Complexity Theory

Turing Machines and Languages

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Computational Logic

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Deterministic Turing Machines

A Model for Computation

Clear

To understand computational problems we need to have a formal understanding of what an *algorithm* is.

Example 2.1 (Hilbert's Tenth Problem)

> "Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers." (Wikipedia)

Question

How can we model the notion of an algorithm?

Answer

With Turing machines.

Turing Machines

Let us fix a blank symbol $\square$.

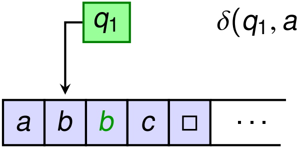
Definition 2.2

A (deterministic) *Turing Machine* $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$ consists of

- a finite set Q of *states*,
- an *input alphabet* Σ not containing $\square$,
- a *tape alphabet* Γ such that $\Gamma \supseteq \Sigma \cup \{\square\}$.
- a *transition function* $\delta\colon Q \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}\}$
- an *initial state* $q_0 \in Q$,
- an *accepting state* $q_{\text{accept}} \in Q$, and
- an *rejecting state* $q_{\text{reject}} \in Q$ such that $q_{\text{accept}} \neq q_{\text{reject}}$.

Turing Machines

Example 2.3



$\delta(q_1, a) = (q_2, b, \mathsf{L})$

- The tape is bounded on the left, but unbounded on the right; the content of the tape is a finite word over Γ, followed by an infinite sequence of $\Box$.
- The head of the machine is at exactly one position of the tape
- The head can read only one symbol at a time
- The head moves and writes according to the transition function δ; the current state also changes accordingly
- The head will stay put when attempting to cross the left tape end

Configurations

Observation: to describe the current step of a computation of a TM it is enough to know

- the content of the tape,
- the current state, and
- the position of the head

Definition 2.4

A *configuration* of a TM $\mathcal{M}$ is a word uqv such that

- $q \in Q$,
- $uv \in \Gamma^*$

Some special configurations:

- The *start configuration* for some input word $w \in \Sigma^*$ is the configuration $q_0 w$
- A configuration uqv is *accepting* if $q = q_{\text{accept}}$.
- A configuration uqv is *rejecting* if $q = q_{\text{reject}}$.

Computation

We write

- $C \vdash_{\mathcal{M}} C'$ only if C' can be reached from C by one computation step of $\mathcal{M}$;
- $C \vdash^*_{\mathcal{M}} C'$ only if C' can be reached from C in a finite number of computation steps of $\mathcal{M}$.

We say that $\mathcal{M}$ *halts* on input *w* if and only if there is a finite sequence of configurations

$$C_0 \vdash_{\mathcal{M}} C_1 \vdash_{\mathcal{M}} \cdots \vdash_{\mathcal{M}} C_\ell$$

such that C_0 is the start configuration of $\mathcal{M}$ on input *w* and C_ℓ is an accepting or rejecting configuration. Otherwise $\mathcal{M}$ *loops* on input *w*.

We say that $\mathcal{M}$ *accepts* the input *w* only if $\mathcal{M}$ halts on input *w* with an accepting configuration.

Recognizability and Decidability

Recognizability and Decidability

Definition 2.5

Let $\mathcal{M}$ be a Turing machine with input alphabet Σ. The *language accepted by* $\mathcal{M}$ is the set

$$\mathcal{L}(\mathcal{M}) := \{\, w \in \Sigma^* \mid \mathcal{M} \text{ accepts } w \,\}.$$

A language $\mathcal{L} \subseteq \Sigma^*$ is called *Turing-recognizable* (*recursively enumerable*) if and only if there exists a Turing machine $\mathcal{M}$ with input alphabet Σ^* such that $\mathcal{L} = \mathcal{L}(\mathcal{M})$. In this case we say that $\mathcal{M}$ *recognizes* $\mathcal{L}$.

A language $\mathcal{L} \subseteq \Sigma^*$ is called *Turing-decidable* (*decidable*, *recursive*) if and only if there exists a Turing machine $\mathcal{M}$ such that $\mathcal{L} = \mathcal{L}(\mathcal{M})$ and $\mathcal{M}$ halts on every input. In this case we say that $\mathcal{M}$ *decides* $\mathcal{L}$.

Example

Claim

The language $\mathcal{L} := \{\, 0^{2^n} \mid n \geq 0 \,\}$ is decidable.

Proof

A Turing machine $\mathcal{M}$ that decides $\mathcal{L}$ is

$\mathcal{M}$:= On input w, where w is a string

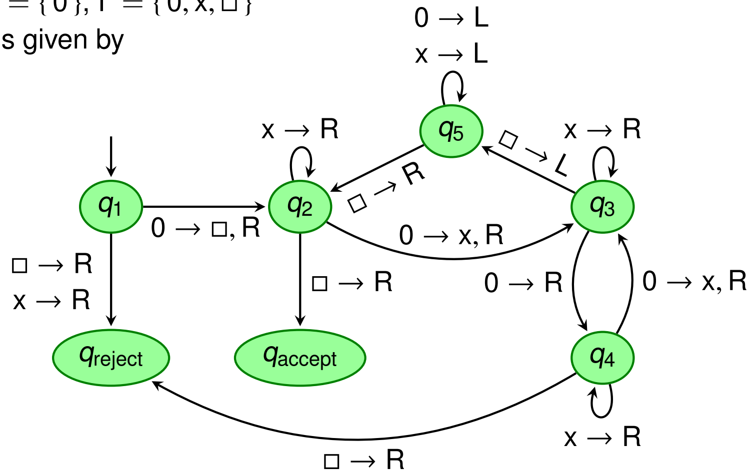
- Go from left to right over the tape and cross off every other 0
- If in the first step the tape contained a single 0, *accept*
- If in the first step the number of 0s on the tape was odd, *reject*
- Return the head the beginning of the tape
- Go to the first step

Example (cont'd)

Formally, $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_1, q_{\text{accept}}, q_{\text{reject}})$, where

- $Q = \{ q_1, q_2, q_3, q_4, q_5, q_{\text{accept}}, q_{\text{reject}} \}$
- $\Sigma = \{ 0 \}$, $\Gamma = \{ 0, \text{x}, \square \}$

and δ is given by



Problems as Languages

Observation

- Languages can be used to model computational problems.
- For this, a suitable *encoding* is necessary
- TMs must be able to decode the encoding

Example 2.6 (Graph-Connectedness)

The question whether a graph is connected or not can be seen as the *word problem* of the following language

$$\text{GCONN} := \{ \langle G \rangle \mid G \text{ is a connected graph} \},$$

where $\langle G \rangle$ is (for example) the adjacency matrix encoded in binary.

Notation

The encoding of objects $\mathcal{O}_1, \ldots, \mathcal{O}_n$ we denote by $\langle \mathcal{O}_1, \ldots, \mathcal{O}_n \rangle$.

The Church-Turing Thesis

It turns out that Turing-machines are *equivalent* to a number of formalizations of the intuitive notion of an *algorithm*

- λ-calculus
- while-programs
- μ-recursive functions
- Random-Access Machines
- ...

Because of this it is believed that Turing-machines completely capture the intuitive notion of an algorithm. $\rightsquigarrow$ *Church-Turing Thesis*:

> *"A function on the natural numbers is intuitively computable if and only if it can be computed by a Turing machine."*

(→ Wikipedia: Church-Turing Thesis)

Variants of Turing Machines

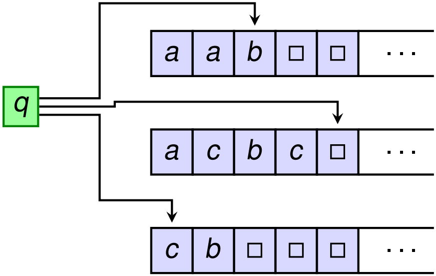
Variations of Turing-Machines

It has also been shown that deterministic, single-tape Turing machines are equivalent to a wide range of other forms of Turing machines:

- Multi-tape Turing machines
- Nondeterministic Turing machines
- Turing machines with doubly-infinite tape
- Multi-head Turing machines
- Two-dimensional Turing machines
- Write-once Turing machines
- Two-stack machines
- Two-counter machines
- …

Multi-Tape Turing Machines

k-tape Turing machines are a variant of Turing machines that have *k* tapes.



Multi-Tape Turing Machines

Definition 2.7

Let $k \in \mathbb{N}$. Then a (deterministic) *k-tape Turing machine* is a tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where

- Q, Σ, Γ, q_0, q_{accept}, q_{reject} are as for TMs
- δ is a transition function for k tapes, i.e.,

$$\delta\colon Q \times \Gamma^k \to Q \times \Gamma^k \times \{\, \mathrm{L}, \mathrm{R}, \mathrm{N} \,\}^k$$

Running M on input $w \in \Sigma^*$ means to start M with the content of the first tape being w and all other tapes blank.

The notions of a *configuration* and of the *language accepted by M* are defined analogously to the single-tape case.

Multi-Tape Turing Machines

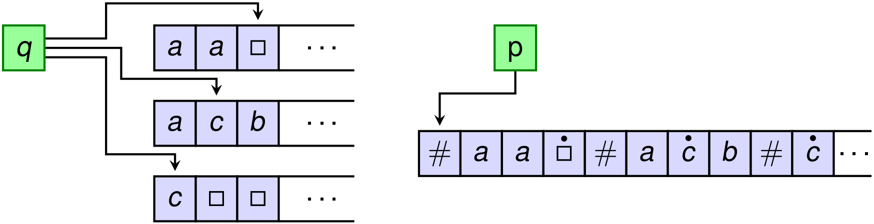
Theorem 2.8

Every multi-tape Turing machine has an equivalent single-tape Turing machine.

Proof.

Let M be a k-tape Turing machine. Simulate M with a single-tape TM S by

- keeping the content of all k tapes on a single tape, separated by #
- marking the positions of the individual heads using special symbols



Multi-Tape Turing Machines

$S :=$ On input $w = w_1 \dots w_n$

- Format the tape to contain the word

$$\#\dot{w}_1 w_2 \dots w_n \# \dot{\square} \# \dot{\square} \# \dots \#$$

- Scan the tape from the first # to the $(k+1)$-th # to determine the symbols below the markers.
- Update all tapes according to *M*'s transition function with a second pass over the tape; if any head of *M* moves to some previously unread portion of its tape, insert a blank symbol at the corresponding position and shift the right tape contents by one cell
- Repeat until the accepting or rejection state is reached.

□

Nondeterministic Turing Machines

Goal

Allow transitions to be *nondeterministic*.

Approach

Change transition function from

$$\delta\colon Q \times \Gamma \to Q \times \Gamma \times \{\, \mathsf{L}, \mathsf{R} \,\}$$

to

$$\delta\colon Q \times \Gamma \to \mathfrak{P}(Q \times \Gamma \times \{\, \mathsf{L}, \mathsf{R} \,\}).$$

The notions of *accepting* and *rejecting computations* are defined accordingly. Note: there may be more than one or no computation of a nondeterministic TM on a given input.
A nondeterministic TM M *accepts* an input w if and only if *there exists* some accepting computation of M on input w.

Nondeterministic Turing Machines

Theorem 2.9

Every nondeterministic TM has an equivalent deterministic TM.

Proof.

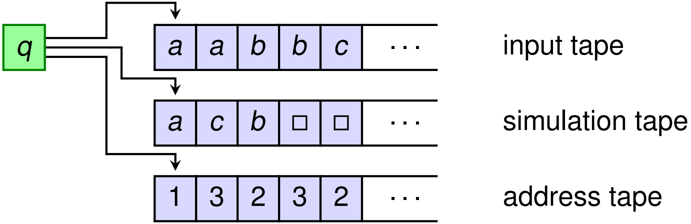
Let N be a nondeterministic TM. We construct a deterministic TM D that is equivalent to N, i.e., $\mathcal{L}(N) = \mathcal{L}(D)$.

Idea

- D deterministically traverses in breath-first order the tree of configuration of N, where each branch represents a different possibility for N to continue.
- For this, successively try out all possible choices of transitions allowed by N.

Nondeterministic Turing Machines

Sketch of D:



Let b be the maximal number of choices in δ, i.e.,

$$b := \max\left\{ |\delta(q, x)| \;\middle|\; q \in Q, x \in \Gamma \right\}.$$

Nondeterministic Turing Machines

D works as follows:

(1) Start: input tape contains input *w*, simulation and address tape empty
(2) Copy *w* to the simulation tape and initialize the address tape with 0.
(3) Simulate one finite computation of *N* on *w* on the simulation tape.
   - Interpret the address tape as a list of choices to make during this computation.
   - If a choice is invalid, abort simulation.
   - If an accepting configuration is reached at the end of the simulation, *accept*.
(4) Increment the content of the address tape, considered as a number in base *b*, by 1. Go to step 2.

□

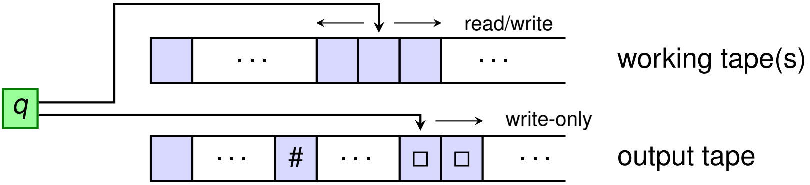
Enumerators

Definition 2.10

A multi-tape Turing machine M is an *enumerator* if

- M has a designated write-only *output-tape* on which a symbol, once written, can never be changed and where the head can never move left;
- M has a *marker symbol* # separating words on the output tape.

We define the *language generated by* M to be the set $\mathcal{G}(M)$ of all words that eventually appear between two consecutive # on the output tape of M when started on the empty word as input.



Enumerators

Theorem 2.11

A language $\mathcal{L}$ is Turing-recognizable if and only if there exists some enumerator E such that $\mathcal{G}(E) = \mathcal{L}$.

Proof.

Let E be an enumerator for $\mathcal{L}$. Then the following TM accepts $\mathcal{L}$:

$\mathcal{M}$:= On input w

- Simulate E on the empty input. Compare every string output by E with w
- If w appears in the output of E, *accept*

Enumerators

Let $\mathcal{L} = \mathcal{L}(\mathcal{M})$ for some TM M, and let $s_1, s_2, \ldots$ be an enumeration of Σ^*. Then the following enumerator $\mathcal{E}$ enumerates $\mathcal{L}$:

$\mathcal{E}$:= Ignore the input.

- Repeat for $i = 1, 2, 3, \ldots$
  - Run M for i steps on each input $s_1, s_2, \ldots, s_i$
  - If any computation accepts, print the corresponding s_j followed by #

□

Theorem 2.12

If $\mathcal{L}$ is Turing-recognizable, then there exists an enumerator for $\mathcal{L}$ that prints each word of $\mathcal{L}$ exactly once.

Enumerators

Theorem 2.13

A language $\mathcal{L}$ is decidable if and only if there exists an enumerator for $\mathcal{L}$ that outputs exactly the words of $\mathcal{L}$ in some order of non-decreasing length.

Proof.

Suppose $\mathcal{L}$ to be decidable, and let M be a TM that decides $\mathcal{L}$.

- Define a TM M' that generates, on some scratch tape, all words over Σ in some order of non-decreasing length. (Exercise!)
- For each word w thus generated, simulate M on w_i. If M accepts w, then M' prints w followed by #.

Then M' enumerates exactly the words of $\mathcal{L}$ in some order of non-decreasing length.

Enumerators

Now suppose $\mathcal{L}$ can be enumerated by some TM $\mathcal{E}$ in some order of non-decreasing length.

- If $\mathcal{L}$ is finite, then $\mathcal{L}$ is accepted by a finite automaton.
- If $\mathcal{L}$ is infinite, then we define a decider $\mathcal{M}$ for it as follows.

$\mathcal{M}$:= On input *w*

- Simulate $\mathcal{E}$ until it either outputs *w* or some word longer than *w*
- If $\mathcal{E}$ outputs *w*, then *accept*, else *reject*.

Observation: since $\mathcal{L}$ is infinite, for each $w \in \Sigma^*$ the TM $\mathcal{E}$ will eventually generate *w* or some word longer than *w*. Therefore, $\mathcal{M}$ always halts and thus decides $\mathcal{L}$.

□