The classification of homomorphism homogeneous tournaments

Thomas Feller[a,1], Christian Pech, Maja Pech[b,2]

[a] *TU Dresden, Germany*
[b] *University of Novi Sad, Department of Mathematics and Informatics*

Abstract

The notion of homomorphism homogeneity was introduced by Cameron and Nešetřil as a natural generalization of the classical model-theoretic notion of homogeneity. A relational structure is called homomorphism homogeneous (HH) if every homomorphism between finite substructures extends to an endomorphism. It is called polymorphism homogeneous (PH) if every finite power of the structure is homomorphism homogeneous. Despite the similarity of the definitions, the HH and PH structures lead a life quite separate from the homogeneous structures. While the classification theory of homogeneous structure is dominated by Fraïssé-theory, other methods are needed for classifying HH and PH structures. In this paper we give a complete classification of HH countable tournaments (with loops allowed). We use this result in order to derive a classification of countable PH tournaments. The method of classification is designed to be useful also for other classes of relational structures. Our results extend previous research on the classification of finite HH and PH tournaments by Ilić, Mašulović, Nenadov, and the first author.

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Email addresses: `thomas.feller@tu-dresden.de` (Thomas Feller), `cpech@freenet.de` (Christian Pech), `maja@dmi.uns.ac.rs` (Maja Pech)

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1. Introduction

A relational structure **A** is called *homomorphism homogeneous* if every homomorphism between finite substructures of **A** extends to an endomorphism of **A** (when we talk about substructures, we always mean *induced* substructures, that is, substructures in the sense of model theory). It is called *k-polymorphism homogeneous* if every homomorphism from a finite substructure of $\mathbf{A}^k$ to **A** extends to a homomorphism from $\mathbf{A}^k$ to **A** (the latter kind of homomorphisms are called *k-ary polymorphisms* of **A**). It is called *polymorphism homogeneous* if it is k-polymorphism homogeneous, for every $k \in \mathbb{N} \setminus \{0\}$. The notion of homomorphism homogeneity was introduced by Cameron and Nešetřil in [1], while the notion of polymorphism homogeneity stems from [2] (see Section 4 for some more information about the notion of polymorphism homogeneity in relation with other topics of research). Both notions are related to the well-known notion of *homogeneity* from Fraïssé-theory (a relational structure is called *homogeneous* if every isomorphism between finite substructures extends to an automorphism of the structure in question). While on the surface these three concepts look rather similar, their respective classification problems are quite different, each having its own beauties and difficulties.

When classifying countably infinite homogeneous structures of a given type, the main tool is Fraïssé-theory. Namely, by Fraïssé's Theorem every countable homogeneous structure is determined up to isomorphism by its *age* (i.e., by the class of finitely generated substructures, taken up to isomorphism). Moreover, a class of finite structures of a given type is the age of a homogeneous structure if and only if it has the hereditary property (HP), the joint embedding property (JEP), and the amalgamation property (AP). Such classes are called *amalgamation classes*. Thus, the classification of countable homogeneous structures is "reduced" to the classification problem of amalgamation classes. Alas, the ages of relational structures are not nearly as important when classifying homomorphism homogeneous structures. On the one hand we know from [3, Theorem 3.5] that a class of finite structures is the age of a homomorphism homogeneous structure if and only if it has the HP, the JEP, and the homomorphism amalgamation property (HAP) (a.k.a one point homomorphism extension property (1PHEP), cf. [4]). On the other hand, in general there may be several non-isomorphic homomorphism homogeneous structures with the same age. E.g, it was shown in [5, Theorem 2.1] that every chain $(C, \leq)$ is homomorphism homogeneous, but all infinite chains have the same age (the class of finite chains). When we talk about classifications of homomorphism homogeneous structures, we mean to give a transparent finitary description together with a rich source of examples possibly covering all homomorphism homogeneous structures in question.

The goal of this paper is to characterize all countable homomorphism homogeneous tournaments with loops allowed, and, starting from there, to identify those among them that are polymorphism homogeneous at the same time. Here a *tournament with loops allowed* is just a nonempty binary relational structure $\mathbf{A} = (A, \varrho)$, where

- ϱ is *total*, i.e., $\forall x, y \in A: x \neq y \Rightarrow (x, y) \in \varrho \vee (y, x) \in \varrho$,
- ϱ is *antisymmetric*, i.e., $\forall x, y \in A: (x, y) \in \varrho \wedge (y, x) \in \varrho \Rightarrow x = y$.

For a binary relational structure $\mathbf{A} = (A, \varrho)$, a vertex $x \in A$ is called a *loop* if $(x, x) \in \varrho$, otherwise it is called a *non-loop*. If a tournament contains no loops then we call it *loopless* or *irreflexive*. Otherwise it is called *loopy*. In addition, x is called a *source* if whenever $(y, x) \in \varrho$, then $x = y$. Dually, it is called a *sink* if whenever $(x, y) \in \varrho$, then $x = y$. For two subsets B_1 and B_2 of A we write $B_1 \to B_2$ if $(x, y) \in \varrho$ for all $x \in B_1$ and $y \in B_2$. Instead of $\{b_1\} \to B_2$, $B_1 \to \{b_2\}$, and $\{b_1\} \to \{b_2\}$ we write $b_1 \to B_2$, $B_1 \to b_2$, and $b_1 \to b_2$, respectively.

It should be mentioned that the classification of the finite homomorphism homogeneous and polymorphism homogeneous tournaments was carried out already in [6] and in [7], respectively. Our task is to extend these results to the case of countably infinite tournaments.

2. Homomorphism homogeneous tournaments

We start by a rough classification of tournaments that helps to divide the task at hand into digestible pieces: Let $\mathbf{A} = (A, \varrho)$ be a tournament with loops allowed. Then exactly one of the following holds:

1. $\mathbf{A}$ is loopless,
2. $\mathbf{A}$ is loopy and ϱ is intransitive,
3. $\mathbf{A}$ is loopy and ϱ is transitive.

The case that can be resolved most quickly is the first one. Observe that every homomorphism between finite substructures of a loopless tournament is an isomorphism. Moreover, every endomorphism of a loopless tournament is a self-embedding.

Proposition 2.1. *A countable loopless tournament is homomorphism homogeneous if and only if it is homogeneous.*

Proof. A straightforward back-and-forth argument. ☐

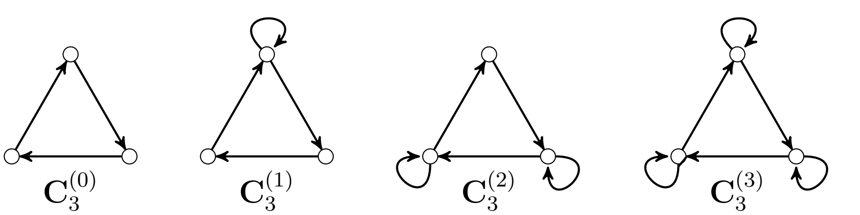
The countable homogeneous tournaments were completely classified by Lachlan and Woodrow:

Theorem 2.2 ([8, 9])**.** *The countable homogeneous tournaments are:*

- *the trivial one-vertex tournament,*
- *the oriented cycle of length* 3*,*
- $(\mathbb{Q}, <)$*,*
- *the countable circular tournament (a.k.a. the countable dense local order),*
- *the countable universal homogeneous tournament.*

We are not going to describe these tournaments in detail as they are part of the folklore of Fraïssé-theory. The interested reader may consult [10] for further information and for additional references. Thus the case of the loopless homomorphism homogeneous tournaments is done.

Let us come now to the case of loopy but intransitive homomorphism homogeneous tournaments. Clearly, every loopy intransitive tournament contains one of the following tournaments as a substructure:



It is not hard to see that only $\mathbf{C}_3^{(0)}$ and $\mathbf{C}_3^{(3)}$ are homomorphism homogeneous. So from now on let us concentrate only on loopy intransitive tournaments with at least 4 vertices. To settle this case, we show:

Proposition 2.3. *A loopy intransitive tournament with at least* 4 *vertices is not homomorphism homogeneous.*

Before coming to the proof we need to collect some auxiliary tools. This is the moment to introduce a simple but essential notion in classification theory of homomorphism homogeneous structures, the *witnesses*. There is no sense to define witnesses just for tournaments or even just for binary relational structures. Their natural habitat are general *relational structures* in the sense of model theory (cf. [11]). For us a *relational signature* is a model theoretic signature that has no function symbols and no constant symbols, but only relational symbols. Relational structures are denoted by $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$. Their domains are denoted by $A, B, C, \dots$. If $\mathbf{A}$ is a relational structure and if $B \subseteq A$, then by $\langle B \rangle_{\mathbf{A}}$ we denote the substructure of $\mathbf{A}$ induced by B. All the terms that we previously introduced for binary relational structures have an obvious definition for general relational structures.

Definition 2.4. Let $\mathbf{A}$ be a relational structure. A *witness in* $\mathbf{A}$ is a quadruple $(\mathbf{B}_1, \mathbf{B}_2, f, c)$, such that

- $\mathbf{B}_1$ and $\mathbf{B}_2$ are finite substructures of $\mathbf{A}$,
- $c \in B_1$,
- $f\colon \langle B_1 \setminus \{c\} \rangle_{\mathbf{B}_1} \twoheadrightarrow \mathbf{B}_2$ is a surjective homomorphism,
- f cannot be extended to a homomorphism from $\mathbf{B}_1$ to $\mathbf{A}$.

Moreover, a quadruple $(\mathbf{B}_1, \mathbf{B}_2, f, c)$ is called a *witness* if it is a witness in some structure $\mathbf{A}$.

Clearly, if a relational structure contains a witness, then it is not homomorphism homogeneous. On the other hand an easy inductive argument shows that if a countable relational structure contains no witnesses, then it is homomorphism homogeneous. It should also be noted that in any witness $(\mathbf{B}_1, \mathbf{B}_2, f, c)$ we have that $B_1 \setminus \{c\}$ and B_2 are non-empty.

Our next step is to show that homomorphism homogeneous tournaments cannot contain certain types of subconfigurations. Recall that a *monomorphism* is an injective homomorphism.

Lemma 2.5. *Let* $\mathbf{T}$ *be a tournament that contains a loop that is not a sink. If* $\mathbf{\Delta}_1$ *monomorphically maps to* $\mathbf{T}$ *(see Figure 1), then* $\mathbf{T}$ *is not homomorphism homogeneous.*

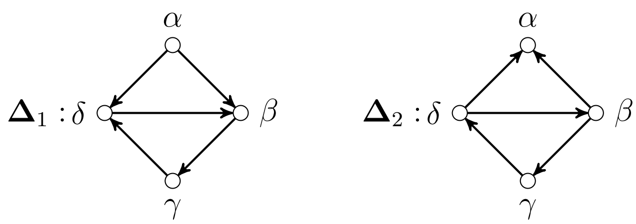


Figure 1: Two forbidden configurations

Proof. Let $u \in T$ be a loop that is not a sink. Let $v \in T \setminus \{u\}$, such that $u \to v$. Further let $\mu \colon \mathbf{\Delta}_1 \rightarrowtail \mathbf{T}$ be a monomorphism, say $\mu = \left(\begin{smallmatrix} \alpha & \beta & \gamma & \delta \\ a & b & c & d \end{smallmatrix}\right)$. Let $\mathbf{B}_1 := \langle\{a,b,c,d\}\rangle_{\mathbf{T}}$, $\mathbf{B}_2 := \langle\{u,v\}\rangle_{\mathbf{T}}$. Define $f \colon B_1 \setminus \{b\} \to B_2$ by $f = \left(\begin{smallmatrix} a & c & d \\ u & u & v \end{smallmatrix}\right)$. We claim that $(\mathbf{B}_1, \mathbf{B}_2, f, b)$ is a witness in $\mathbf{T}$. It is easy to check that $f \colon \langle B_1 \setminus \{b\}\rangle_{\mathbf{B}_1} \twoheadrightarrow \mathbf{B}_2$ is indeed a surjective homomorphism. We just need to show that f cannot be extended to a homomorphism of $\mathbf{B}_1$ to $\mathbf{T}$. Aiming at a contradiction, suppose that f has such an extension $\hat{f}$. Then we must have $\hat{f}(b) = u$, since $a \to b \to c$ implies $\hat{f}(a) \to \hat{f}(b) \to \hat{f}(c)$, whence $u \to \hat{f}(b) \to u$. Note now that $d \to b$, while $\hat{f}(b) = u \to v = \hat{f}(d)$, a contradiction. It follows that $\mathbf{T}$ is not homomorphism homogeneous. □

Dual to Lemma 2.5 we have:

Lemma 2.6. *Let* $\mathbf{T}$ *be a tournament that contains a loop that is not a source. If* $\mathbf{\Delta}_2$ *monomorphically maps to* $\mathbf{T}$ *(see Figure 1), then* $\mathbf{T}$ *is not homomorphism homogeneous.*

Proof. Analogous to the proof of Lemma 2.5. □

Now we are ready to prove Proposition 2.3:

Proof of Proposition 2.3. Let $b, c, d \in T$, such that $b \to c \to d \to b$. We distinguish the following three cases:

(1) $\mathbf{T}$ contains a loop that is not a source and a loop that is not a sink,
(2) all loops in $\mathbf{T}$ are sources,
(3) all loops in $\mathbf{T}$ are sinks.

In **case 1** let $a \in T \setminus \{b, c, d\}$. By the pigeon hole principle, either there are at least two arrows from elements of $\{b, c, d\}$ pointing to a or there are at least two arrows from a pointing to elements of $\{b, c, d\}$. In the former case we can assume without loss of generality that $b \to a$ and $d \to a$. But then $\mu\colon \Delta_2 \to T$ given by $\mu = \left(\begin{smallmatrix} \alpha & \beta & \gamma & \delta \\ a & b & c & d \end{smallmatrix}\right)$ is a monomorphism from $\mathbf{\Delta}_2$ to $\mathbf{T}$. Since $\mathbf{T}$ contains a loop that is not a source, by Lemma 2.6, $\mathbf{T}$ is not homomorphism homogeneous.

The case that two arrows point from a to elements of $\{b, c, d\}$ is handled analogously, using Lemma 2.5.

Note that cases 2 and 3 both entail that $\mathbf{T}$ has exactly one loop. In the following let us denote this loop by a.

Case 2: Let $b, c, d \in T$, such that $b \to c \to d \to b$. Clearly, $a \notin \{b, c, d\}$. Note now that $\mu\colon \Delta_1 \to T$ given by $\mu = \left(\begin{smallmatrix} \alpha & \beta & \gamma & \delta \\ a & b & c & d \end{smallmatrix}\right)$ is a monomorphism from $\mathbf{\Delta}_1$ to $\mathbf{T}$. Since a is not a sink, it follows from Lemma 2.5 that $\mathbf{T}$ is not homomorphism homogeneous.

Case 3 is handled analogously to case 2, using Lemma 2.6. □

The following corollary sums up the results we have so far about loopy, intransitive homomorphism homogeneous tournaments:

Corollary 2.7. *Let* $\mathbf{T}$ *be an intransitive, loopy, homomorphism homogeneous tournament. Then* $\mathbf{T}$ *is isomorphic to* $\mathbf{C}_3^{(3)}$.

Proof. This follows immediately from the observation that the only intransitive loopy tournament on three vertices is $\mathbf{C}_3^{(3)}$, in conjunction with Proposition 2.3. □

Note how the previous result is completely independent from the cardinality of the vertex sets of the tournaments under consideration. What remains to do be done is to treat the case of loopy, transitive homomorphism homogeneous tournaments. These are basically chains in the sense of order theory in which some elements are comparable to themself (the loops) and some are not (the non-loops). This allows us to use order theoretic terminology like *upper bound*, *lower bound*, *minimum*, *maximum*, etc. We employ the convention that $x \to y$ reads as "x is less than (or equal to) y".

Our strategy is to filter out all transitive loopy tournaments that contain no witness. In order to see that this strategy is feasible, we use that witnesses carry some structure that equips them with an implicational theory. This technique was used for the first time in [12]. Here we give a self-contained account of the relevant details. The first step to be made is to observe that witnesses may be classified into isomorphism-types:

Definition 2.8. Witnesses $T_1 = (\mathbf{B}_1, \mathbf{C}_1, f_1, c_1)$ and $T_2 = (\mathbf{B}_2, \mathbf{C}_2, f_2, c_2)$ are called *isomorphic* if there exist isomorphisms $h_1 \colon \mathbf{B}_1 \to \mathbf{B}_2$ and $h_2 \colon \mathbf{C}_1 \to \mathbf{C}_2$, such that

- $h_1(c_1) = c_2$,
- for all $x \in B_1 \setminus \{c_1\}$ we have $f_2(h_1(x)) = h_2(f_1(x))$.

Given now a witness T for L-structures and an L-structure $\mathbf{A}$, we say that $\mathbf{A}$ *realizes* T if $\mathbf{A}$ has a witness isomorphic to T. Otherwise, we say that $\mathbf{A}$ *avoids* T.

Example 2.9. Let $T = (\mathbf{D}_1, \mathbf{D}_2, f, c)$ be a witness for tournaments. We may depict T as an unlabeled structure by drawing a picture that consists of three parts:

1. an unlabeled graphical representation of $\mathbf{D}_1$ on the left hand side in which the vertex c is colored differently from the others (we do this by drawing "usual" vertices in white and c in black),
2. an unlabeled graphical representation of $\mathbf{D}_2$ on the right hand side,
3. arrows $\mapsto$ connecting vertices from $\mathbf{D}_1$ with vertices from $\mathbf{D}_2$ indicating the action of f.

Obviously, from such a picture, the witness T may be reconstructed, up to isomorphism. In Figure 2 some witnesses for loopy tournaments are depicted.

The second step in our venture to classify witness-free structures is to observe that the class of witnesses comes naturally equipped with an implicational theory:

Definition 2.10. Let $\mathcal{K}$ be a class of L-structures, let $\mathbb{T}$ be a set of witnesses, and let T be a witness for L-structures. We say that $\mathbb{T}$ *entails* T *with respect to* $\mathcal{K}$ (written $\mathbb{T} \models_{\mathcal{K}} T$) if all structures from $\mathcal{K}$ that avoid all witnesses from $\mathbb{T}$ also avoid T. $\mathbb{T}$ is called a *complete set of witnesses for* $\mathcal{K}$ if $\mathbb{T}$ entails every witness that is realized by a structure from $\mathcal{K}$.

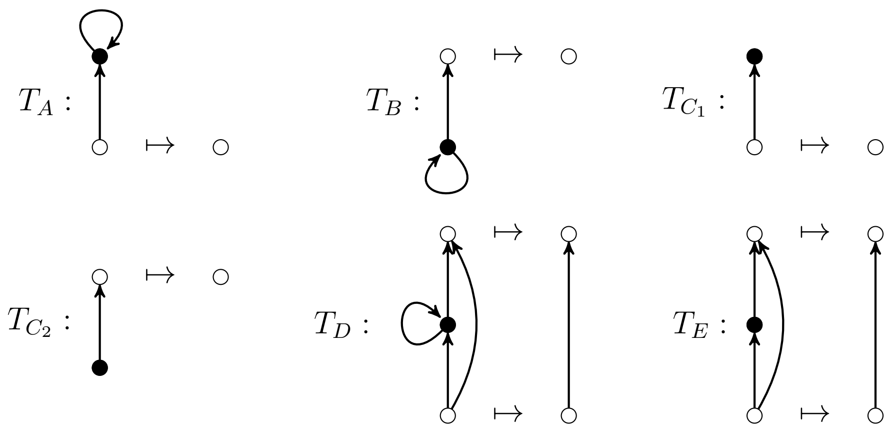
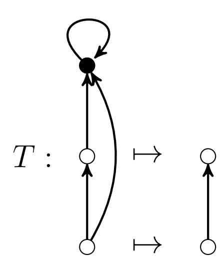


Figure 2: Witnesses for transitive, loopy tournaments

Example 2.11. To see that the entailment relation on witnesses is non-trivial, consider the following witness for tournaments with loops allowed:



Let $\mathcal{K}$ be the class of transitive tournaments with loops allowed. We claim that $\{T_A\} \models_{\mathcal{K}} T$. Indeed, let $\mathbf{A} \in \mathcal{K}$ be a tournament with loops allowed that realizes T. Let $W = (\mathbf{B}_1, \mathbf{B}_2, f, c)$ be a witness isomorphic to T in $\mathbf{A}$. Then $B_1 = \{b_{11}, b_{12}, c\}$ with $b_{11} \to b_{12}$, $B_2 = \{b_{21}, b_{22}\}$ with $b_{21} \to b_{22}$. Moreover, $\{b_{11}, b_{12}\} \to c$, $f(b_{11}) = b_{21}$, and $f(b_{12}) = b_{22}$. There is no loopy vertex above b_{22}, for if there was such a vertex d with $b_{22} \to d$, then, by transitivity, also $b_{21} \to d$, which is a contradiction to the assumption that W is a witness in $\mathbf{A}$. Thus, $W' := (\langle\{b_{12}, c\}\rangle_{\mathbf{B}_1}, \langle\{b_{22}\}\rangle_{\mathbf{B}_2}, f\restriction_{\{b_{12}\}}, c)$ is a witness in $\mathbf{A}$. Clearly, $W' \cong T_A$.

Proposition 2.12. *The witnesses T_A, T_B, T_{C_1}, T_{C_2}, T_D, T_E given in Figure 2 form a complete set of witnesses for the class of transitive tournaments with loops allowed.*

Proof. Let $\mathbf{A}$ be any transitive tournament with loops allowed, let $W = (\mathbf{B}_1, \mathbf{B}_2, f, c)$ be a witness in $\mathbf{A}$. Let $L_c := \{b \in B_1 \mid b \to c, b \neq c\}$, $U_c := \{b \in B_1 \mid c \to b, b \neq c\}$. We distinguish the following three cases:

(1) $L_c = \emptyset$, $U_c \neq \emptyset$,
(2) $L_c \neq \emptyset$, $U_c = \emptyset$,
(3) $L_c \neq \emptyset$, $U_c \neq \emptyset$.

Case 1: Let $u := \min(U_c)$. Let $\mathbf{B}_1' := \langle\{u, c\}\rangle_{\mathbf{B}_1}$, $\mathbf{B}_2' := \langle\{f(u)\}\rangle_{\mathbf{B}_2}$, $f' := f\restriction_{\{u\}}$. We claim that $W' := (\mathbf{B}_1', \mathbf{B}_2', f', c)$ is a witness in $\mathbf{A}$. In order to show this, we need to show that f' cannot be extended to a homomorphism from $\mathbf{B}_1'$ to $\mathbf{A}$. Suppose now on the contrary that f' has a homomorphic extension $\hat{f}'$ to $\mathbf{B}_1'$. Let $d := \hat{f}'(c)$. Define $\hat{f} \colon B_1 \to A$ according to

$$\hat{f} \colon x \mapsto \begin{cases} f(x) & x \neq c, \\ d & x = c. \end{cases}$$

We claim that $\hat{f} \colon \mathbf{B}_1 \to \mathbf{A}$ is a homomorphism. Since f is a homomorphism, in order to show that $\hat{f}$ is a homomorphism, too, it is enough to show that it maps arcs that start or end in c onto arcs. If $c \to c$ then also $\hat{f}(c) \to \hat{f}(c)$, since $\hat{f}(c) = d = \hat{f}'(c)$ and since $\hat{f}'$ is a homomorphism. All other arcs involving c are of the shape $c \to x$ for some $x \in U_c$. If $x = u$, then $\hat{f}(c) \to \hat{f}(x)$, since $\hat{f}(c) = \hat{f}'(c)$ and since $\hat{f}(u) = \hat{f}'(u)$, and since $\hat{f}'$ is a homomorphism. If $x \neq u$, then $u \to x$, since u is the minimum of U_c. As we already saw we have $\hat{f}(c) \to \hat{f}(u)$. Since $\hat{f}(u) = f(u)$ and $\hat{f}(x) = f(x)$, and since f is a homomorphism, we also have $\hat{f}(u) \to \hat{f}(x)$. Since $\mathbf{A}$ is a transitive tournament, it follows $\hat{f}(c) \to \hat{f}(x)$. Thus, $\hat{f}$ is a homomorphism, a contradiction to the assumption that W is a witness in $\mathbf{A}$. It follows that our assumption was false and that W' is indeed a witness in $\mathbf{A}$. An immediate consequence of this is that $f'(u)$ cannot be a loop in $\mathbf{A}$, because otherwise we could extend f' to $\langle\{u, c\}\rangle_{\mathbf{A}}$ by mapping c to $f'(u)$, contradictory to W' being a witness in $\mathbf{A}$. It follows that u is a non-loop, too. Thus the isomorphism type of W' depends only on whether c is a loop or not. In the former case we have $W' \cong T_B$ and in the latter case we have $W' \cong T_{C_2}$.

Case 2: This case is handled analogously to case 1, leading to a witness W' in $\mathbf{A}$ isomorphic either to T_A or to T_{C_1}. In this case u is defined to be the maximum of L_c.

Case 3: Let $u := \min(U_c)$ and let $l := \max(L_c)$. Similarly as above it can be shown that with $\mathbf{B}_1' = \langle\{l, u, c\}\rangle_{\mathbf{B}_1}$, $\mathbf{B}_2' = \langle\{f(l), f(u)\}\rangle_{\mathbf{B}_2}$, $f' = f\restriction_{\{l,u\}}$

we get that $W' := (\mathbf{B}'_1, \mathbf{B}'_2, f', c)$ is a witness in $\mathbf{A}$. This implies that l and u both are non-loops. Depending on whether c is a loop or not we get that $W' \cong T_D$ or $W' \cong T_E$.

Altogether we showed that whenever $\mathbf{A}$ has a witness, then it has also a witness isomorphic to one of the witnesses from Figure 2. In other words, whenever a transitive tournament with loops allowed avoids all witnesses from Figure 2, then it avoids all witnesses altogether. $\square$

At this point the classification of countable homomorphism homogeneous tournaments with loops allowed is almost done. We have a finitary description of such tournaments: A countable homomorphism homogeneous tournament is either

- isomorphic to one of the countable homogeneous loopless tournaments, or
- it is isomorphic to $C_3^{(3)}$, or
- it is transitive and avoids all witnesses from Figure 2.

What is still missing, at least for the third part of the classification, is transparence. In the following we refine the description of the countable homomorphism homogeneous transitive tournaments. Note that there are two ways in which a structure may avoid a witness:

Lemma 2.13. *Let $\mathbf{A}$ be an L-structure, and let $T = (\mathbf{D}_1, \mathbf{D}_2, f, c)$ be a witness for L structures. Then $\mathbf{A}$ avoids T if and only if either*

(1) *$\mathbf{D}_1$ does not embed into $\mathbf{A}$, or*

(2) *$\mathbf{D}_1$ embeds into $\mathbf{A}$, and for every embedding $\iota\colon \mathbf{D}_2 \hookrightarrow \mathbf{A}$ there exists some $d \in A$, such that the function $\hat{f}_d\colon D_1 \to A$ defined by*

$$\hat{f}_d\colon x \mapsto \begin{cases} (\iota \circ f)(x) & x \in D_1 \setminus \{c\}, \\ d & x = c \end{cases} \tag{$*$}$$

is a homomorphism from $\mathbf{D}_1$ to $\mathbf{A}$.

Proof. "$\Rightarrow$" Suppose $\mathbf{A}$ avoids T but neither Condition 1 nor Condition 2 is satisfied. Let $\kappa\colon \mathbf{D}_1 \hookrightarrow \mathbf{A}$, and let $\iota\colon \mathbf{D}_2 \hookrightarrow \mathbf{A}$, such that for no $d \in A$ the function $\hat{f}_d$ defined in $(*)$ is a homomorphism. Define $\mathbf{B}_1 := \langle \kappa(D_1) \rangle_{\mathbf{A}}$, $\mathbf{B}_2 := \langle \iota(D_2) \rangle_{\mathbf{A}}$, and let $h\colon B_1 \setminus \{\kappa(c)\} \twoheadrightarrow B_2$ be given by $h\colon x \mapsto (\iota \circ f)(\kappa^{-1}(x))$.

Since B_2 is the image of ι and since f is surjective, it follows that h is surjective, too. Moreover, since κ and ι are embeddings and since f is a homomorphism, it follows that $h\colon \langle B_1 \setminus \{\kappa(c)\}\rangle_{\mathbf{A}} \twoheadrightarrow \mathbf{B}_2$ is a homomorphism. By construction we have that $W := (\mathbf{B}_1, \mathbf{B}_2, h, \kappa(c))$ is a witness isomorphic to T. Since $\mathbf{A}$ avoids T, we have that W can not be a witness in $\mathbf{A}$. Let $\hat{h}\colon \mathbf{B}_1 \to \mathbf{A}$ be an extension of h. Let $d := \hat{h}(\kappa(c))$. Then we may confirm pointwise that $\hat{f}_d = \hat{h} \circ \kappa$. In particular, $\hat{f}_d$ is a homomorphism, a contradiction to the assumption that $\mathbf{A}$ violates Condition 2. Hence our initial assumption was wrong and one of the Conditions 1 and 2 is satisfied.

"$\Leftarrow$" Every witness $(\mathbf{B}_1, \mathbf{B}_2, h, c')$ in $\mathbf{A}$ isomorphic to T must have that $\mathbf{B}_1 \cong \mathbf{D}_1$. Thus, surely, if $\mathbf{D}_1$ does not embed into $\mathbf{A}$, then $\mathbf{A}$ avoids T. So suppose that Condition 2 holds but that $\mathbf{A}$ has a witness $W = (\mathbf{B}_1, \mathbf{B}_2, h, c')$ isomorphic to T. That is, there are isomorphisms $\kappa\colon \mathbf{D}_1 \to \mathbf{B}_1$ and $\iota\colon \mathbf{D}_2 \to \mathbf{B}_2$, such that for all $x \in D_1 \setminus \{c\}$ we have $(h \circ \kappa)(x) = (\iota \circ f)(x)$ and such that $\kappa(c) = c'$. Let $d \in A$ be such that the function defined in $(*)$ is a homomorphism from $\mathbf{D}_1$ to $\mathbf{A}$. Define $\hat{h}\colon B_1 \to A$ according to

$$\hat{h}\colon x \mapsto \begin{cases} h(x) & x \in B_1 \setminus \{c'\}, \\ d & x = c'. \end{cases}$$

Then it can be checked pointwise that $\hat{h} \circ \kappa = \hat{f}$. Since $\hat{f}$ is a homomorphism and since κ is an isomorphism, it follows that $\hat{h}\colon \mathbf{B}_1 \to \mathbf{A}$ is a homomorphism that extends h, a contradiction to the assumption that W is a witness in $\mathbf{A}$. Thus the assumption that $\mathbf{A}$ realizes T was wrong and $\mathbf{A}$ avoids T. $\square$

In Figure 3 the left hand sides of the witnesses from Figure 2 are given. Let us, for reasons of convenience, state what Lemma 2.13 means for transitive tournaments with loops allowed:

Corollary 2.14. *Let* $\mathbf{A}$ *be a transitive tournament with loops allowed then:*

- $\mathbf{A}$ *avoids* T_A *iff either* $\mathbf{S}_A$ *does not embed into* $\mathbf{A}$ *or above every loopless vertex in* $\mathbf{A}$ *there is a loopy one,*
- $\mathbf{A}$ *avoids* T_B *iff either* $\mathbf{S}_B$ *does not embed into* $\mathbf{A}$ *or below every loopless vertex in* $\mathbf{A}$ *there is a loopy one,*
- $\mathbf{A}$ *avoids* T_{C_1} *iff either* $\mathbf{S}_C$ *does not embed into* $\mathbf{A}$ *or above every loopless vertex in* $\mathbf{A}$ *there is a vertex (loopless or loopy),*

- $\mathbf{A}$ *avoids* T_{C_2} *iff either* $\mathbf{S}_C$ *does not embed into* $\mathbf{A}$ *or below every loopless vertex in* $\mathbf{A}$ *there is a vertex (loopless or loopy),*

- $\mathbf{A}$ *avoids* T_D *iff either* $\mathbf{S}_D$ *does not embed into* $\mathbf{A}$ *or in-between any two loopless vertices in* $\mathbf{A}$ *there is a loopy vertex,*

- $\mathbf{A}$ *avoids* T_E *iff either* $\mathbf{S}_E$ *does not embed into* $\mathbf{A}$ *or in-between any two loopless vertices in* $\mathbf{A}$ *there is a vertex (loopless or loopy).*

Proof. This is a direct consequence of Lemma 2.13. □

The finer classification of transitive homomorphism homogeneous tournaments depends on whether they embed one of $\mathbf{S}_A, \dots, \mathbf{S}_E$ or not (see Figure 3).

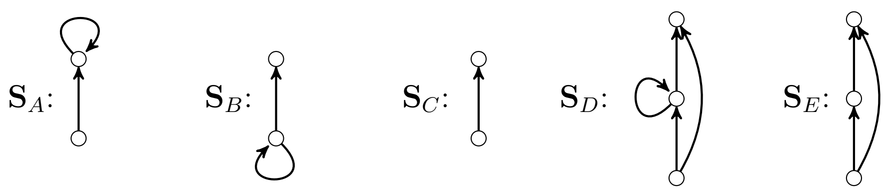


Figure 3: Left hand sides of witnesses

Theorem 2.15. *Let* $\mathbf{T}$ *be a countable tournament. Then* $\mathbf{T}$ *is homomorphism homogeneous if and only if there exist nonempty subsets* M, M_1, M_2 *of* $\mathbb{Q}$*, such that exactly one of the following is true:*

(1) $\mathbf{T} \cong \mathbf{C}_3^{(0)}$*;*
(2) $\mathbf{T}$ *is the countable universal homogeneous tournament;*
(3) $\mathbf{T}$ *is the countable circular tournament (a.k.a the countable dense local order);*
(4) $\mathbf{T} \cong \mathbf{C}_3^{(3)}$*;*
(5) $\mathbf{T} \cong (\{0\}, <)$*;*
(6) $\mathbf{T} \cong (\{-\infty\}, <) + (M, \leq)$*;*
(7) $\mathbf{T} \cong (M, \leq) + (\{+\infty\}, <)$*;*
(8) $\mathbf{T} \cong (M_1, \leq) + (\{0, 1\}, <) + (M_2, \leq)$*;*
(9) $\mathbf{T} \cong (\mathbb{Q}, <)$*;*

(10) $\mathbf{T} \cong (\mathbb{Q} \cup \{+\infty\}, <) + (M, \leq)$;
(11) $\mathbf{T} \cong (\mathbb{Q}, <) + (M, \leq)$;
(12) $\mathbf{T} \cong (M, \leq) + (\mathbb{Q} \cup \{-\infty\}, <)$;
(13) $\mathbf{T} \cong (M, \leq) + (\mathbb{Q}, <)$;
(14) $\mathbf{T} \cong (M_1, \leq) + (\mathbb{Q}, <) + (M_2, \leq)$;
(15) $\mathbf{T} \cong (M_1, \leq) + (\mathbb{Q} \cup \{-\infty\}, <) + (M_2, \leq)$;
(16) $\mathbf{T} \cong (M_1, \leq) + (\mathbb{Q} \cup \{+\infty\}, <) + (M_2, \leq)$;
(17) $\mathbf{T} \cong (M_1, \leq) + (\mathbb{Q} \cup \{-\infty, +\infty\}, <) + (M_2, \leq)$;
(18) $\mathbf{T}$ *has the following properties:*

- *below every loopless vertex there is at least one loopy vertex,*
- *above every loopless vertex there is at least one loopy vertex,*
- *in-between any two loopless vertices there is at least one loopy vertex.*

Proof. We only need to treat the transitive case. The intransitive case was treated before. In particular, the irreflexive case (corresponding to (1), (2), and (3)) was handled in Proposition 2.1 in conjunction with Theorem 2.2, while the intransitive loopy case (corresponding to (4)) was done in Corollary 2.7.

For the transitive case we use Proposition 2.12. It is not hard to see that tournaments of shapes 5–18 avoid all witnesses depicted in Figure 2. Thus, by Proposition 2.12, these tournaments contain no witnesses. In particular all of them are homomorphism homogeneous.

In order to show that the given classification of countable homomorphism homogeneous transitive tournaments is complete, we need to show that every such tournament $\mathbf{T}$ belongs to one of the given classes. To this end we distinguish the cases given in Table 1 according to whether the structures $\mathbf{S}_A, \dots, \mathbf{S}_E$ from Figure 3 are embeddable into $\mathbf{T}$ or not. On the surface it looks like that we should have to distinguish $2^5 = 32$ cases. Observe however that $\mathbf{S}_A$, $\mathbf{S}_B$, and $\mathbf{S}_C$ embed into $\mathbf{S}_D$, and that $\mathbf{S}_C$ embeds into $\mathbf{S}_E$. This decimates the number of cases to be distinguished to 14.

Case 1: In this case $\mathbf{T}$ is either reflexive or it consists of exactly one loopless vertex. In the former case $\mathbf{T}$ is of shape (18), while in the latter case it is of shape (5).

Case 2: In this case $\mathbf{T}$ contains exactly one loopless and at least one loopy vertex. Moreover, this loopless vertex is the least element of $\mathbf{T}$. In other words, $\mathbf{T}$ is of shape (6).

	$\mathbf{S}_A \hookrightarrow \mathbf{T}$	$\mathbf{S}_B \hookrightarrow \mathbf{T}$	$\mathbf{S}_C \hookrightarrow \mathbf{T}$	$\mathbf{S}_D \hookrightarrow \mathbf{T}$	$\mathbf{S}_E \hookrightarrow \mathbf{T}$
Case 1	−	−	−	−	−
Case 2	+	−	−	−	−
Case 3	−	+	−	−	−
Case 4	+	+	−	−	−
Case 5	−	−	+	−	−
Case 6	+	−	+	−	−
Case 7	−	+	+	−	−
Case 8	+	+	+	−	−
Case 9	+	+	+	+	−
Case 10	−	−	+	−	+
Case 11	+	−	+	−	+
Case 12	−	+	+	−	+
Case 13	+	+	+	−	+
Case 14	+	+	+	+	+

Table 1: Forced and forbidden substructures of $\mathbf{T}$

Case 3: $\mathbf{T}$ contains at least one loopy vertex and exactly one loopless vertex that is on the top. Thus $\mathbf{T}$ is of shape (7).

Case 4: $\mathbf{T}$ contains exactly one loopless vertex. Above and below this vertex there is at least one loopy vertex. Thus, $\mathbf{T}$ is of shape (18).

Case 5: $\mathbf{T}$ consists of exactly two loopless vertices. In particular, it is irreflexive. By Proposition 2.1 it should be homogeneous. However, by Theorem 2.2, the two-element tournament is not homogeneous. It follows that no homomorphism homogeneous transitive tournament falls under case 5.

Case 6: $\mathbf{T}$ has exactly two loopless vertices u and v and at least one loopy vertex. The loopless vertices are at the bottom of $\mathbf{T}$. Without loss of generality $u \to v$. Clearly, $(\langle\{u,v\}\rangle_{\mathbf{T}}, \langle\{u\}\rangle_{\mathbf{T}}, f, u)$ with $f\colon v \mapsto u$ is a witness in $\mathbf{T}$ isomorphic to T_{C_2}. This is a contradiction with the assumption that $\mathbf{T}$ is homomorphism-homogeneous. It follows that no homomorphism homogeneous transitive tournament falls under case 6.

Case 7: This case is dual to case 6. In particular, in contradiction to the assumption that **T** is homomorphism homogeneous, it can be shown that **T** realizes T_{C_1}. It follows that no homomorphism homogeneous transitive tournament falls under case 7.

Case 8: **T** has exactly two loopless vertices u and v. These vertices are consecutive. Above and below u and v there is at least one loopy vertex. Thus, **T** is of the shape (8).

Case 9: **T** has exactly two loopless vertices u and v. Between, above, and below u and v there is at least one loopy vertex. Thus, **T** is of shape (18).

Case 10: **T** has at least three loopless vertices and no loopy one. In particular **T** is irreflexive. By Proposition 2.1, **T** is homogeneous. It follows from Theorem 2.2 that **T** is of shape (9).

Case 11: **T** has at least three loopless and at least one loopy vertex. Each loopy vertex is above all loopless vertices. Since **T** avoids T_{C_2}, **T** should not have a smallest element. Since **T** avoids T_E, the subtournament induced by the non-loops is a dense chain (it might have a greatest element). Altogether we have that **T** is either of shape (10) or (11).

Case 12: This case is dual to case 11. Thus, **T** is either of shape (12) or (13).

Case 13: **T** has at least three loopless and at least two loopy vertices. No loopy vertex is in-between two loopless vertices. Thus **T** decomposes into three parts. On the bottom there is a reflexive part, in the middle there is a loopless part and on the top there is another reflexive part. Since **T** avoids T_E, the loopless part needs to be a dense chain (it might have a smallest element, a greatest element, or both). Thus, **T** is either of shape (14), or (15), or (16), or (17).

Case 14: **T** has at least three loopless and at least four loopy vertices. Since **T** avoids T_A, above each non-loop there is at least one loop. Dually, since **T** avoids T_B, below every non-loop there is at least one loop. Finally, since **T** avoids T_D, between any two non-loops there should be at least one loop. Thus, **T** is of the shape (18). □

3. Polymorphism homogeneous tournaments

From the definition of polymorphism homogeneity it follows immediately that every polymorphism homogeneous structure is also homomorphism homogeneous. Our goal is to filter the list of the eighteen modes of structures from Theorem 2.15 for polymorphism homogeneous tournaments. In order to be able to use similar techniques (notably witnesses), we recall a simple but very helpful observation from [2]:

Proposition 3.1 ([2, Proposition 2.1])**.** *A structure $\mathbf{A}$ is k-polymorphism homogeneous if and only if $\mathbf{A}^k$ is homomorphism homogeneous; hence a structure is polymorphism homogeneous iff all of its finite powers are homomorphism homogeneous.*

To reiterate a previous observation, a structure $\mathbf{A}$ is *not* polymorphism homogeneous if for some $k \in \mathbb{N} \setminus \{0\}$ the structure $\mathbf{A}^k$ contains a witness.

During the classification of homomorphism homogeneous tournaments it proved useful to separate the intransitive case from the transitive one. So let us start by identifying the countable intransitive polymorphism homogeneous tournaments. Looking at Theorem 2.15 we see that there are four possible candidates:

1. the countable universal homogeneous tournament,
2. the countable circular tournament,
3. $\mathbf{C}_3^{(3)}$,
4. $\mathbf{C}_3^{(0)}$.

Lemma 3.2. *Let $\mathbf{T}$ be a loopless tournament. If the the oriented graph $\mathbf{\Delta}_3$ from Figure 4 monomorphically maps to $\mathbf{T}$, then $\mathbf{T}$ is not 2-polymorphism homogeneous.*

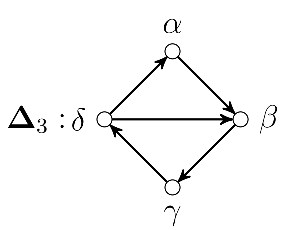
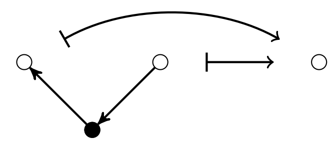


Figure 4: A forbidden configuration for polymorphism homogeneous tournaments

Proof. Let $\mu\colon \Delta_3 \rightarrowtail \mathbf{T}$ be a monomorphism, say $\mu = \left(\begin{smallmatrix} \alpha & \beta & \gamma & \delta \\ a & b & c & d \end{smallmatrix}\right)$. Let $\mathbf{B}_1 := \langle\{(a,c),(b,d),(d,b)\}\rangle_{\mathbf{T}^2}$, $\mathbf{B}_2 = \langle\{(b,d)\}\rangle_{\mathbf{T}^2}$, $f\colon B_1 \setminus \{(a,c)\} \to B_2$ be the function that maps every element of B_1 to (b,d). It is easy to check that

$(\mathbf{B}_1, \mathbf{B}_2, f, (a, c))$ is a witness in $\mathbf{T}^2$ of the isomorphism type depicted below.



□

Corollary 3.3. *The countable universal homogeneous tournament is not* 2-*polymorphism homogeneous.*

Proof. Every countable oriented loopless graph monomorphically maps to the universal homogeneous tournament. Thus the claim follows from Lemma 3.2. □

Corollary 3.4. *The countable circular tournament is not* 2*-polymorphism homogeneous.*

Proof. The countable circular tournament is up to isomorphism the unique countable homogeneous intransitive tournament into which the tournaments given in Figure 5 do not embed (this is a direct consequence of [8, Theorem 4.3] where it is shown that the rationals and the circular tournament are up to isomorphism the only two countable homogeneous tournaments in which the successors of any point are linearly ordered). It follows that there is a

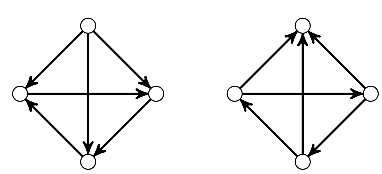


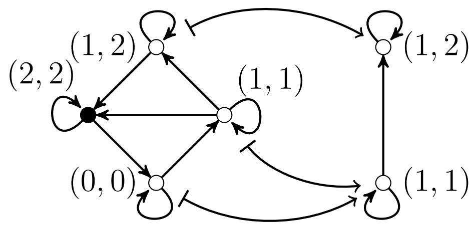
Figure 5: Two forbidden subtournaments in the circular tournament

monomorphism from $\mathbf{\Delta}_3$ into the countable circular tournament. Thus, the claim follows from Lemma 3.2. □

Proposition 3.5 ([7, Proposition 5.2])**.** $\mathbf{C}_3^{(3)}$ *is not* 2*-polymorphism homogeneous.*

Proof. Let $\{0, 1, 2\}$ be the vertex set of $\mathbf{C}_3^{(3)}$, such that $0 \to 1 \to 2 \to 0$. Let $\mathbf{B}_1 := \langle\{(0,0), (1,1), (1,2), (2,2)\}\rangle_{(\mathbf{C}_3^{(3)})^2}$, $\mathbf{B}_2 := \langle\{(1,1), (1,2)\}\rangle_{(\mathbf{C}_3^{(3)})^2}$,

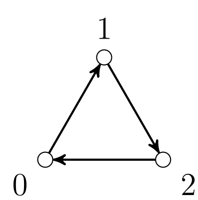
$f\colon B_1 \setminus \{(2,2)\} \to B_2$ given by $f := \begin{pmatrix} (0,0) & (1,1) & (1,2) \\ (1,1) & (1,1) & (1,2) \end{pmatrix}$. We claim that $W = (\mathbf{B}_1, \mathbf{B}_2, f, (2,2))$ is a witness in $(\mathbf{C}_3^{(3)})^2$ of the following type:



To prove this claim observe that any homomorphic extension $\hat{f}$ of f to $\mathbf{B}_1$ needs to map $(2,2)$ to $(1,1)$, since $(1,1) \to (2,2) \to (0,0)$, whence $\hat{f}(1,1) = (1,1) \to \hat{f}(2,2) \to (1,1) = \hat{f}(0,0)$. However, this leads into trouble, since $(1,2) \to (2,2)$, while $\hat{f}(2,2) = (1,1) \to (1,2) = \hat{f}(1,2)$. Thus, a homomorphic extension $\hat{f}$ of f to $\mathbf{B}_1$ does not exist. □

Proposition 3.6 ([7, Proposition 5.4])**.** $\mathbf{C}_3^{(0)}$ *is polymorphism homogeneous.*

Proof. Let $\mathbf{T} = (T, \varrho) \cong \mathbf{C}_3^{(0)}$ be the tournament given in the following picture:



Observe that $\mathbf{T}^k \cong 3^{k-1} \times \mathbf{T}$, where the structure $n \times \mathbf{T}$ is defined to be $(n \times T, n \times \varrho)$, with

$$((i,x),(j,y)) \in (n \times \varrho) :\iff (i = j) \wedge (x,y) \in \varrho.$$

If we show that $n \times \mathbf{T}$ is homomorphism homogeneous, then it follows that $\mathbf{T}$ is polymorphism homogeneous. So let $\mathbf{A}$ be a substructure of $n \times \mathbf{T}$, let $h\colon \mathbf{A} \to n \times \mathbf{T}$ be a homomorphism, and let (i,u) be a vertex of $n \times \mathbf{T}$. If we can show that h can be extended to a homomorphism $\hat{h}\colon \langle A \cup \{(i,u)\}\rangle_{n\times\mathbf{T}} \to n \times \mathbf{T}$, then we are done.

If $(i,u) \in A$, then we can obviously choose $\hat{h} = h$.

If $(i,u) \notin A$ and no element of A is of the shape (i,v), then we can define $\hat{h}(i,u) := (i,u)$.

If $(i,u) \notin A$ and some element of A is of the shape (i,v), then either $u \equiv v+1 \pmod 3$ or $v \equiv u+1 \pmod 3$. Suppose $h(i,v) = (j,w)$. In the former case we define $\hat{h}(i,u) := (j, (w+1) \mod 3)$. In the latter case we define $\hat{h}(i,u) := (j,(w-1) \mod 3)$.

A routine check shows that in this way $\hat{h}\colon \langle A \cup \{(i,u)\}\rangle_{n\times\mathbf{T}} \to n \times \mathbf{T}$ is indeed a homomorphism. Thus, $\mathbf{C}_3^{(0)}$ is indeed polymorphism homogeneous. □

To sum up our findings, the only countable intransitive polymorphism homogeneous tournament is $\mathbf{C}_3^{(0)}$. It remains to sift through the list of transitive homomorphism homogeneous tournaments from Theorem 2.15. Alas, the complete set of witnesses for transitive tournaments that we used in the previous section is not sufficient for the test for polymorphism homogeneity because the arc-relation of the finite powers of a transitive tournament with loops allowed are transitive and antisymmetric, but not necessarily total.

The next task at hand is to find a complete set of witnesses for binary relational structures whose basic relation is antisymmetric and transitive. To this end, let $\mathbf{A}$ be any such structure and let $W = (\mathbf{B}_1, \mathbf{B}_2, f, c)$ be a witness in $\mathbf{A}$, such that $|B_1|$ is as small as possible. Let us first sketch our general strategy. We split the analysis of witnesses into several claims each of which will be proved below:

Claim 1: $f\colon B_1 \setminus \{c\} \to B_2$ is a bijection,
Claim 2: For each $b \in B_1 \setminus \{c\}$ we have either $b \to c$ or $c \to b$,

Let $B_{11} := \{b \in B_1 \mid b \to c, b \neq c\}$, $B_{12} := \{b \in B_1 \mid c \to b, b \neq c\}$. Moreover, let $B_{21} := f(B_{11})$ and $B_{22} := f(B_{12})$.

Claim 3: B_{21} and B_{22} induce antichains in $\mathbf{A}$,
Claim 4: B_{11} and B_{12} induce antichains in $\mathbf{A}$,
Claim 5: for each $i \in \{1,2\}$, if B_{2i} is a singleton, then its unique element is loopless,
Claim 6: for each $i \in \{1,2\}$, if B_{1i} is a singleton, then its unique element is loopless,
Claim 7: for each $i \in \{1,2\}$, if B_{1i} contains a loopless vertex, then it has at most two elements,
Claim 8: if c has a loop, then B_{11} and B_{12} have at most two elements each.

About claim 1: Suppose that f is not bijective. Let $\tilde{B}_1$ be a transversal of the kernel of f. Let $\tilde{f} := f\restriction_{\tilde{B}_1}$. Then $\tilde{f}\colon \tilde{B}_1 \to B_2$ is a bijection. Observe that $(\langle \tilde{B}_1 \cup \{c\}\rangle_{\mathbf{A}}, \mathbf{B}_2, \tilde{f}, c)$ is a witness in $\mathbf{A}$. Since f is not bijective, we have that $|\tilde{B}_1 \cup \{c\}| < |B_1|$, a contradiction with the minimality of $|B_1|$.

About claim 2: Suppose there exists some $b \in B_1$ such that neither $b \to c$ nor $c \to b$. Let $\tilde{B}_1 := B_1 \setminus \{b\}$, let $\tilde{B}_2 := f(\tilde{B}_1 \setminus \{c\})$, and let $\tilde{f} := f\restriction_{\tilde{B}_1\setminus\{c\}}$. By the minimality of $|B_1|$, $(\langle \tilde{B}_1\rangle_{\mathbf{A}}, \langle \tilde{B}_2\rangle_{\mathbf{A}}, \tilde{f}, c)$ is not a witness in $\mathbf{A}$. Let $\tilde{f}' \colon \langle \tilde{B}_1\rangle_{\mathbf{A}} \to \mathbf{A}$ be a homomorphic extension of $\tilde{f}$. Now define $\hat{f} \colon B_1 \to A$ according to

$$\hat{f} \colon x \mapsto \begin{cases} f(x) & x \in B_1 \setminus \{c\}, \\ \tilde{f}'(x) & x = c. \end{cases}$$

It is not hard to see that in fact $\hat{f} \colon \mathbf{B}_1 \to \mathbf{A}$ is a homomorphism. Thus, W is not a witness in $\mathbf{A}$, a contradiction.

About claim 3: Suppose that B_{21} is not an antichain in $\mathbf{A}$. Let $\tilde{B}_{21}$ be the set of maximal elements in B_{21}. Let $\tilde{B}_2 := \tilde{B}_{21} \cup B_{22}$. Let $\tilde{B}_1 := f^{-1}(\tilde{B}_2) \cup \{c\}$, and let $\tilde{f} := f\restriction_{\tilde{B}_1\setminus\{c\}}$. Then $(\langle \tilde{B}_1\rangle_{\mathbf{A}}, \langle \tilde{B}_2\rangle_{\mathbf{A}}, \tilde{f}, c)$ is again a witness in $\mathbf{A}$, for, if it is not, then there exists some $x \in A$, such that $\tilde{B}_{21} \to x \to B_{22}$. But then we also have $B_{21} \to x \to B_{22}$, a contradiction to the assumption that $(\mathbf{B}_1, \mathbf{B}_2, f, c)$ is a witness in $\mathbf{A}$. Since $|\tilde{B}_2| < |B_2|$ and since $\tilde{f}$ is bijective, it follows that $|\tilde{B}_1| < |B_1|$, and we arrive at a contradiction with the minimality of $|B_1|$. The claim for B_{22} is proved analogously.

About claim 4: If B_{11} does not induce an antichain in $\mathbf{A}$, then neither does $f(B_{11}) = B_{21}$, a contradiction. The argument for B_{12} goes analogously.

About claim 5: Suppose that $B_{21} = \{b\}$ and that b is a loop. Define $\hat{f} \colon B_1 \to B_2$, such that $\hat{f}\restriction_{B_1\setminus\{c\}} = f$ and such that $\hat{f}(c) = b$. It is not hard to see that $\hat{f} \colon \mathbf{B}_1 \to \mathbf{B}_2$ is a homomorphism, a contradiction. It follows that b is a non-loop. The claim for B_{22} is proved analogously.

About claim 6: Suppose that $B_{11} = \{b\}$ and that b is a loop. Then also $B_{21} = \{f(b)\}$ is a singleton and its element is a loop, a contradiction. The claim for B_{12} is shown analogously.

About claim 7: Suppose that B_{11} contains a loopless vertex but that it has more than two elements, say, $B_{11} = \{b_1, \dots, b_{k+1}\}$ for some $k \geq 2$. Without loss of generality, let b_1 be a non-loop. Let $B'_{11} := \{b_1, \dots, b_k\}$, and let $B'_{21} := f(B'_{11})$. By the minimality of $|B_1|$, $(\langle B'_{11} \cup B_{12} \cup \{c\}\rangle_{\mathbf{A}}, \langle B'_{21} \cup B_{22}\rangle_{\mathbf{A}}, f\restriction_{B'_{11}\cup B_{12}}, c)$ is not a witness in $\mathbf{A}$. Let $f' \colon \langle B'_{11} \cup B_{12} \cup \{c\}\rangle_{\mathbf{A}} \to \mathbf{A}$ be an extension of f, and let $d := f'(c)$. Let $\tilde{B}_1 := \{b_1, b_{k+1}\} \cup B_{12} \cup \{c\}$,

$\tilde{B}_2 := \{d, f(b_{k+1})\} \cup B_{22}$. Further define $\tilde{f} : \tilde{B}_1 \setminus \{c\} \to \tilde{B}_2$ according to

$$\tilde{f}\colon x \mapsto \begin{cases} f(x) & x \in B_{12} \cup \{b_{k+1}\}, \\ d & x = b_1. \end{cases}$$

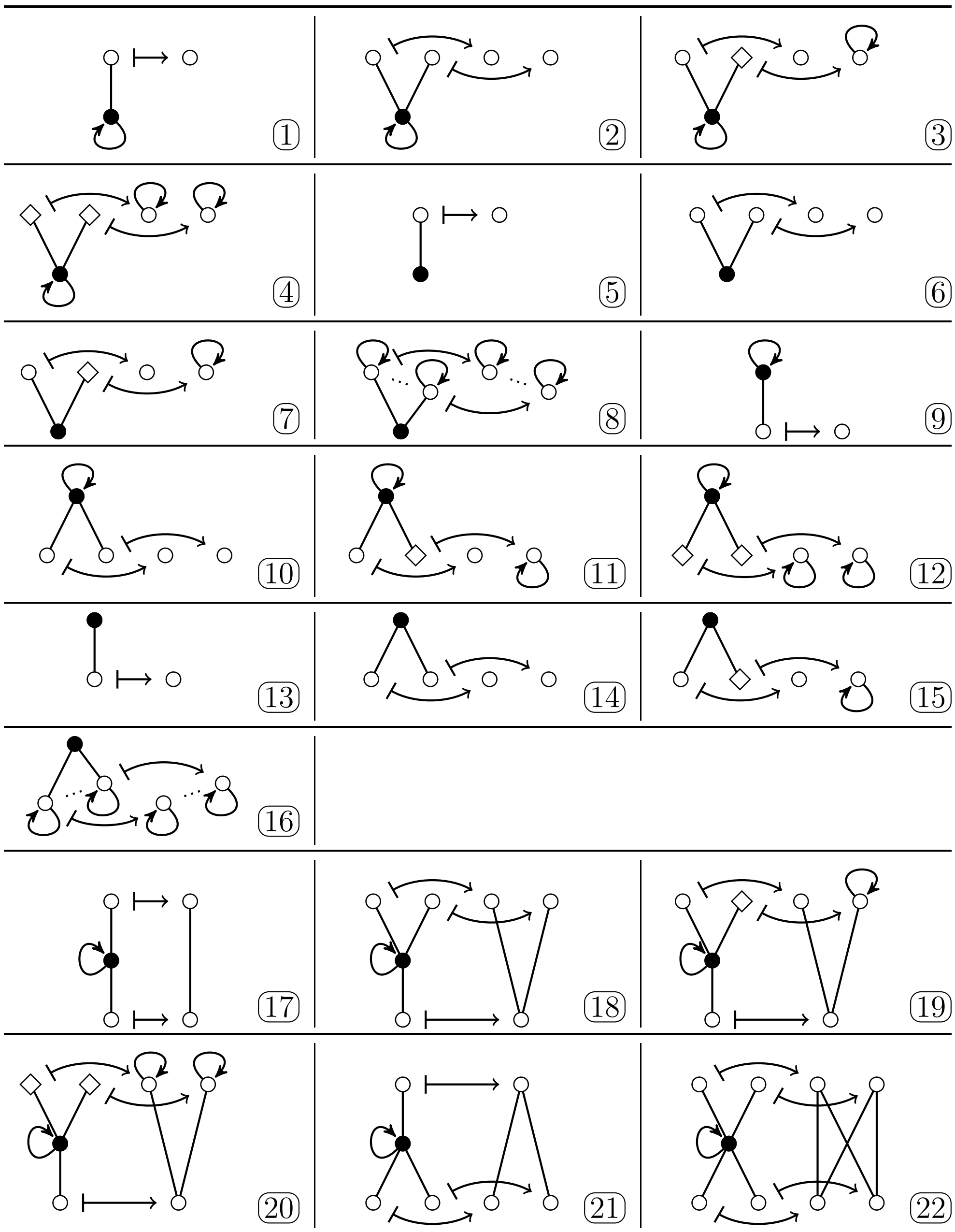
Then $\tilde{f}\colon \langle \tilde{B}_1 \setminus \{c\}\rangle_{\mathbf{A}} \to \langle \tilde{B}_2\rangle_{\mathbf{A}}$ is a homomorphism. Moreover we have that $(\langle \tilde{B}_1\rangle_{\mathbf{A}}, \langle \tilde{B}_2\rangle_{\mathbf{A}}, \tilde{f}, c)$ is a witness in $\mathbf{A}$. However, $|\tilde{B}_1| < |B_1|$, a contradiction with the minimality of $|B_1|$. It follows that B_{11} has at most two elements. The proof for B_{12} goes analogously.

About claim 8: Suppose that c has a loop but that B_{11} has more than two elements, say, $B_{11} = \{b_1, \ldots, b_{k+1}\}$ for some $k \geq 2$. Let $B'_{11} := \{b_1, \ldots, b_k\}$, and let $B'_{21} := f(B'_{11})$. By the minimality of $|B_1|$, $(\langle B'_{11} \cup B_{12} \cup \{c\}\rangle_{\mathbf{A}}, \langle B'_{21} \cup B_{22}\rangle_{\mathbf{A}}, f\restriction_{B'_{11}\cup B_{12}}, c)$ is not a witness in $\mathbf{A}$. Let $f'\colon \langle B'_{11} \cup B_{12} \cup \{c\}\rangle_{\mathbf{A}} \to \mathbf{A}$ be an extension of f, and let $d := f'(c)$. Note that d has a loop. Let $\tilde{B}_1 := \{b_1, b_{k+1}\} \cup B_{12} \cup \{c\}$, $\tilde{B}_2 := \{d, f(b_{k+1})\} \cup B_{22}$. Further define $\tilde{f} : \tilde{B}_1 \setminus \{c\} \to \tilde{B}_2$ according to

$$\tilde{f}\colon x \mapsto \begin{cases} f(x) & x \in B_{12} \cup \{b_{k+1}\}, \\ d & x = b_1. \end{cases}$$

Then $\tilde{f}\colon \langle \tilde{B}_1 \setminus \{c\}\rangle_{\mathbf{A}} \to \langle \tilde{B}_2\rangle_{\mathbf{A}}$ is a homomorphism. Moreover we have that $(\langle \tilde{B}_1\rangle_{\mathbf{A}}, \langle \tilde{B}_2\rangle_{\mathbf{A}}, \tilde{f}, c)$ is a witness in $\mathbf{A}$. However, $|\tilde{B}_1| < |B_1|$, a contradiction with the minimality of $|B_1|$. It follows that B_{11} has at most two elements. The proof for B_{12} goes analogously.

Now that all the claims are proved, it remains to enumerate all witnesses that have all the postulated properties in order to have a complete set of witnesses. A complete list is given in Table 2. For better readability, all structures in this table are depicted as Hasse diagrams. Moreover, to preserve some space one and the same picture may represent several witnesses. This is achieved by introducing a new type of vertex symbol, the diamond. A vertex that is depicted as a diamond may be interpreted as a loop or as a non-loop. We are going to refer to these pictures as *shapes* of witnesses. Each shape has an index given in Table 2. Note that there are shapes that represent infinitely many witnesses. E.g., shape number 16 has on the left hand side an antichain of length n consisting of loopy vertices, together with a loopless joint upper bound. On the right hand side it has just an antichain of n loopy vertices. Here n may vary through $\mathbb{N} \setminus \{0, 1\}$.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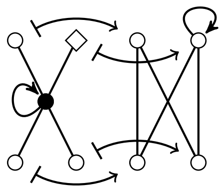
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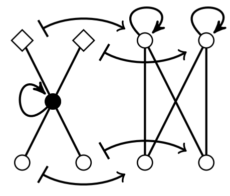
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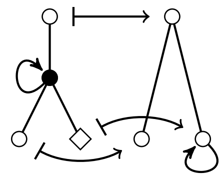
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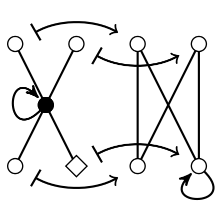
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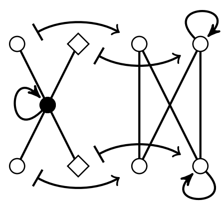
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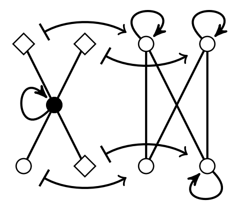
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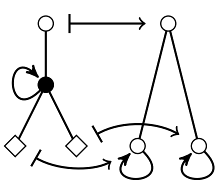
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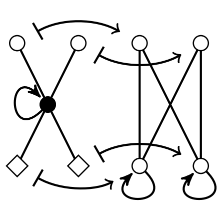
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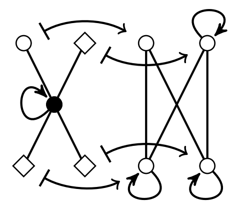
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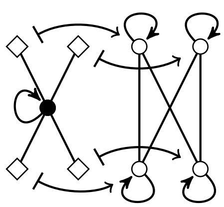
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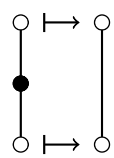
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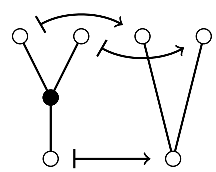
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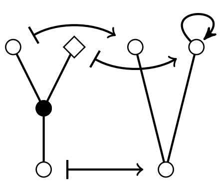
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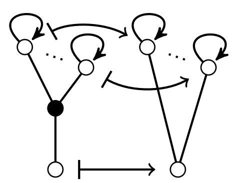
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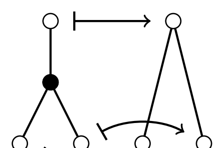
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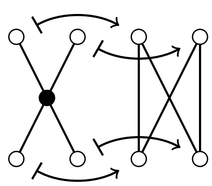
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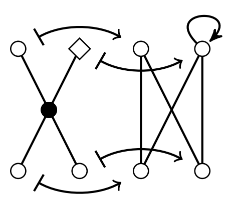
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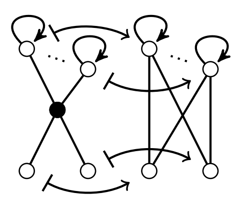
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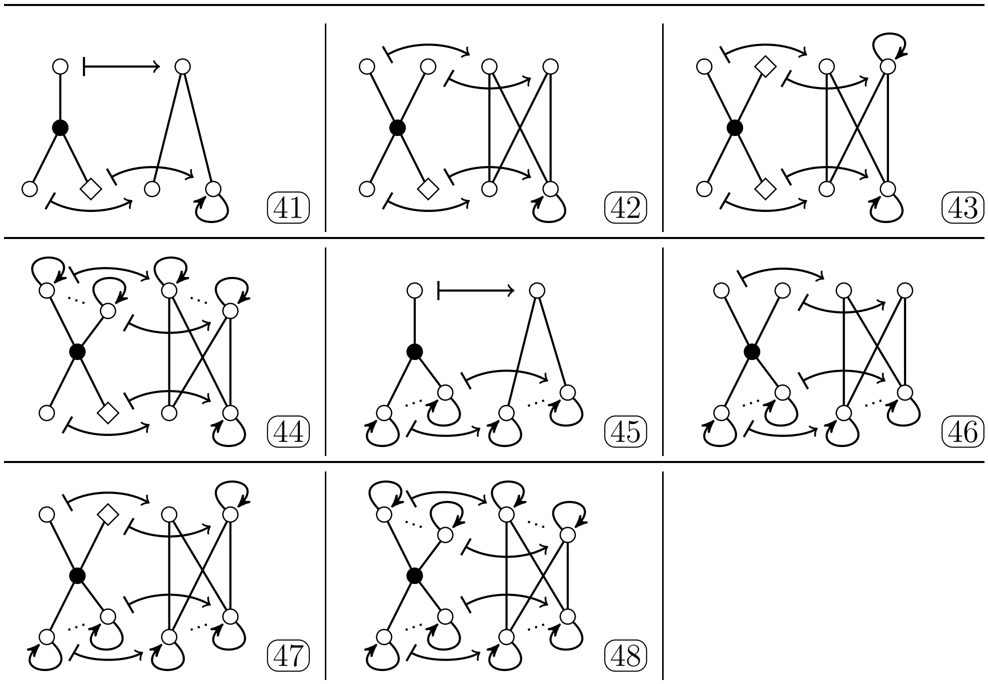

Table 2: A complete set of witnesses for antisymmetric transitive relations

This is still not the end of the story. In order to test whether a homomorphism homogeneous transitive tournament with loops allowed is polymorphism homogeneous, we should in principle check for each of its finite powers that it avoids all witnesses from Table 2. Of course, this is not feasible to do in a brute force manner. Instead we identify some properties of transitive antisymmetric relations that are stable under finite powers and that, each for itself, imply that certain shapes of witnesses are avoided:

Definition 3.7. Let $\mathbf{A} = (A, \varrho)$ be a binary relational structure, such that ϱ is antisymmetric and transitive. We say that $\mathbf{A}$ satisfies Condition

(A) if every finite nonempty set of vertices has a lower bound (loopy or not),

(B) if every finite nonempty set of vertices has an upper bound (loopy or not),

(C) if for all finite nonempty sets B_1, B_2 of vertices there exists some vertex c (loopy or not), such that if $B_1 \to B_2$, then $B_1 \to c \to B_2$,

(D) if every finite nonempty set of vertices has a loopy lower bound,
(E) if every finite nonempty set of vertices has a loopy upper bound,
(F) if for all finite nonempty sets B_1, B_2 of vertices there exists some loopy vertex c, such that if $B_1 \to B_2$, then $B_1 \to c \to B_2$,
(G) if every finite nonempty set of loopy vertices has a loopy lower bound,
(H) if every finite nonempty set of loopy vertices has a loopy upper bound,
(I) if for all finite nonempty sets B_1, B_2 of loopy vertices there exists some loopy vertex c, such that if $B_1 \to B_2$, then $B_1 \to c \to B_2$,
(J) $\mathbf{S}_B$ does not embed into $\mathbf{A}$,
(K) $\mathbf{S}_A$ does not embed into $\mathbf{A}$,
(L) if for every finite nonempty set B_1 of vertices and for every finite nonempty set B_2 of loopy vertices there exists a loopy vertex c, such that if $B_1 \to B_2$, then $B_1 \to c \to B_2$.
(M) if for every finite nonempty set B_1 of loopy vertices and for every finite nonempty set B_2 of vertices there exists a loopy vertex c, such that if $B_1 \to B_2$, then $B_1 \to c \to B_2$.
(N) $\mathbf{S}_C$ does not embed into $\mathbf{A}$,
(O) $\mathbf{S}_D$ does not embed into $\mathbf{A}$.

Lemma 3.8. *Conditions A–O are stable under finite powers. I.e., if some* $\mathbf{A}$ *satisfies one of the conditions, then so does every finite power of* $\mathbf{A}$.

Proof. Straightforward. □

The next step is to analyze which witnesses from Table 2 are avoided, given that certain conditions from Definition 3.7 are satisfied:

Lemma 3.9. *Let* $\mathbf{A} = (A, \varrho)$ *be a binary relational structure such that* ϱ *is antisymmetric and transitive. Then the following is true:*

(a) *if* $\mathbf{A}$ *satisfies (A), then* $\mathbf{A}$ *avoids all witnesses of the shapes 5 to 8,*
(b) *if* $\mathbf{A}$ *satisfies (B), then* $\mathbf{A}$ *avoids all witnesses of the shapes 13 to 16,*
(c) *if* $\mathbf{A}$ *satisfies (C), then* $\mathbf{A}$ *avoids all witnesses of the shapes 33 to 48,*
(d) *if* $\mathbf{A}$ *satisfies (D), then* $\mathbf{A}$ *avoids all witnesses of shapes 1 to 8,*
(e) *if* $\mathbf{A}$ *satisfies (E), then* $\mathbf{A}$ *avoids all witnesses of shapes 9 to 16,*
(f) *if* $\mathbf{A}$ *satisfies (F), then* $\mathbf{A}$ *avoids all witnesses of shapes 17 to 48,*
(g) *if* $\mathbf{A}$ *satisfies (G), then* $\mathbf{A}$ *avoids all witnesses of shapes 4 and 8,*
(h) *if* $\mathbf{A}$ *satisfies (H), then* $\mathbf{A}$ *avoids all witnesses of shapes 12 and 16,*

(i) *if* **A** *satisfies (I), then* **A** *avoids all witnesses of shapes 32 and 48,*
(j) *if* **A** *satisfies (J), then* **A** *avoids all witnesses of shapes 1 to 3, 16 to 19, 21 to 23, 25 to 27, 29 to 31, 41 to 43 and 45 to 48,*
(k) *if* **A** *satisfies (K), then* **A** *avoids all witnesses of shapes 8 to 11, 17 to 28, 35, 36, 39, 40, 43, 44 and 48,*
(l) *if* **A** *satisfies (L), then* **A** *avoids all witnesses of shapes 20, 24, 28, 32, 36, 40, 44 and 48,*
(m) *if* **A** *satisfies (M), then* **A** *avoids all witnesses of shapes 29 to 32 and 45 to 48,*
(n) *if* **A** *satisfies (N), then* **A** *avoids all witnesses of shapes 5 to 7, 13 to 15, 17 to 19, 21 to 23, 25 to 27 and 33 to 47,*
(o) *if* **A** *satisfies (O), then* **A** *avoids all witnesses of shapes 17 to 19, 21 to 23 and 25 to 27.*

Proof. **About (a):** In order to avoid witnesses of shape 5, it is sufficient that every vertex in **A** has a lower bound. In order to avoid witnesses of shapes 6 and 7, it is sufficient that any two vertices in **A** have a lower bound. Finally, in order to avoid witnesses of shape 8, it is sufficient that every finite antichain M of loopy vertices in **A** has a lower bound. Clearly, all this is entailed by Condition A.

The other parts of the lemma have similar straightforward proofs. Therefore they are omitted. □

Theorem 3.10. *Let* **T** *be a countable tournament. Then* **T** *is polymorphism homogeneous if and only if there exist nonempty subsets* M, M_1, M_2 *of* $\mathbb{Q}$*, such that exactly one of the following is true:*

(1) $\mathbf{T} \cong \mathbf{C}_3^{(0)}$;
(2) $\mathbf{T} \cong (\{0\}, <)$;
(3) $\mathbf{T} \cong (\{-\infty\}, <) + (M, \leq)$;
(4) $\mathbf{T} \cong (M, \leq) + (\{+\infty\}, <)$;
(5) $\mathbf{T} \cong (\mathbb{Q}, <)$;
(6) $\mathbf{T} \cong (\mathbb{Q} \cup \{+\infty\}, <) + (M, \leq)$;
(7) $\mathbf{T} \cong (\mathbb{Q}, <) + (M, \leq)$;
(8) $\mathbf{T} \cong (M, \leq) + (\mathbb{Q} \cup \{-\infty\}, <)$;
(9) $\mathbf{T} \cong (M, \leq) + (\mathbb{Q}, <)$;
(10) $\mathbf{T} \cong (M_1, \leq) + (\mathbb{Q}, <) + (M_2, \leq)$;
(11) $\mathbf{T} \cong (M_1, \leq) + (\mathbb{Q} \cup \{-\infty\}, <) + (M_2, \leq)$;

(12) $\mathbf{T} \cong (M_1, \leq) + (\mathbb{Q} \cup \{+\infty\}, <) + (M_2, \leq)$;

(13) $\mathbf{T} \cong (M_1, \leq) + (\mathbb{Q} \cup \{-\infty, +\infty\}, <) + (M_2, \leq)$;

(14) $\mathbf{T}$ *has the following properties:*

- *below every loopless vertex there is at least one loopy vertex,*
- *above every loopless vertex there is at least one loopy vertex,*
- *in-between any two loopless vertices there is at least one loopy vertex.*

Proof. Let us start by proving that all the given tournaments are indeed polymorphism homogeneous. For the transitive tournaments it is sufficient to show that they avoid all witnesses of shapes from Table 2, since these form a complete set for antisymmetric transitive relations.

about (1): The polymorphism homogeneity of $\mathbf{C}_3^{(0)}$ was already shown in Proposition 3.6.

about (2): $(\{0\}, <)$ is isomorphic to each of its finite powers. Since it is homomorphism homogeneous, it is also polymorphism homogeneous, by Proposition 3.1.

about (3): Tournaments of this kind satisfy Conditions B, C, E–J and L–O. From Lemmas 3.8 in conjunction with Lemma 3.9 it follows that these tournaments avoid all witnesses from Table 2.

about (4): This case is dual to case (3).

about (5): $(\mathbb{Q}, <)$ satisfies Conditions A–C, G–M and O. From Lemmas 3.8 in conjunction with Lemma 3.9 it follows that this tournament avoids all witnesses from Table 2.

about (6): Tournaments of this kind satisfy Conditions A–C, E, G–J, L, M and O. From Lemmas 3.8 in conjunction with Lemma 3.9 it follows that these tournaments avoid all witnesses from Table 2.

about (7): Analogous to case (6).

about (8): Dual to case (6).

about (9): Dual to case (7).

about (10): Tournaments of this kind satisfy Conditions A–E, G–I, L, M and O. From Lemmas 3.8 in conjunction with Lemma 3.9 it follows that these tournaments avoid all witnesses from Table 2.

about (11): Analogous to case (10).

about (12): Dual to case (11).

about (13): Analogous to case (10).

about (14): Tournaments of this kind satisfy Conditions A–I, L and M. From Lemmas 3.8 in conjunction with Lemma 3.9 it follows that these tournaments avoid all witnesses from Table 2.

At the end we go through the remaining homomorphism homogeneous tournaments from Theorem 2.15:

It follows from Corollaries 3.3, 3.4, and from Proposition 3.5 that neither the countable universal tournament nor the circular tournament, nor $\mathbf{C}_3^{(3)}$ are polymorphism homogeneous. It remains to show that tournaments of the shape $\mathbf{T} = (M_1, \leq) + (\{0,1\}, <) + (M_2, \leq)$ (cf. Theorem 2.15(8)) are not polymorphism homogeneous: Let $u \in M_1$, $v \in M_2$. Let $B_1 = \{(0,u),(v,1),(1,0)\}$, $B_2 := \{(0,0),(1,1)\}$, $f\colon B_1 \setminus \{(1,0)\} \to B_2$ given by $f\colon (0,u) \mapsto (0,0)$, $(v,1) \mapsto (1,1)$. Let $c := (1,0)$. Then $(\langle B_1\rangle_{\mathbf{T}^2}, \langle B_2\rangle_{\mathbf{T}^2}, f, c)$ is a witness of shape 33 from Table 2 in $\mathbf{T}^2$. Consequently, $\mathbf{T}$ is not 2-polymorphism homogeneous. □

4. Concluding remarks

Related research

The present paper does not stand alone. Ever since the seminal paper [1] by Cameron and Nešetřil the classification theory of homomorphism homogeneous structures has been actively studied. Already in [1] the problem to classify homomorphism homogeneous graphs was posed and solved in the finite case. The countably infinite case was partially solved there, too, but until today it defies all efforts for a complete classification (cf. [13, 14]). Examples where a complete classification was reached, include strict and non-strict partial orders (cf. [5, 15]), lattices (cf. [16]), monounary algebras (cf. [17]). Examples where a complete classification of the finite homomorphism homogeneous structures was reached include tournaments with loops allowed ([6]), L-colored graphs over chains ([18]), uniform oriented graphs ([19]). There is no hope that the classification problem of finite homomorphism homogeneous structures can be solved by the currently known methods. The reason is that Rusinov and Schweitzer showed in [13] that the problem to decide whether a given finite loopy graph is homomorphism homogeneous is coNP-complete. However, so far all classification results in this area entailed a polynomial test of homomorphism homogeneity for the finite structures in question.

Other modes of homogeneity

Meanwhile a whole spectrum of homogeneity conditions was introduced by Lockett and Truss in [20]. They form a hierarchy of 18 conditions including homogeneity and homomorphism homogeneity. The classification theory for these conditions became itself a topic of research (cf. [14, 21]).

Polymorphism homogeneity

The notion of polymorphism homogeneity entered the stage through a different door. The notion arose as a byproduct in the research on weakly oligomorphic structures with the goal to get a version of the Theorem of Engeler, Ryll-Nardzewski and Svenonius for endomorphism monoids and for polymorphism clones of countable structures (cf. [3, 12, 22, 23]). Just like that homogeneous ω-categorical structures admit elimination of quantifiers, countable weakly oligomorphic polymorphism homogeneous structures admit quantifier elimination for primitive positive formulae. This feature was essential in the proofs of several results on polymorphism clones of countable homogeneous structures. E.g., in [24] polymorphism homogeneity of certain homogeneous structures is used in order to show automatic homeomorphicity for their polymorphism clones (see [25] for more information on this line of research). The classification theory of PH structures was initiated in [2, 26]. It soon turned out that the classification problem of polymorphism homogeneous structures is amenable even in cases where the case of HH structures makes problems (e.g., in [2] the countable polymorphism homogeneous graphs are classified). Despite this, classifying polymorphism homogeneous structures is far from easy. Up till now we only know that the problem whether a given finite structure is PH is decidable (cf. [2, Corollary 5.5]). Nothing more specific is known about the complexity of this decision problem. However, it should be mentioned that up till now we never came upon a class of finite relational structures where the problem to decide polymorphism homogeneity is intractable.

A bonus result

In Table 2 a complete set of witnesses for binary relational structures with an antisymmetric, transitive relation is given. Such a set generally can be used to devise an algorithm that tests whether a finite structure of this type is homomorphism homogeneous. Modulo some small consideration about those shapes that represent infinitely many witnesses it is easy to see that the polymorphism homogeneity for finite, binary, antisymmetric, transitive relational structures is decidable in polynomial time. This observation extends even further to the class of finite binary, transitive relational structures, because, as it was shown in [12, Section 3.2.2], such a structure is homomorphism homogeneous if and only if its maximal antisymmetric retract is.

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