Concurrency Theory

Lecture 9: Petri Nets

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Knowledge-Based Systems Group

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Warm-up: Something Useful

For $k \in \mathbb{N}$, $(\mathbb{N}^k, \leq)$ is a **well partial order** (antisymmetric wqo).

Dickson's Lemma

For every infinite sequence $(a_i)_{i\in\mathbb{N}}$ ($a_j \in \mathbb{N}^k$ for each $j \in \mathbb{N}$), there is an infinite increasing subsequence, that is $a_{n_0} \leq a_{n_1} \leq a_{n_2} \leq \ldots$ with $n_0 < n_1 < n_2 < \ldots$.

Proof.

a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	$\cdots$

$$A = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \ldots\}$$

$\min A = a$ and there is an $n_0 \geq 0$, such that $a_{n_0} = a$

$\square$

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Proof.

$\dots$	a_{n_0-2}	a_{n_0-1}	a_{n_0}	a_{n_0+1}	a_{n_0+2}	a_{n_0+3}	a_{n_0+4}	a_{n_0+5}	a_{n_0+6}	a_{n_0+7}	$\dots$

$$B = \{a_{n_0+1}, a_{n_0+2}, a_{n_0+3}, a_{n_0+4}, a_{n_0+5}, a_{n_0+6}, \dots\}$$

$\min B = b$ and there is an $n_1 > n_0$, such that $a_{n_1} = b$

a_{n_0} and n_0

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$$C = \{a_{n_1+1}, a_{n_1+2}, a_{n_1+3}, a_{n_1+4}, a_{n_1+5}, a_{n_1+6}, \ldots\}$$

$\min C = c$ and there is an $n_2 \geq n_1$, such that $a_{n_2} = c$

$a_{n_0} \leq a_{n_1} \leq a_{n_2} \leq \ldots$ and $n_0 < n_1 < n_2 < \ldots$

$\square$

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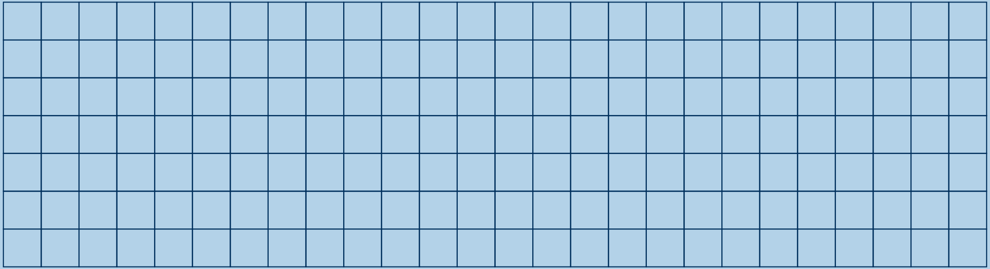

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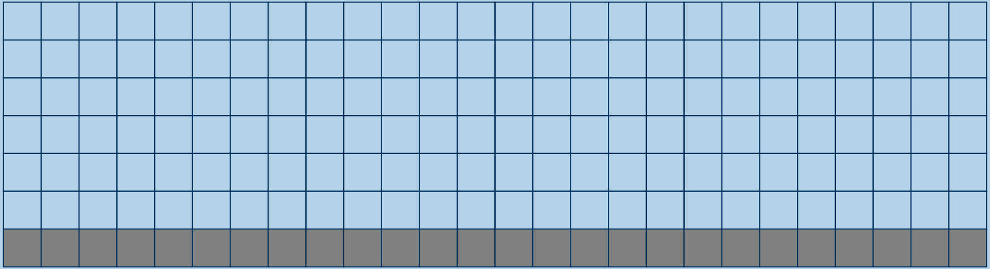

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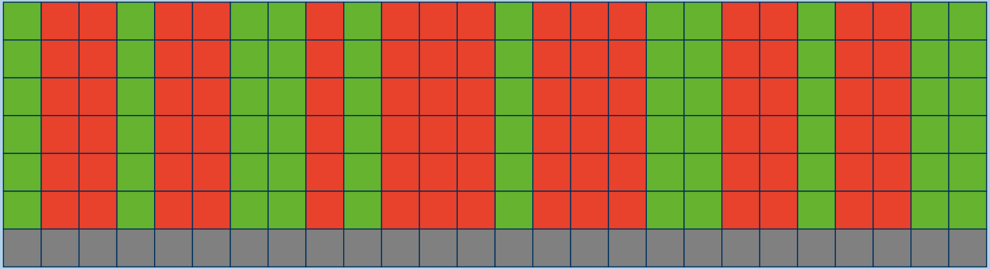

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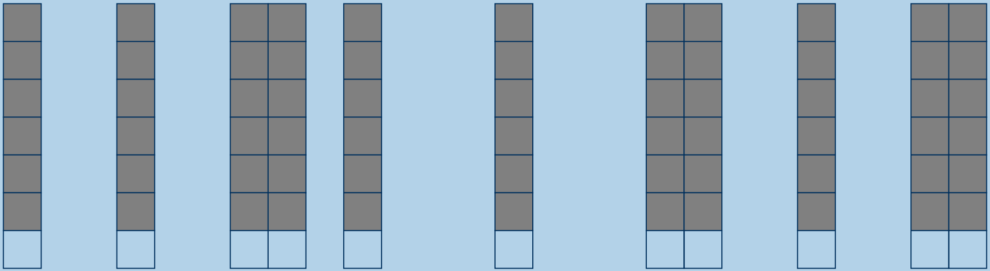

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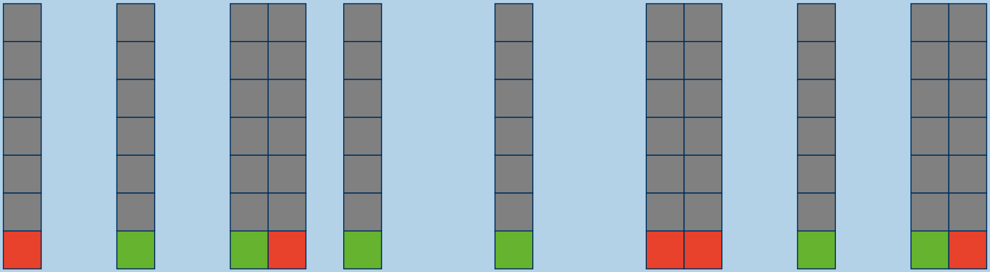

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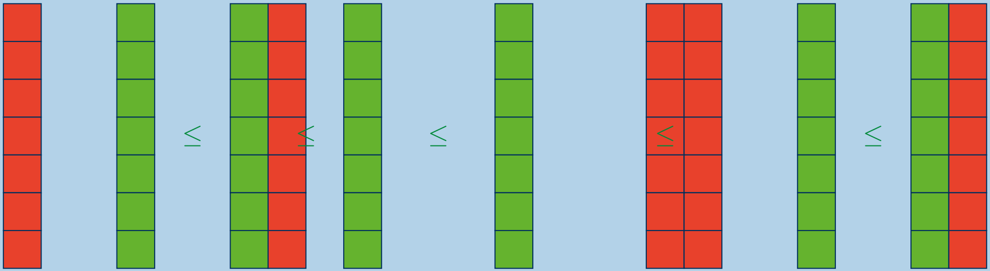


Disclaimer

I will break with any conventions you may have heard of …

(e. g., P/T nets or S/T nets, elementary net systems, net systems, Petri nets, … will all be called **Petri nets**)

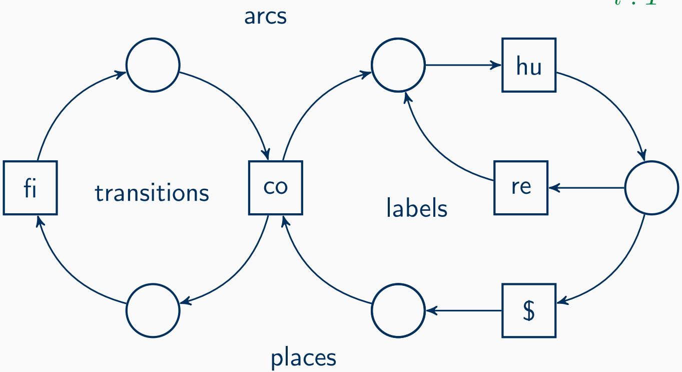
Net Structure

(P, T, F, l)

P, T disjoint and finite sets

$F \subseteq (P \times T) \cup (T \times P)$

$l : T \to \Sigma$ (Σ is an alphabet)



Markings and the Token Game

$N = (P, T, F, l, m_0)$

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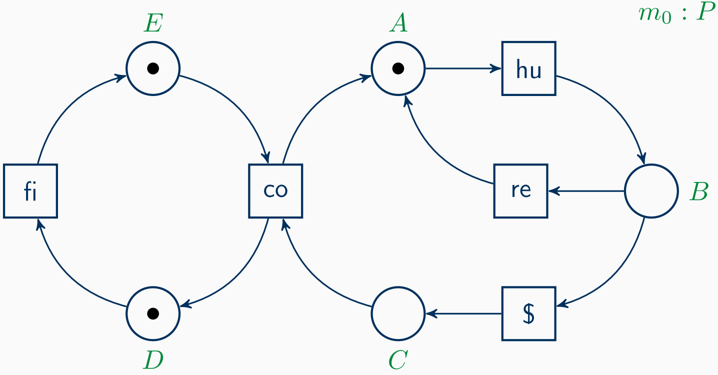
$m_0 : P \to \mathbb{N}$ (multiset)

$$
\begin{array}{lll}
m: & & \\
A & \mapsto & 1 \\
B & \mapsto & 0 \\
C & \mapsto & 0 \\
D & \mapsto & 1 \\
E & \mapsto & 1
\end{array}
$$



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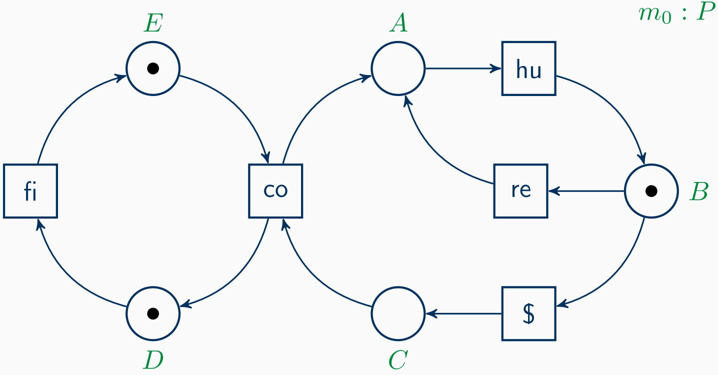
$m:$

$$\begin{array}{lcl} A & \mapsto & 0 \\ B & \mapsto & 1 \\ C & \mapsto & 0 \\ D & \mapsto & 1 \\ E & \mapsto & 1 \end{array}$$



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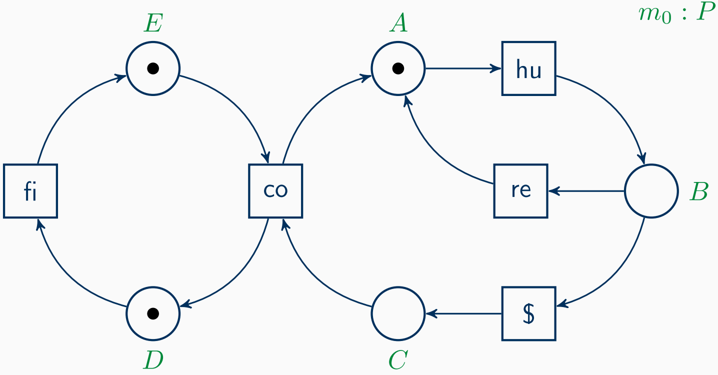

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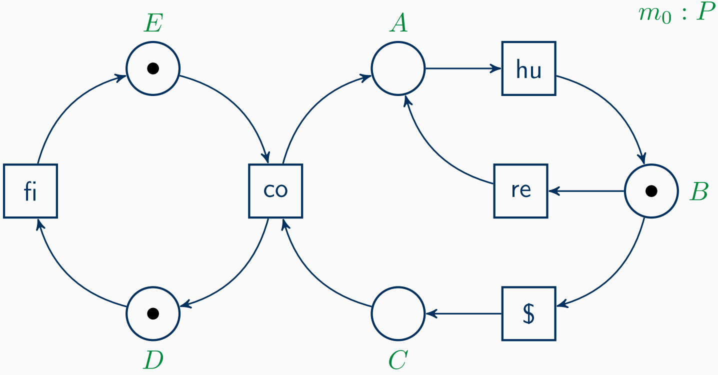

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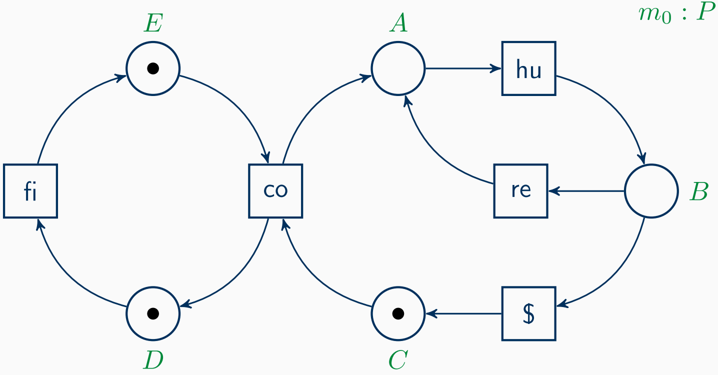

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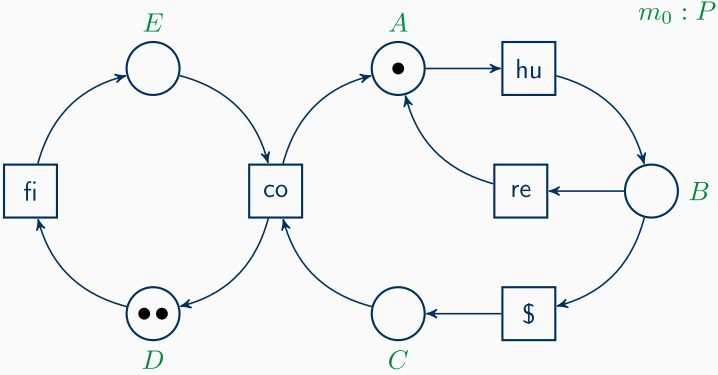
$m_0 : P \to \mathbb{N}$ (multiset)

$$m : \begin{array}{ccc} A & \mapsto & 1 \\ B & \mapsto & 0 \\ C & \mapsto & 0 \\ D & \mapsto & 2 \\ E & \mapsto & 0 \end{array}$$



Definitions and Observations

Definition 4.1 (Net Structure)

Let Σ be an alphabet. A **(Σ-labeled) net structure** is a quadruple (P, T, F, l) with disjoint finite sets P of places and T of transitions, $F \subseteq (P \times T) \cup (T \times P)$, and $l : T \to \Sigma$.

For nodes $v \in P \cup T$, ${}^\bullet v := \{u \mid (u, v) \in F\}$ and $v^\bullet := \{w \mid (v, w) \in F\}$.

Definition 4.2 (Marking, Firing Rule)

For (labeled) net structure $N = (P, T, F, l)$, we call a multiset m over P a **marking** of N. A transition $t \in T$ is **enabled under marking** m if ${}^\bullet t \leq m$. An enabled transition t under marking m may fire, producing the successor marking m' such that for all $p \in P$,

$$m(p) := \begin{cases} m(p) - 1 & \text{if } p \in {}^\bullet t \setminus t^\bullet \\ m(p) + 1 & \text{if } p \in t^\bullet \setminus {}^\bullet t \\ m(p) & \text{otherwise.} \end{cases}$$

We also write $m \xrightarrow{t} m'$ or even $m \xrightarrow{l(t)} m'$.

Definitions and Observations

Definition 4.3 (Petri net, reachability graph)

A (Σ-labeled) Petri net is a quintuple $N = (P, T, F, l, m_0)$ where (P, T, F, l) is a labeled net structure and m_0 is a marking for it (**initial marking**).
The set of **reachable markings of** N $[N\rangle$ is defined inductively by (1) $m_0 \in [N\rangle$ and (2) $m \in [N\rangle$ and $m \xrightarrow{t} m'$ implies $m' \in [N\rangle$.
The **reachability graph of** N $\mathcal{R}(N)$ is induced by the set of reachable markings $[N\rangle$ as the set of nodes and $(\xrightarrow{t})_{t \in T}$ forming the edge relation.

We sometimes needs $[N, m\rangle$ for arbitrary markings m of N to be the set of reachable markings of N where m_0 is replaced by m. **Special case:** $[N, m_0\rangle = [N\rangle$.

The Boundedness Problem

Given a Petri net $N = (P, T, F, l, m_0)$, is $[N\rangle$ finite?

Definition 4.4 (Bounded Petri net)

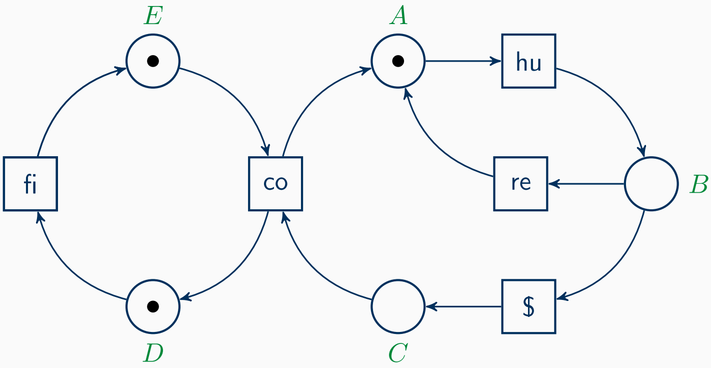
Let $k \in \mathbb{N}$. A Petri net $N = (P, T, F, l, m_0)$ is k-**bounded** if for all $m \in [N\rangle$ and all places $p \in P$, $m(p) \leq k$. N is **bounded** if there is a k, such that N is k-bounded. If no such k exists, N is **unbounded**.

Lemma 4.5

The following statements are equivalent for Petri nets $N = (P, T, F, l, m_0)$*:*

1. $[N\rangle$ *is infinite.*
2. N *is unbounded.*
3. *There are markings* m_1, m_2 *of* N*, such that*
   (a) $m_1 \in [N\rangle$*, (b)* $m_2 \in [N, m_1\rangle$*, (c)* $m_1 \leq m_2$*, and (d)* $m_1(p) < m_2(p)$ *for some* $p \in P$*.*

Bounded and Unbounded Nets



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3. *There are markings m_1, m_2 of N, such that*
   (a) $m_1 \in [N\rangle$, (b) $m_2 \in [N, m_1\rangle$, (c) $m_1 \leq m_2$, and (d) $m_1(p) < m_2(p)$ for some $p \in P$.

From 3 to 2

For Petri net $N = (P, T, F, l, m_0)$, let m_1, m_2 be markings, such that
(a) $m_1 \in [N\rangle$, (b) $m_2 \in [N, m_1\rangle$, (c) $m_1 \leq m_2$, and (d) $m_1(p) < m_2(p)$ for some $p \in P$.

$m_2 = m_1 + s$ for some non-empty marking s! In particular, $s(p) > 0$

$$m_0 \longrightarrow \cdots \longrightarrow m_1 \underbrace{\longrightarrow \cdots \longrightarrow}_{\sigma} m_2$$

Lemma 4.6 (Monotonicity)

For Petri net $N = (P, T, F, l, m_0)$, $t \in T$, and markings m, m', s of N, $m \xrightarrow{t} m'$ implies $m + s \xrightarrow{t} m' + s$.

For every $k \in \mathbb{N}$, repeat transition sequence σ $k + 1$ times, reaching a marking m^k with $m^k(p) > k$.

Thus, N is unbounded.

From 1 to 3

Let $N = (P, T, F, l, m_0)$ be a Petri net, such that $[N\rangle$ is infinite.

1. As $[N\rangle$ is infinite, $\mathcal{R}(G)$ is infinite.
2. For every $m \in [N\rangle$, the number of successors of m in $\mathcal{R}(G)$ is bounded by $|T|$.
3. Hence, there is an infinite simple path $m_0 \to m_1 \to m_2 \to \ldots$ (by König's Lemma)
4. $m_0 m_1 m_2 \ldots$ is an infinite sequence of markings or, equivalently vectors from $\mathbb{N}^{|P|}$.
5. Due to Dickson's Lemma, there is an infinite chain $n_0 < n_1 < n_2 < \ldots$ of indices, such that $m_{n_0} \leq m_{n_1} \leq m_{n_2} \leq \ldots$.
6. Set $m_1 = m_{n_0}$ and $m_2 = m_{n_1}$.
7. By construction (a) $m_1 \in [N\rangle$, (b) $m_2 \in [N, m_1\rangle$, and (c) $m_1 \leq m_2$.
8. As m_1 and m_2 stem from a simple path, there is at least one place $p \in P$ with $m_2(p) > m_1(p)$.

Theorem: Boundedness is Decidable

Start constructing $\mathcal{R}(N)$ by BFS:

- either the construction terminates (**bounded**), or
- a marking m_2 is constructed with a respective marking $m_1 \leq m_2$ earlier on a path from m_0, such that $m_1(p) < m_2(p)$ for some $p \in P$ (**unbounded**).

Many more decidable problems:

- Reachability
- Coverability
- Deadlock-freedom
- Liveness
- Language inclusion/equivalence (?)
- Bisimilarity (?)

Yes to both (?), but not for labeled Petri nets!

The Equivalence Problem(s)

The **(prefix) language** $\mathcal{L}(N)$ **of a labeled Petri net** $N = (P, T, F, l, m_0)$ is the set of all words $w \in \Sigma^*$, such that $w = \varepsilon$ or $m_0 \xrightarrow{t_1} \xrightarrow{t_2} \cdots \xrightarrow{t_{|w|}}$ such that $l^*(t_1 t_2 \ldots t_{|w|}) = w$.

Two Petri nets N_1, N_2 are **language equivalent** if $\mathcal{L}(N_1) = \mathcal{L}(N_2)$.

Theorem 4.1: Language equivalence is undecidable for labeled Petri nets.

We reduce from the halting problem of Minsky machines with two counters.

Petri nets are not Turing-complete!
$\rightsquigarrow$ weak simulation of Turing machines/Minsky machines

Minsky Machines

A **Minsky machine** is a pair $\langle P, \{c_1, c_2, \ldots, c_k\}\rangle$, where $c_1, \ldots, c_k$ are counters and P is a finite sequence of commands $l_1 l_2 \ldots l_n$, such that $l_n =$ **HALT** and l_i $(i = 1, \ldots, n-1)$ is

1. i: c_j`:=`c_j`+1; goto` k, or
2. i: `if` c_j`=0 then goto` k_1 `else` c_j`:=`c_j`-1; goto` k_2

Example 4.7

We consider two counter c_1 and c_2.

1: `if` c_2`=0 then goto 3 else` c_2`:=`c_2`-1; goto 2`

2: c_1`:=`c_1`+1; goto 1`

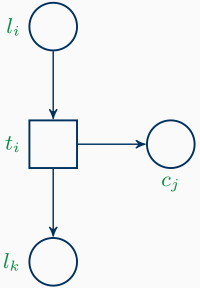
3: `HALT`

If c_1 and c_2 are initialized with m and n, then the program halts with value $m + n$ in c_1.

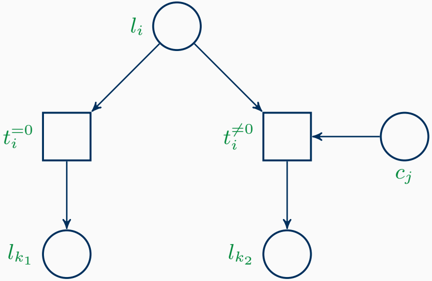
Constructing a Petri Net

For Minsky machine $\mathcal{M} = \langle l_1 l_2 \dots l_m, \{c_1, \dots, c_n\}\rangle$,
$N(\mathcal{M}) = (\{l_1, \dots, l_m, c_1, \dots, c_n\}, T, F, l, m_0)$ where for each $i \in \{1, \dots, m-1\}$:

$l_i = i$: c_j := c_j+1; goto l_k:



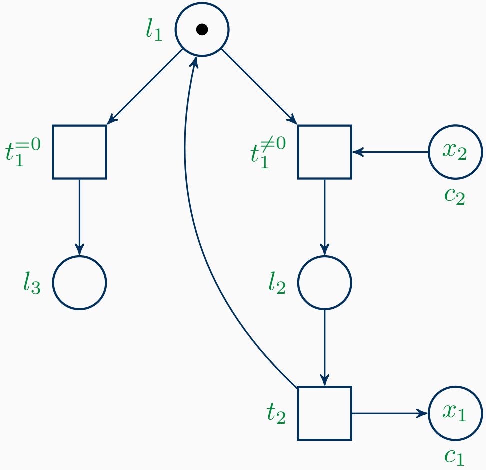
$l_i = i$: if c_j=0 then goto k_1 else c_j := c_j-1; goto k_2:



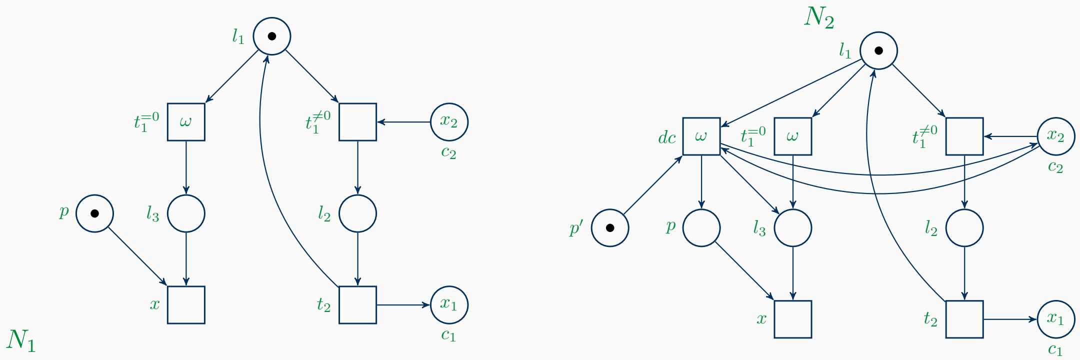
The labeling can be arbitrary but injective.

For input $x_1, \dots, x_n \in \mathbb{N}$, define $m_0 = \{c_1 \mapsto x_1, \dots, c_n \mapsto x_n, l_1 \mapsto 1\}$.

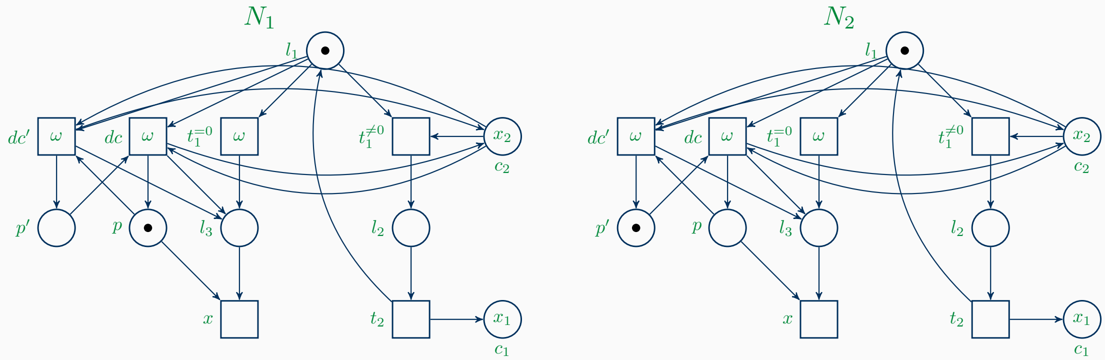
Petri Net Construction by Example



Undecidability of Language Equivalence: The Reduction



Undecidability of Bisimilarity: The Reduction



The Coverability Graph

Definition 4.8 (ω-marking)

For a net (P, T, F), $m : P \to \mathbb{N} \cup \{\omega\}$ is called an ω-marking.

Note, $\omega > n$ and $\omega + / - n = \omega$ for all $n \in \mathbb{N}$.

For directed graph $G = (V, E)$ and $v \in V$, defined $v \Downarrow$ to be the smallest set, such that (1) $v \in v \Downarrow$ and (2) if $w \in v \Downarrow$ and $u \to w$, then $u \in v \Downarrow$.

Definition 4.9

Let $N = (P, T, F, m_0, l)$ be a (labeled) Petri net. The coverability graph (of N) is the graph $\mathcal{C}(N) = (V, E)$, such that

1. $m_0 \in V$;
2. if $m \in V$ and $m \xrightarrow{t} m'$, then $\omega(m') \in V$ and $(m, \omega(m')) \in E$ such that for all $p \in P$,

$$\omega(m')(p) = \begin{cases} \omega & \text{if } m'' \in m \Downarrow \text{ with } m''(p) < m'(p) \\ m'(p) & \text{otherwise.} \end{cases}$$

Properties of the Coverability Graph

Theorem 4.2: The coverability graph $\mathcal{C}(N)$ of a Petri net N is finite.

$\rightsquigarrow$ follows the same argument as for the decidability proof of the boundedness problem.

Properties of the Coverability Graph

Theorem 4.3: The coverability problem — given a Petri net N and a marking m, is there a reachable marking m', such that $m \leq m'$? — is decidable.

1. Construct $C(N)$
2. Check if there is an ω-marking m^ω with $m \leq m^\omega$
3. Consider the path $m_0 \xrightarrow{t_1} \ldots \xrightarrow{t_n} m^\omega$ and the marking m' reached after firing the sequence $t_1 \ldots t_n$
4. If $m \leq m'$, witness found.
5. If $m \not\leq m'$, then there is at least one ω in m^ω and there are markings on the path from m_0 to m^ω that led to the addition of ω
6. Repeat the respective firing sequences until a covering marking is reached.
7. Hence, it is sufficient to check only m^ω.
8. If m is not coverable, then there is no marking m' in the coverability graph with $m \leq m'$.

Equivalence of Unlabeled Nets

Theorem 4.4: Bisimilarity and language equivalence of Petri nets is decidable for unlabeled Petri nets.

- Given are $N_1 = (P_1, T_1, F_1, m_0^1, l_1)$ and $N_2 = (P_2, T_2, F_2, m_0^2, l_2)$ ($P_1 \cap P_2 = \emptyset = T_1 \cap T_2$).
- Construct $N_1 + N_2 = (P_1 \cup P_2, T_1 \cup T_2, F_1 \cup F_2, m_0^1 + m_0^2)$.
- Each transition $t \in T_1 \cup T_2$ is duplicated to t' with the same in-/outputs and label as t.
- Add a fresh place p and add $\{p\} \times (T_1 \cup T_2)$ and $\{t' \mid t' \text{ is a duplicate}\} \times \{p\}$ to the arc relation.
- For each label $a \in \Sigma$, add places p_1^a, p_2^a and for $t \in T_i$ with $l_i(t) = a$, add arcs $(t, p_j^a), (p_j^a, u')$ for transition duplicate u' with $l_j(u) = a$.
- If the nets are language equivalent, then every transition firing of $t \in T_i$ can be reproduced in N_j by u', such that $l_i(t) = l_j(u)$.
- If the nets are not language equivalent, then there is a shortest word w of $L(N_i) \setminus L(N_j)$. After firing the last transition of w in N_i, no duplicate can be fired in N_j.
- Unlabledness is important to not leave N_j the chance to use more clever a-labeled transitions.