heuristics can dramatically reduce search cost
- Greedy best-first search expands lowest $h$
  - incomplete and not always optimal
- Dynamic programming
  - complete and optimal
  - time and space consuming
  - how to define the sub-problems?
- A* search expands lowest $g + h$
  - complete and optimal
  - also optimally efficient
- Admissible heuristics can be derived from exact solution of relaxed problems
References

Zbigniew Michalewicz and David B. Fogel.  

Stuart J. Russell and Peter Norvig.  