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DEDUCTION SYSTEMS

Optimizations for Tableau Procedures

Sebastian Rudolph

DRESDEN
concept
Exzellenz aus
Wissenschaft
und Kultur

Agenda

- Optimizations
  - Unfolding
  - Absorption
  - Dependency-Directed Backtracking
  - Further Optimizations
- Classification
- Summary

Optimizations

- Naïve implementation not performant enough
  - $\mathcal{T}$-regel adds one disjunction per axiom to the corresponding node
  - ontologies may contain > 1.000 axioms and tableaux may contain thousands of nodes

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    - (Lazy) unfolding
    - Absorbtion
    - Dependency directed backtracking
    - Simplification and Normalization
    - Caching
    - Heuristics
    - …

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Unfolding

- $\mathcal{T}$-rule is not necessary if $\mathcal{T}$ is unfoldable, i.e., every axiom is:
  - definitorial: form $A \sqsubseteq C$ or $A \equiv C$ for A a concept name ($A \equiv C$ corresponds to $A \sqsubseteq C$ and $C \sqsubseteq A$)
  - acyclic: C uses A neither directly nor indirectly
  - unique: only one such axiom exists for every concept name A

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  - acyclic: C uses A neither directly nor indirectly
  - unique: only one such axiom exists for every concept name A
- If $\mathcal{T}$ is unfoldable, the TBox can be (unfolded) into a concept

Unfolding Example

- We check satisfiability of A w.r.t. the TBox $\mathcal{T}$

$\mathcal{T}$:

$$A \sqsubseteq B \sqcap \exists r.C$$
$$B \equiv C \sqcup D$$
$$C \sqsubseteq \exists r.D$$

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\rightsquigarrow\ & A \sqcap B \sqcap \exists r.C \\
\rightsquigarrow\ & A \sqcap (C \sqcup D) \sqcap \exists r.C \\
\rightsquigarrow\ & A \sqcap ((C \sqcap \exists r.D) \sqcup D) \sqcap \exists r.(C \sqcap \exists r.D)
\end{aligned}$$

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A &\sqsubseteq B \sqcap \exists r.C \\
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$\mathcal{T}$:

$$\begin{aligned} A &\sqsubseteq B \sqcap \exists r.C \\ B &\equiv C \sqcup D \\ C &\sqsubseteq \exists r.D \end{aligned}$$

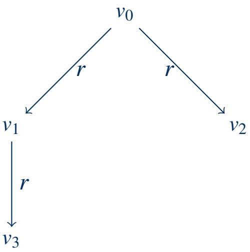
- A is satisfiable w.r.t. $\mathcal{T}$ iff

$$A \sqcap ((C \sqcap \exists r.D) \sqcup D) \sqcap \exists r.(C \sqcap \exists r.D)$$

is satisfiable w.r.t. the empty TBox

Tableau Algorithm Example with Unfolding

We obtain the following contradiction-free tableau for the satisfiability of $U = A \sqcap ((C \sqcap \exists r.D) \sqcup D) \sqcap \exists r.(C \sqcap \exists r.D)$:



$$
\begin{aligned}
L(v_0) = \{&U, A, (C \sqcap \exists r.D) \sqcup D,\\
&\exists r.(C \sqcap \exists r.D), C \sqcap \exists r.D,\\
&C, \exists r.D\}\\
L(v_1) = \{&C \sqcap \exists r.D, C, \exists r.D\}\\
L(v_2) = \{&D\}\\
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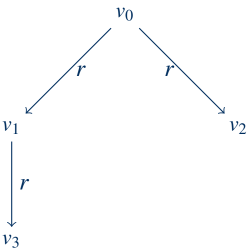

$$\begin{aligned} L(v_0) = \{&U, A, (C \sqcap \exists r.D) \sqcup D, \\ &\exists r.(C \sqcap \exists r.D), C \sqcap \exists r.D, \\ &C, \exists r.D\} \\ L(v_1) = \{&C \sqcap \exists r.D, C, \exists r.D\} \\ L(v_2) = \{&D\} \\ L(v_3) = \{&D\} \end{aligned}$$

Only one disjunctive decision left!

Lazy Unfolding

- computation of NNF together with unfolding may decrease performance, e.g.:
  - satisfiability of $C \sqcap \neg C$ w.r.t. $\mathcal{T} = \{C \sqsubseteq A \sqcap B\}$
  - unfolding: $C \sqcap A \sqcap B \sqcap \neg(C \sqcap A \sqcap B)$
  - NNF + unfolding: $C \sqcap A \sqcap B \sqcap (\neg C \sqcup \neg A \sqcup \neg B)$

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  - NNF + unfolding: $C \sqcap A \sqcap B \sqcap (\neg C \sqcup \neg A \sqcup \neg B)$
- better: apply NNF and unfolding if needed, via corresponding tableau rules:
  - $A \equiv C \rightsquigarrow A \sqsubseteq C$ and $A \sqsupseteq C$

$\sqsubseteq$-rule: For $v \in V$ such that $A \sqsubseteq C \in \mathcal{T}$, $A \in L(v)$ and $C \notin L(v)$
let $L(v) := L(v) \cup C$.

$\sqsupseteq$-rule: For $v \in V$ such that $A \sqsupseteq C \in \mathcal{T}$, $\neg A \in L(v)$ and $\neg C \notin L(v)$
let $L(v) := L(v) \cup \{\neg C\}$.

$\neg$-rule: For $v \in V$ such that $\neg C \in L(v)$ and NNF$(\neg C) \notin L(v)$,
let $L(v) := L(v) \cup \{$NNF$(\neg C)\}$.

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  - Separate $\mathcal{T}$ into $\mathcal{T}_u$ (unfoldable part) and $\mathcal{T}_g$ (GCIs, not unfoldable)
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- absorption decreases $\mathcal{T}_g$ and increases $\mathcal{T}_u$
  1. take an axiom from $\mathcal{T}_g$, e.g., $A \sqcap B \sqsubseteq C$
  2. transform the axiom: $A \sqsubseteq C \sqcup \neg B$
  3. if $\mathcal{T}_u$ contains an axiom of the form $A \equiv D$ $\quad$ ($A \sqsubseteq D$ and $D \sqsupseteq A$), then $A \sqsubseteq C \sqcup \neg B$ cannot be absorbed; $A \sqsubseteq C \sqcup \neg B$ remains in $\mathcal{T}_g$
  4. otherwise, if $\mathcal{T}_u$ contains an axiom of the form $A \sqsubseteq D$, then absorb $A \sqsubseteq C \sqcup \neg B$ resulting in $A \sqsubseteq D \sqcap (C \sqcup \neg B)$
  5. otherwise move $A \sqsubseteq C \sqcup \neg B$ to $\mathcal{T}_u$

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- If $A \equiv D \in \mathcal{T}_u$, try rewriting/absorption with other axioms in $\mathcal{T}_u$
- nondeterministic: $B \sqsubseteq C \sqcup \neg A$ also possible

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Dependency-Directed Backtracking

- despite those optimizations, search space often to big
- let $v \in V$ with $(C_1 \sqcup D_1) \sqcap \ldots \sqcap (C_n \sqcup D_n) \sqcap \exists r.\neg A \sqcap \forall r.A \in L(v)$

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- exponentially big search space is traversed

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- backjumping works roughly as follows:
  - concepts in the node label are tagged with a set of integers (dependency set) allowing to identify the concept's "origin"
  - initially, all concepts are tagged with $\emptyset$
  - tableau rules combine and extend these tags
  - $\sqcup$-rule adds the tag $\{d\}$ to the existing tag, where d is the $\sqcup$-depth (number of $\sqcup$-rules applied by now)
  - when encountering a contradiction, the labels alow to identify the origin of the concepts causing the contradiction
  - jump back to the last relevant application of a $\sqcup$-rule

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  - jump back to the last relevant application of a $\sqcup$-rule
- irrelevant part of the search space is not considered

Dependency-Directed Backtracking Example

$(C_1 \sqcup D_1) \sqcap \ldots \sqcap (C_n \sqcup D_n) \sqcap \exists r.\neg A \sqcap \forall r.A \in L(v)$ tagged with $\emptyset$

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v $\sqcap$-rule L(v) $:= L(v) \cup \{(C_1 \sqcup D_1), \ldots, (C_n \sqcup D_n),$
$\exists r.\neg A, \forall r.A\}$ all with $\emptyset$

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Dependency-Directed Backtracking
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Dependency-Directed Backtracking Example

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$\sqcup$-rule	L(v)	:=	$L(v) \cup \{C_1\}$	C_1 tagged with $\{1\}$
$\vdots$	$\vdots$		$\vdots$	
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- None of the $\sqcup$-rules has contributed to the cotradiction
- Output false (unsatisfiable)

Agenda

- Optimizations
    - Unfolding
    - Absorption
    - Dependency-Directed Backtracking
    - Further Optimizations
- Classification
- Summary

Further Optimizations

- Simplification and Normalization
  - quick recognition of trivial contradictions
  - normalization, z.B., $A \sqcap (B \sqcap C) \equiv \sqcap\{A, B, C\}$, $\forall r.C \equiv \neg\exists r.\neg C$
  - simplification, e.g., $\sqcap\{A, \ldots, \neg A, \ldots\} \equiv \bot$, $\exists r.\bot \equiv \bot$, $\forall r.\top \equiv \top$

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Optimizing Classification

One of the most wide-spread tasks for automated reasoning is classification

- compute all subclass relationships between atomic concepts in $\mathcal{T}$

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- check for $\mathcal{T} \models C \sqsubseteq D$ can be reduced to checking satisfiability of $\mathcal{T}$ together with the ABox $(C \sqcap \neg D)(a)$ (or, equivalenty: $C(a), (\neg D)(a)$)
  - ⇝ if $\top$ is satisfiable: subsumption does not hold (as we have constructed a counter-model)
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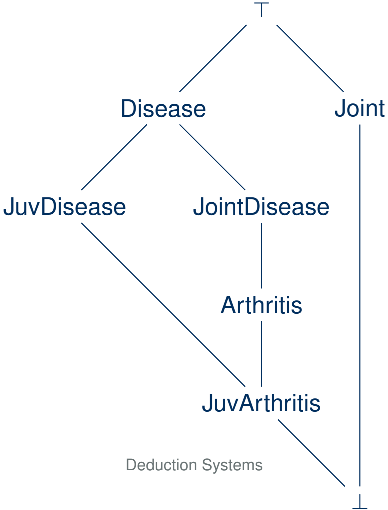
  - $\rightsquigarrow$ if $\top$ is satisfiable: subsumption does not hold (as we have constructed a counter-model)
  - $\rightsquigarrow$ if $\top$ is unsatisfiable: subsumption holds (no counter-model exists)
- naïve approach needs n^2 subsumption checks for n concept names
- normally cached in the concept hierarchy graph

Concept Hierarchy Graph



$\top$

Disease

Joint

JuvDisease

JointDisease

Arthritis

JuvArthritis

$\bot$

Optimizing Classification

most wide-spread technique is called enhanced traversal

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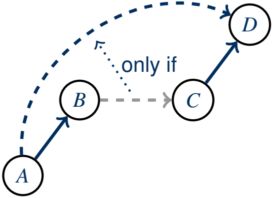
Optimizing Classification

most wide-spread technique is called enhanced traversal

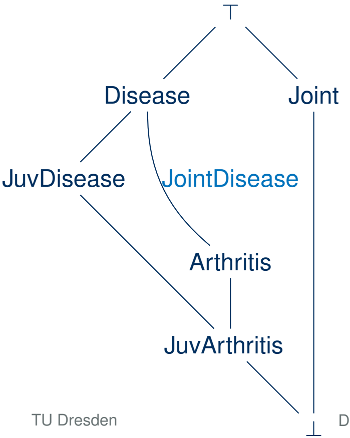
- hierarchy is created incrementally by introducing concept after concept
- top-down phase: recognize direct superconcepts
- bottom-up phase: recognize direct subconcepts
- transitivity of $\sqsubseteq$ used to save checks



- If $A \sqsubseteq B$ and $C \sqsubseteq D$ hold,
- then $B \sqsubseteq C \longrightarrow A \sqsubseteq D$
- and $A \not\sqsubseteq D \longrightarrow B \not\sqsubseteq C$

Enhanced Traversal Example

already created hierarchy:


$\top$
Disease
Joint
JuvDisease
JointDisease
Arthritis
JuvArthritis
$\bot$

Goal: insertion of JointDisease

Top-Down Phase:

Bottom-Up Phase:

Enhanced Traversal Example

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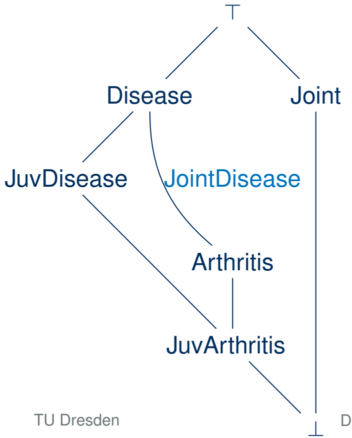

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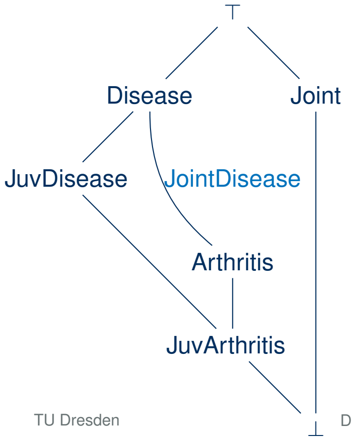
Top-Down Phase:

- JointDisease $\sqsubseteq^?$ Disease

Bottom-Up Phase:

Enhanced Traversal Example

already created hierarchy:



Goal: insertion of JointDisease

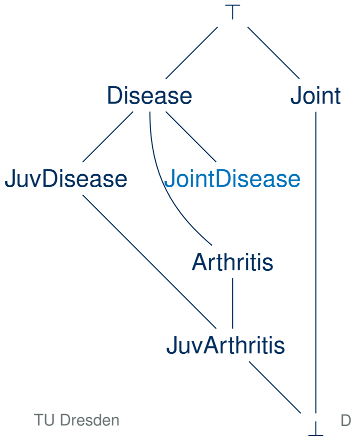
Top-Down Phase:

- JointDisease $\sqsubseteq$ Disease
- JointDisease $\sqsubseteq^?$ JuvDisease

Bottom-Up Phase:

Enhanced Traversal Example

already created hierarchy:



Goal: insertion of JointDisease

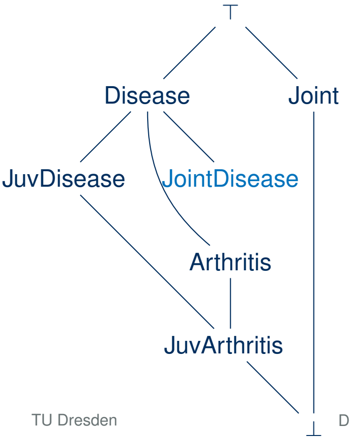
Top-Down Phase:

- JointDisease $\sqsubseteq$ Disease
- JointDisease $\not\sqsubseteq$ JuvDisease
- JointDisease $\sqsubseteq^?$ Arthritis

Bottom-Up Phase:

Enhanced Traversal Example

already created hierarchy:



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Bottom-Up Phase:

Enhanced Traversal Example

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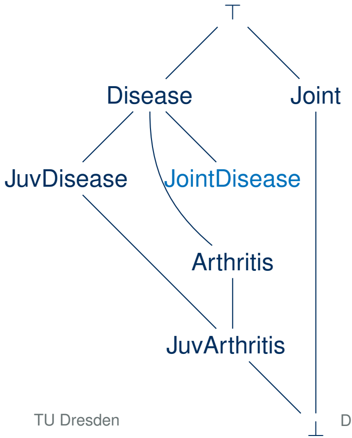

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Bottom-Up Phase:

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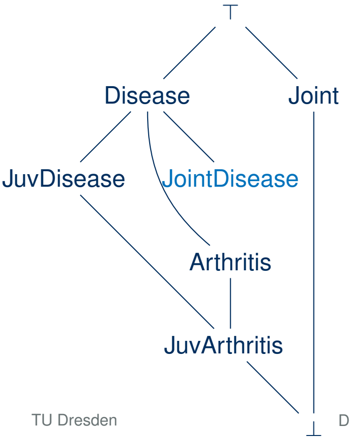

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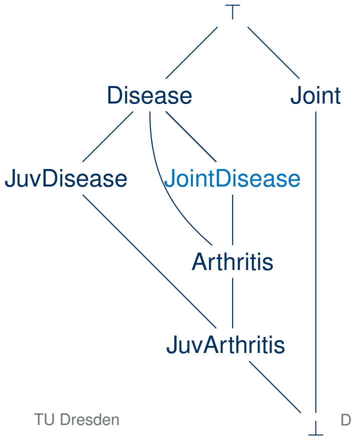

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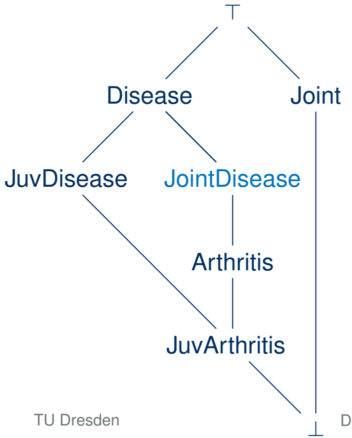

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Summary

- we have a tableau algorithm for $\mathcal{ALCIF}$ knowledge bases
    - ABox treated like for $\mathcal{ALC}$
    - number restrictions are treated similar to functionality and existential quantifiers
- termination via cycle detection
    - becomes harder as the logic becomes more expressive
- naive tableau algorithm not sufficiently performant
- diverse optimizations improve average case
- specific methods for classification
    - enhanced traversal
- tableaux algorithms or variants modifications thereof are the basis of OWL reasoners