Happy Ever After: Temporally Attributed Description Logics
Extended version with appendix

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Abstract. Knowledge graphs are based on graph models enriched with (sets of) attribute-value pairs, called annotations, attached to vertices and edges. Many application scenarios of knowledge graphs crucially rely on the frequent use of annotations related to *time*. Based on recently proposed attributed logics, we design description logics enriched with temporal annotations whose values are interpreted over discrete time. Investigating the complexity of reasoning in this new formalism, it turns out that reasoning in our temporally attributed description logic $\mathcal{ALCH}^{\mathbb{T}}_{@}$ is highly undecidable; thus we establish restrictions where it becomes decidable, and even tractable.

1 Introduction

Graph-based data formats play an essential role in modern information management, since they offer schematic flexibility, ease information re-use, and simplify data integration. Ontological knowledge representation has been shown to offer many benefits to such data-intensive applications, e.g., by supporting integration, querying, error detection, or repair. However, practical *knowledge graphs*, such as Wikidata [25] or YAGO2 [13], are based on *enriched* graphs where edges are augmented with additional annotations. To model these enriched graphs, *attributed logics* have been proposed as a way of integrating annotations with logical reasoning [21, 14, 15]. Other formalisms for reasoning over annotated relations have been studied in the context of data modelling [6].

Annotations in practical knowledge graphs have many purposes, such as recording provenance, specifying context, or encoding *n*-ary relations. One of their most important uses, however, is to encode *temporal validity* of statements. In Wikidata, e.g., *start/end time* and *point in time* are among the most frequent annotations, used in 5.4 million statements overall.[1] In YAGO2, time and space are the main types of annotations considered.

Reasoning with time clearly requires an adequate semantics, and many approaches were proposed. Validity time points and intervals are a classical topic in data management [9, 10], and similar models of time have also been studied in ontologies [2, 17]. However, researchers in ontologies have most commonly focussed on abstract models of time as

[1] As of October 2018, the only more common annotations are *reference* (provenance) and *determination method* (context); see `https://tools.wmflabs.org/sqid/#/browse?type=properties&sortpropertyqualifiers=fa-sort-desc`

used in temporal logics [20, 26, 5]. Temporal reasoning in $\mathcal{ALC}$ with concrete domains was proposed by Lutz et. al [19]. It is known that satisfiability of $\mathcal{ALC}$ with a concrete domain consisting of a dense domain and containing the predicates = and < is ExpTime-complete [18]. In the same setting but for *discrete time*, the complexity of the satisfiability problem is open, a criterion which only guarantees decidability has been proposed by Carapelle and Turhan [7]. None of these approaches has been considered for attributed logics yet, and indeed support for temporal reasoning for knowledge graphs, such as Wikidata and YAGO2, is still missing today. In this paper, we address this shortcoming by endowing attributed description logics with a temporal semantics for annotations. Indeed, annotations are already well-suited for representing time-related data.

Example 1. The fact that Johannes Gutenberg died in Mainz in 1468 could be encoded in attributed DLs as:

$$\mathsf{diedIn}(\mathsf{Gutenberg}, \mathsf{Mainz})\,@\,[\mathsf{time}\colon 1468]$$

Not all annotations are temporal, and we can also annotate concept assertions, e.g., to state that he lived in Strasbourg:

$$\mathsf{Lived}(\mathsf{Gutenberg})\,@\,[\mathsf{loc}\colon \mathsf{Strasbourg}]$$

Gutenberg's early life is less certain, and we only know that he was born between 1394 and 1404 in Mainz. Such uncertainty about precise dates is very common in practice. Nevertheless, we would like to record the information available, which could be expressed as follows:

$$\mathsf{bornIn}(\mathsf{Gutenberg}, \mathsf{Mainz})\,@\,[\mathsf{between} : [1394, 1404]]$$

To deal with such temporally annotated data in a semantically adequate way and to specify temporal background knowledge, we propose the temporally attributed description logic $\mathcal{ALCH}^{\mathbb{T}}_{@}$, enabling reasoning and querying support for such information. Beyond defining syntax and semantics of $\mathcal{ALCH}^{\mathbb{T}}_{@}$, this paper's contributions are the following:

- We show that the full formalism is highly undecidable using an encoding of a recurring tiling problem.
- We present three ways (of increasing reasoning complexity) for regaining decidability: disallowing variables altogether (ExpTime), disallowing the use of variables only for temporal attributes (2ExpTime), or disallowing the use of temporal attributes referencing time points in the future (3ExpTime).
- Finally we single out a lightweight case based on the description logic $\mathcal{EL}$ which features PTime reasoning.

This work corresponds to the eponymous workshop publication [22], extended with an additional appendix that contains details on some long proofs.

2 Temporally Attributed DLs

We first present the syntax and underlying intuition of temporally attributed description logics. In DL, a true fact corresponds to the membership of an element in a class, or of a pair of elements in a binary relation. Attributed DLs further allow each true fact to carry a finite set of annotations [14] , given as attribute-value pairs. As suggested in Example 1, the same relationship may be true with several different annotation sets, e.g., in case Gutenberg also lived elsewhere.

We define our description logic $\mathcal{ALCH}_{@}^{\mathbb{T}}$ as a multi-sorted version of the attributed DL $\mathcal{ALCH}_{@}$, thereby introducing datatypes for time points and intervals. Elements of the different types are represented by members of mutually disjoint sets of *(abstract) individual names* $\mathsf{N_I}$, *time points* $\mathsf{N_T}$, and *time intervals* $\mathsf{N_T^2}$. We represent time points by natural numbers, and assume that elements of $\mathsf{N_T}$ ($\mathsf{N_T^2}$) are (pairs of) numbers in *binary* encoding. We write $[k, \ell]$ for a pair of numbers k, ℓ in $\mathsf{N_T^2}$. Moreover, we require that there are the following seven special individual names, called *temporal attributes*: $\mathsf{time}, \mathsf{before}, \mathsf{after}, \mathsf{until}, \mathsf{since}, \mathsf{during}, \mathsf{between} \in \mathsf{N_I}$.

The intuitive meaning of temporal attributes is as one might expect: time describes individual times at which a statement is true, while the others describe (half-open) intervals. The meaning of before, after, and between is existential in that they require the statement to hold only at some time in the interval, while until, since, and during are universal and require something to be true throughout an interval.

Axioms of $\mathcal{ALCH}_{@}^{\mathbb{T}}$ are further based on sets of *concept names* $\mathsf{N_C}$, *role names* $\mathsf{N_R}$, and *(set) variables* $\mathsf{N_V}$. Attributes are represented by individual names, and we associate a *value type* $\mathsf{vt}(a)$ with each individual $a \in \mathsf{N_I}$ for this purpose: during and between have value type $\mathsf{N_T^2}$, all other temporal attributes have value type $\mathsf{N_T}$, and all other individuals have value type $\mathsf{N_I}$. An *attribute-value pair* is an expression $a\colon v$ where $a \in \mathsf{N_I}$ and $v \in \mathsf{vt}(a)$. Now, concept and role assertions of $\mathcal{ALCH}_{@}^{\mathbb{T}}$ have the following form:

$$C(a)@[a_1\colon v_1, \ldots, a_n\colon v_n] \qquad r(a, b)@[a_1\colon v_1, \ldots, a_n\colon v_n]$$

where $C \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, $a, b \in \mathsf{N_I}$, and $a_i\colon v_i$ are attribute-value pairs.

Role and concept inclusion axioms of $\mathcal{ALCH}_{@}^{\mathbb{T}}$ introduce additional expressive power to refer to partially specified and variable annotation sets. An *(annotation set) specifier* can be a set variable $X \in \mathsf{N_V}$, a *closed specifier* $[a_1\colon v_1, \ldots, a_n\colon v_n]$, or an *open specifier* $\lfloor a_1\colon v_1, \ldots, a_n\colon v_n\rfloor$, where $a_i \in \mathsf{N_I}$ and either $v_i \in \mathsf{vt}(a_i)$ or $v_i = X.b$ with $X \in \mathsf{N_V}$, $b \in \mathsf{N_I}$, and $\mathsf{vt}(a_i) = \mathsf{vt}(b)$. Intuitively speaking, closed specifiers define specific annotation sets whereas open specifiers merely provide lower bounds. The notation $X.b$ is used to copy all of the zero or more b-values of annotation set X to a new annotation set. The set of all specifiers is denoted $\mathbf{S}$. A specifier is *ground* if it does not contain variables. $\mathcal{ALCH}_{@}^{\mathbb{T}}$ *role expressions* have the form $r@S$ with $r \in \mathsf{N_R}$ and $S \in \mathbf{S}$. $\mathcal{ALCH}_{@}^{\mathbb{T}}$ *concept expressions* C, D are defined recursively:

$$C, D ::= \top \mid A@S \mid \neg C \mid (C \sqcap D) \mid \exists R.C \tag{1}$$

with $A \in \mathsf{N_C}$, $S \in \mathbf{S}$ and R an $\mathcal{ALCH}_{@}^{\mathbb{T}}$ role expression. We use abbreviations $(C \sqcup D)$, $\bot$, and $\forall R.C$ for $\neg(\neg C \sqcap \neg D)$, $\neg\top$, and $\neg(\exists R.\neg C)$, respectively.

$\mathcal{ALCH}_{@}^{\mathbb{T}}$ axioms are essentially just DL inclusions between $\mathcal{ALCH}_{@}^{\mathbb{T}}$ role and concept expressions, which may, however, share variables.

Example 2. In an ontology containing biographical information, we might want to make sure that children cannot be born before their parents. This can be expressed by the axiom

$$\exists \mathsf{bornIn}@X.\top \sqsubseteq \neg\exists \mathsf{hasChild}@\lfloor\,\rfloor.\exists \mathsf{bornIn}@\lfloor \mathsf{before}\colon X.\mathsf{time}\rfloor.\top.$$

Similar axioms can be used, e.g., to state that nobody has more than one birthday ("is born before being born").

It is sometimes useful to represent annotations by variables while also specifying some further constraints on their possible values. This can be accommodated by adding such constraints as (optional) prefixes to axioms. Hence we define an $\mathcal{ALCH}^{\mathbb{T}}_{@}$ *concept inclusion* as an expression of the form

$$X_1 : S_1, \ldots, X_n : S_n \quad (C \sqsubseteq D), \tag{2}$$

where C, D are $\mathcal{ALCH}^{\mathbb{T}}_{@}$ concept expressions, $S_1, \ldots, S_n \in \mathbf{S}$ are closed or open specifiers, and $X_1, \ldots, X_n \in \mathsf{N_V}$ are set variables occurring in C, D or in $S_1, \ldots, S_n$. $\mathcal{ALCH}^{\mathbb{T}}_{@}$ *role inclusions* are defined analogously, but with role expressions instead of the concept expressions. An $\mathcal{ALCH}^{\mathbb{T}}_{@}$ *ontology* is a set of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ assertions, and role and concept inclusions. To simplify notation, we sometimes omit the specifier $\lfloor\,\rfloor$ (meaning "any annotation set") in role or concept expressions. In this sense, any $\mathcal{ALCH}$ axiom is also an $\mathcal{ALCH}^{\mathbb{T}}_{@}$ axiom.

3 Formal Semantics

We first recall the general semantics of attributed DLs without temporal attributes. The semantics of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ can then be obtained as a multi-sorted extension that imposes additional restrictions on the interpretation of time.

An *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of attributed logic consists of a non-empty domain $\Delta^{\mathcal{I}}$ and a function $\cdot^{\mathcal{I}}$. Individual names $a \in \mathsf{N_I}$ are interpreted as elements $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. To interpret annotation sets, we use the set $\Phi^{\mathcal{I}} := \mathcal{P}_{\mathsf{fin}}\left(\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}\right)$ of all finite binary relations over $\Delta^{\mathcal{I}}$. Now each concept name $C \in \mathsf{N_C}$ is interpreted as a set $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ of elements with annotations, and each role name $r \in \mathsf{N_R}$ is interpreted as a set $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \times \Phi^{\mathcal{I}}$ of pairs of elements with annotations. Note that each element (pair of elements) may appear with multiple different annotations. $\mathcal{I}$ *satisfies* a concept assertion $C(a)@[a_1 : v_1, \ldots, a_n : v_n]$ if $(a^{\mathcal{I}}, \{(a_1^{\mathcal{I}}, v_1^{\mathcal{I}}), \ldots, (a_n^{\mathcal{I}}, v_n^{\mathcal{I}})\}) \in C^{\mathcal{I}}$, and likewise for role assertions. Expressions with free set variables are interpreted using variable assignments $\mathcal{Z} : \mathsf{N_V} \to \Phi^{\mathcal{I}}$. A specifier $S \in \mathbf{S}$ is interpreted as a set $S^{\mathcal{I},\mathcal{Z}} \subseteq \Phi^{\mathcal{I}}$ of matching annotation sets. We set $X^{\mathcal{I},\mathcal{Z}} := \{\mathcal{Z}(X)\}$ for variables $X \in \mathsf{N_V}$. The semantics of closed specifiers is defined as follows:

(i) $[a : b]^{\mathcal{I},\mathcal{Z}} := \{\{(a^{\mathcal{I}}, b^{\mathcal{I}})\}\}$
(ii) $[a : X.b]^{\mathcal{I},\mathcal{Z}} := \{\{(a^{\mathcal{I}}, \delta) \mid (b^{\mathcal{I}}, \delta) \in \mathcal{Z}(X)\}\}$
(iii) $[a_1 : v_1, \ldots, a_n : v_n]^{\mathcal{I},\mathcal{Z}} := \{\bigcup_{i=1}^{n} F_i\}$ where $\{F_i\} = [a_i : v_i]^{\mathcal{I},\mathcal{Z}}$ for all $i \in \{1, \ldots, n\}$.

$S^{\mathcal{I},\mathcal{Z}}$ therefore is a singleton set for variables and closed specifiers. For open specifiers, however, we define $\lfloor a_1 : v_1, \ldots, a_n : v_n \rfloor^{\mathcal{I},\mathcal{Z}}$ to be the set

$$\{F \in \Phi^{\mathcal{I}} \mid F \supseteq G \text{ for } \{G\} = [a_1 : v_1, \ldots, a_n : v_n]^{\mathcal{I},\mathcal{Z}}\}.$$

Now given $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and $S \in \mathbf{S}$, we define:

$$(A@S)^{\mathcal{I},\mathcal{Z}} := \{\delta \mid (\delta, F) \in A^{\mathcal{I}} \text{ for some } F \in S^{\mathcal{I},\mathcal{Z}}\},$$
$$(r@S)^{\mathcal{I},\mathcal{Z}} := \{(\delta, \epsilon) \mid (\delta, \epsilon, F) \in r^{\mathcal{I}} \text{ for some } F \in S^{\mathcal{I},\mathcal{Z}}\}.$$

Further DL expressions are defined as usual: $\top^{\mathcal{I},\mathcal{Z}} = \Delta^{\mathcal{I}}$, $\neg C^{\mathcal{I},\mathcal{Z}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I},\mathcal{Z}}$, $(C \sqcap D)^{\mathcal{I},\mathcal{Z}} = C^{\mathcal{I},\mathcal{Z}} \cap D^{\mathcal{I},\mathcal{Z}}$, and $(\exists R.C)^{\mathcal{I},\mathcal{Z}} = \{\delta \mid \text{there is } (\delta,\epsilon) \in R^{\mathcal{I},\mathcal{Z}} \text{ with } \epsilon \in C^{\mathcal{I},\mathcal{Z}}\}$.

$\mathcal{I}$ *satisfies* a concept inclusion of the form (2) if, for all variable assignments $\mathcal{Z}$ that satisfy $\mathcal{Z}(X_i) \in S_i^{\mathcal{I},\mathcal{Z}}$ for all $1 \leq i \leq n$, we have $C^{\mathcal{I},\mathcal{Z}} \subseteq D^{\mathcal{I},\mathcal{Z}}$. Satisfaction of role inclusions is defined analogously. $\mathcal{I}$ satisfies an ontology if it satisfies all of its axioms. As usual, $\models$ denotes both satisfaction and the induced logical entailment relation.

Adding Time Time points $t \in \mathsf{N_T}$ are encodings of natural numbers, which we denote by $t^{\mathcal{I}}$. Analogously, $v^{\mathcal{I}}$ denotes a pair of numbers for a time interval $v \in \mathsf{N_T^2}$. To represent time, we consider intervals of natural numbers, which can be finite intervals $[k,\ell] = \{n \in \mathbb{N} \mid k \leq n \leq \ell\}$ or infinite intervals $[k,\infty) = \{n \in \mathbb{N} \mid k \leq n\}$. A *temporal domain* $\Delta_T^{\mathcal{I}}$ is a finite or infinite interval such that $t^{\mathcal{I}} \in \Delta_T^{\mathcal{I}}$ for all $t \in \mathsf{N_T}$ and $v^{\mathcal{I}} \in \Delta_T^{\mathcal{I}} \times \Delta_T^{\mathcal{I}}$ for all $v \in \mathsf{N_T^2}$. Note that $\Delta_T^{\mathcal{I}}$ can be finite if $\mathsf{N_T}$ and $\mathsf{N_T^2}$ are (which is always admissible, since any ontology mentions only finitely many time points).

A *time-sorted interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ has a sorted domain $\Delta^{\mathcal{I}}$ that is a disjoint union $\Delta_I^{\mathcal{I}} \cup \Delta_T^{\mathcal{I}} \cup \Delta_{2T}^{\mathcal{I}}$, where $\Delta_I^{\mathcal{I}}$ is the *abstract domain*, $\Delta_T^{\mathcal{I}}$ is a temporal domain, and $\Delta_{2T}^{\mathcal{I}} = \Delta_T^{\mathcal{I}} \times \Delta_T^{\mathcal{I}}$. We interpret individual names $a \in \mathsf{N_I}$ as elements $a^{\mathcal{I}} \in \Delta_I^{\mathcal{I}}$. A pair $(\delta,\epsilon) \in \Delta_I^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ is *well-typed*, if one of the following holds:

(i) $\delta = a^{\mathcal{I}}$ for a temporal attribute a of value type $\mathsf{N_T}$ and $\epsilon \in \Delta_T^{\mathcal{I}}$;
(ii) $\delta = a^{\mathcal{I}}$ for a temporal attribute a of value type $\mathsf{N_T^2}$ and $\epsilon \in \Delta_{2T}^{\mathcal{I}}$; or
(iii) $\delta \neq a^{\mathcal{I}}$ for all temporal attributes a and $\epsilon \in \Delta_I^{\mathcal{I}}$.

Then $\Phi^{\mathcal{I}}$ is the set of all finite sets of well-typed pairs. The remainder of the interpretation function is defined as in the unsorted case, based on this sorted definition of $\Phi^{\mathcal{I}}$.

Time-sorted interpretations can be used to interpret $\mathcal{ALCH}_{@}^{\mathbb{T}}$ ontologies, but they do not take the intended semantics of time into account yet. For example, we might find that $A(c)@[\mathsf{after}\colon 1993]$ holds whereas $A(c)@[\mathsf{time}\colon t]$ does not hold for any time $t \in \mathsf{N_T}$ with $t^{\mathcal{I}} > 1993$. To ensure consistency, we would like to view an interpretation with temporal domain $\Delta_T^{\mathcal{I}}$ as a sequence $(\mathcal{I}_i)_{i \in \Delta_T^{\mathcal{I}}}$ of regular (unsorted) interpretations that define the state of the world at each point in time. Such a sequence represents a *local* view of time as a sequence of events, whereas the time-sorted interpretation represents a *global* view that can explicitly refer to time points. Axioms of $\mathcal{ALCH}_{@}^{\mathbb{T}}$ refer to this global view, but it should be based on an actual sequence of events. To simplify the relationship between local and global views, we assume that the underlying abstract domain $\Delta_I^{\mathcal{I}}$ and interpretation of constants remains the same over time.

Definition 1. *Consider a temporal domain $\Delta_T^{\mathcal{I}}$ and an abstract domain $\Delta_I^{\mathcal{I}}$, and let $(\mathcal{I}_i)_{i \in \Delta_T^{\mathcal{I}}}$ be a sequence of $\mathcal{ALCH}_{@}$ interpretations with domain $\Delta_I^{\mathcal{I}}$, such that, for all $a \in \mathsf{N_I}$, we have $a^{\mathcal{I}_i} = a^{\mathcal{I}_j}$ for all $i, j \in \Delta_T^{\mathcal{I}}$.*

We define a global interpretation *for $(\mathcal{I}_i)_{i \in \Delta_T^{\mathcal{I}}}$ as a multisorted interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ as follows. Let $a^{\mathcal{I}} = a^{\mathcal{I}_i}$ for all $a \in \mathsf{N_I}$. For any finite set $F \in \Phi^{\mathcal{I}}$, let $F_I \coloneqq F \cap (\Delta_I^{\mathcal{I}} \times \Delta_I^{\mathcal{I}})$ denote its abstract part without any temporal attributes. For any $A \in \mathsf{N_C}$, $\delta \in \Delta^{\mathcal{I}}$, and $F \in \Phi^{\mathcal{I}}$ with $F \setminus F_I \neq \emptyset$, we have $(\delta, F) \in A^{\mathcal{I}}$ if and only if*[2] *$(\delta, F_I) \in A^{\mathcal{I}_i}$ for some $i \in \Delta_T^{\mathcal{I}}$, and the following conditions hold for all $(a^{\mathcal{I}}, x) \in F$:*

[2] 'for some $i \in \Delta_T^{\mathcal{I}}$' is useful for attributes which universally quantify time points (e.g., until).

- *if* $a = \text{time}$*, then* $(\delta, F_I) \in A^{\mathcal{I}_x}$*,*
- *if* $a = \text{before}$*, then* $(\delta, F_I) \in A^{\mathcal{I}_j}$ *for some* $j < x$*,*
- *if* $a = \text{after}$*, then* $(\delta, F_I) \in A^{\mathcal{I}_j}$ *for some* $j > x$*,*
- *if* $a = \text{until}$*, then* $(\delta, F_I) \in A^{\mathcal{I}_j}$ *for all* $j \leq x$*,*
- *if* $a = \text{since}$*, then* $(\delta, F_I) \in A^{\mathcal{I}_j}$ *for all* $j \geq x$*,*
- *if* $a = \text{between}$*, then* $(\delta, F_I) \in A^{\mathcal{I}_j}$ *for some* $j \in [x]$*,*
- *if* $a = \text{during}$*, then* $(\delta, F_I) \in A^{\mathcal{I}_j}$ *for all* $j \in [x]$*,*

where $[x]$ *for an element* $x \in \Delta^{\mathcal{I}}_{2T}$ *denotes the finite interval represented by the pair of numbers x, and* $j \in \Delta^{\mathcal{I}}_{T}$*. For roles* $r \in \mathsf{N_R}$*, we define* $(\delta, \epsilon, F) \in r^{\mathcal{I}}$ *analogously.*

In words: in a global interpretation all tuples are consistent with the given sequence of local interpretations. One can see a global interpretation as a snapshot of a local interpretation, with timestamps encoding the information of the temporal sequence. If a global interpretation does not contain temporal attributes the characterization of Definition 1 holds vacuously for any temporal sequence, meaning that without temporal attributes the semantics is essentially the same as for $\mathcal{ALCH}_@$.

Definition 2. *An* interpretation *of* $\mathcal{ALCH}^{\mathbb{T}}_@$ *is a time-sorted interpretation* $\mathcal{I}$ *that is a global interpretation of an interpretation sequence* $(\mathcal{I}_i)_{i \in \Delta^{\mathcal{I}}_T}$ *as in Definition 1.*

A model *of an* $\mathcal{ALCH}^{\mathbb{T}}_@$ *ontology* $\mathcal{O}$ *is an* $\mathcal{ALCH}^{\mathbb{T}}_@$ *interpretation that satisfies* $\mathcal{O}$*, and* $\mathcal{O}$ entails *an axiom* α*, written* $\mathcal{O} \models \alpha$*, if* α *is satisfied by all models of* $\mathcal{O}$*.*

By virtue of the syntax and semantics of $\mathcal{ALCH}^{\mathbb{T}}_@$ we can express background knowledge that helps to maintain integrity of the annotated knowledge and allows us to derive new information from it.

Example 3. Along the lines of Example 2, we can state, e.g., that people cannot live after their death:

$$\mathsf{Lived}@X \sqcap \exists \mathsf{diedIn}@\lfloor \mathsf{before}\colon X.\mathsf{time} \rfloor.\top \sqsubseteq \bot \qquad \exists \mathsf{bornIn}@X.\top \sqsubseteq \mathsf{Lived}@X$$

With these background axioms in place, we can infer from the time-annotated facts in Example 1, e.g.,

$$\mathsf{Lived}(\mathsf{Gutenberg})@[\mathsf{between}\colon[1394, 1468]]$$

Some temporal attributes are closely related. Clearly, time can be captured by using during or between with singleton intervals. Conversely, during can be expressed by specifying all time points in the respective interval explicitly using time, but this incurs an exponential blow-up over the binary encoding of time intervals. Similarly, between could be expressed as a disjunction of statements with specific times. Since time can be infinite, since and after cannot be captured using finite intervals. It may seem as if until and before correspond to during and between using intervals starting at 0. However, it is not certain that 0 is the first element in the temporal domain of an interpretation, and the next example shows that this cannot be assumed in general.

Example 4. The ontology with the two axioms $C(a)@[\mathsf{until}\colon 10]$ and $C@[\mathsf{before}\colon 5] \sqsubseteq \bot$ is satisfiable in $\mathcal{ALCH}^{\mathbb{T}}_@$, but it does not have models that have times before 5. Replacing until: 10 with during: [0, 10] would therefore lead to an inconsistent ontology.

4 Reasoning in $\mathcal{ALCH}^{\mathbb{T}}_{@}$

In this section, we study the expressivity and decidability in $\mathcal{ALCH}^{\mathbb{T}}_{@}$. Our first result, Theorem 1, shows that reasoning is on the first level of the analytical hierarchy and therefore highly undecidable.

Theorem 1. *Satisfiability of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies is Σ^1_1-hard, and thus not recursively enumerable. Moreover, the problem is Σ^1_1-hard even with at most one set variable per inclusion and with only the temporal attributes* time *and* after.

Proof. We reduce from the following tiling problem, known to be Σ^1_1-hard [12]: given a finite set of tile types T with horizontal and vertical compatibility relations H and V, respectively, and $t_0 \in T$, decide whether one can tile $\mathbb{N} \times \mathbb{N}$ with t_0 appearing infinitely often in the first row. We define an $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontology $\mathcal{O}_{T,t_0}$ that expresses this property. In our encoding, we use the following symbols:

- a concept name A, to mark individuals representing a grid position with a time point;
- a concept name P to keep time points associated with previous columns in the grid;
- concept names A_t, for each $t \in T$, to mark individuals with tile types;
- an individual name a, to be connected with the first row of the grid;
- an auxiliary concept name I, to mark the individual a, and a concept name B, used to create the vertical axis;
- role names r, s, to connect horizontally and vertically the elements of the grid, respectively.

We define $\mathcal{O}_{T,t_0}$ as the set of the following $\mathcal{ALCH}^{\mathbb{T}}_{@}$ assertion and concept inclusions. We start encoding the first row of the grid with an assertion $I(a)$ and the concept inclusions:

$$I \sqsubseteq \exists r.A@\lfloor \mathsf{time}\colon 0 \rfloor \text{ and } \exists r.A@X \sqsubseteq \exists r.A@\lfloor \mathsf{after}\colon X.\mathsf{time} \rfloor.$$

Every element in A must be marked in at most one time point (in fact, exactly one):

$$A@X \sqsubseteq \neg A@\lfloor \mathsf{after}\colon X.\mathsf{time} \rfloor \tag{3}$$

Every element representing a grid position can be associated with exactly one tile type at the same time point:

$$A@X \sqsubseteq \bigsqcup_{t \in T} A_t@\lfloor \mathsf{time}\colon X.\mathsf{time} \rfloor,$$
$$\exists r.A_t@X \sqsubseteq \neg \exists r.A_{t'}@\lfloor \mathsf{time}\colon X.\mathsf{time} \rfloor, \text{ for } t \neq t' \in T.$$

We also have:

$$A_t@X \sqsubseteq A@\lfloor \mathsf{time}\colon X.\mathsf{time} \rfloor, \text{ for each } t \in T$$

to ensure that elements are in A_t and A at the same time point (exactly one one, see Eq. 3). The condition that t_0 appears infinitely often in the first row is expressed with:

$$I \sqsubseteq \exists r.(A_{t_0}@\lfloor \mathsf{time}\colon 0 \rfloor \sqcup A_{t_0}@\lfloor \mathsf{after}\colon 0 \rfloor),$$
$$I \sqcap \exists r.A_{t_0}@X \sqsubseteq \exists r.A_{t_0}@\lfloor \mathsf{after}\colon X.\mathsf{time} \rfloor.$$

To vertically connect subsequent rows of the grid, we have:

$$I \sqsubseteq B \text{ and } B \sqsubseteq \exists s.B.$$

We add, for each $t \in T$, the following inclusion to ensure compatibility between vertically adjacent tile types:

$$\exists r.A_t@X \sqsubseteq \forall s.\exists r.(\bigsqcup_{(t,t')\in V} A_{t'}@\lfloor \mathsf{time}\colon X.\mathsf{time}\rfloor)$$

We also have:

$$\exists s.\exists r.A@X \sqsubseteq \exists r.A@\lfloor \mathsf{time}\colon X.\mathsf{time}\rfloor$$

to ensure that the set of time points in each row is the same. We now encode compatibility between horizontally adjacent tile types. We first state that, given a node associated with a time point p, for every sibling node d, if d is associated with a time point after p then we mark d with P and p:

$$\exists r.A@X \sqsubseteq \forall r.(\neg A@\lfloor \mathsf{after}\colon X.\mathsf{time}\rfloor \sqcup P@\lfloor \mathsf{time}\colon X.\mathsf{time}\rfloor).$$

For each node, P keeps the time points associated with previous columns in the grid (finitely many). We also have:

$$\exists r.P@X \sqsubseteq \exists r.A@\lfloor \mathsf{time}\colon X.\mathsf{time}\rfloor \text{ and } P@X \sqsubseteq A@\lfloor \mathsf{after}\colon X.\mathsf{time}\rfloor$$

to ensure that P keeps only those previous time points. Finally, for each $t \in T$, we add to $\mathcal{O}_{T,t_0}$ the inclusion:

$$\exists r.A_t@X \sqsubseteq \forall r.(\neg A@\lfloor \mathsf{after}\colon X.\mathsf{time}\rfloor \sqcup P@\lfloor \mathsf{after}\colon X.\mathsf{time}\rfloor \sqcup \bigsqcup_{(t,t')\in H} A_{t'}).$$

Intuitively, as P keeps the time points associated with previous columns in the grid, only the node representing the horizontally adjacent grid position of a node associated with a time point p will not be marked with P after p.

Theorem 2 shows that even if after is only allowed in assertions reasoning is undecidable, though, in the arithmetical hierarchy [23].

Theorem 2. *Satisfiability of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies with the temporal attributes* time, after *and* before *but* after *only in assertions is Σ^0_1-complete (recall Σ^0_1 stands for* RE*). The problem is Σ^0_1-hard even with at most one set variable per inclusion.*

To recover decidability, we need to restrict $\mathcal{ALCH}^{\mathbb{T}}_{@}$ in some way. A simple approach of doing so is to consider ground $\mathcal{ALCH}^{\mathbb{T}}_{@}$ where we disallow set variables altogether. It is clear from the known complexity of $\mathcal{ALCH}$ that reasoning is ExpTime-hard. We establish a matching membership result by providing a satisfiability-preserving polynomial time translation to $\mathcal{ALCH}$ extended with role conjunctions and disjunctions (denoted $\mathcal{ALCH}b$), where satisfiability is known to be in ExpTime [24].

Theorem 3. *Satisfiability of ground $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies is* ExpTime*-complete.*

Proof. Consider a ground $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontology $\mathcal{O}$, and let $k_0 < \ldots < k_n$ be the ascending sequence of all numbers mentioned (in binary encoding) in time points or in time intervals in $\mathcal{O}$. We define $\mathbb{N}_{\mathcal{O}} := \{k_i \mid 0 \leq i \leq n\} \cup \{k_i + 1 \mid 0 \leq i < n\}$, and let $k_{\mathsf{min}} := \min(\mathbb{N}_{\mathcal{O}})$ and $k_{\mathsf{max}} = \max(\mathbb{N}_{\mathcal{O}})$, where we assume $k_{\mathsf{min}} = k_{\mathsf{max}} = 0$ if $\mathbb{N}_{\mathcal{O}} = \emptyset$. For a finite interval $v \subseteq \mathbb{N}$, let $\mathbb{N}^v_{\mathcal{O}}$ be the set of all finite, non-empty intervals $u \subseteq v$ with end points in $\mathbb{N}_{\mathcal{O}}$. The number of intervals in $\mathbb{N}^v_{\mathcal{O}}$ then is polynomial in the size of $\mathcal{O}$.

We translate $\mathcal{O}$ into an $\mathcal{ALCH}b$ ontology $\mathcal{O}^\dagger$ as follows. First, $\mathcal{O}^\dagger$ contains every axiom from $\mathcal{O}$, with each annotated concept name $A@S$ and each annotated role name $r@S$ replaced by a fresh concept name A_S and a fresh role name r_S, respectively.

Second, given a ground specifier S, we denote by $S(a\colon b)$ the result of removing all temporal attributes from S and adding the pair $a\colon b$. Moreover, let $S_{\mathbb{T}}$ be the set of temporal attribute-value pairs in S. Then, for each A_S and r_S with $S_{\mathbb{T}} \neq \emptyset$, $\mathcal{O}^\dagger$ contains the equivalences (as usual, $\equiv$ refers to bidirectional $\sqsubseteq$ here):

$$A_S \equiv \bigsqcap_{(a:b)\in S_{\mathbb{T}}} (A_{S(a:b)})^\sharp \quad \text{and} \quad r_S \equiv \bigsqcap_{(a:b)\in S_{\mathbb{T}}} (r_{S(a:b)})^\sharp \tag{4}$$

where the concept/role expressions $(H_{S(a:b)})^\sharp$ for $H \in \{A, r\}$ are defined as follows:

- $(H_{S(\mathsf{during}:v)})^\sharp = \bigsqcap_{u\in\mathbb{N}^v_{\mathcal{O}}} H_{S(\mathsf{during}:u)}$
- $(H_{S(\mathsf{between}:v)})^\sharp = \bigsqcup_{k\in(v\cap\mathbb{N}_{\mathcal{O}})} H_{S(\mathsf{during}:[k,k])}$
- $(H_{S(\mathsf{time}:k)})^\sharp = (H_{S(\mathsf{during}:[k,k])})^\sharp$
- $(H_{S(\mathsf{since}:k)})^\sharp = (H_{S(\mathsf{during}:[k,k_{\mathsf{max}}])})^\sharp \sqcap H_{S(\mathsf{since}:k_{\mathsf{max}})}$
- $(H_{S(\mathsf{until}:k)})^\sharp = (H_{S(\mathsf{during}:[k_{\mathsf{min}},k])})^\sharp \sqcap H_{S(\mathsf{until}:k_{\mathsf{min}})}$
- $(H_{S(\mathsf{after}:k)})^\sharp = (H_{S(\mathsf{between}:[k+1,k_{\mathsf{max}}])})^\sharp \sqcup H_{S(\mathsf{after}:k_{\mathsf{max}})}$
- $(H_{S(\mathsf{before}:k)})^\sharp = (H_{S(\mathsf{between}:[k_{\mathsf{min}},k-1])})^\sharp \sqcup H_{S(\mathsf{before}:k_{\mathsf{min}})}$

where $k \neq k_{\mathsf{min}}$ and $k \neq k_{\mathsf{max}}$. If $k \in \{k_{\mathsf{min}}, k_{\mathsf{max}}\}$ then we set $(H_{S(a:k)})^\sharp = H_{S(a:k)}$. Only polynomially many inclusions in the size of $\mathcal{O}$ are introduced by (4) in $\mathcal{O}^\dagger$.

Finally, given attribute-value pairs $a\colon b$ and $c\colon d$ for temporal attributes a and b, we say that $a\colon b$ *implies* $c\colon d$ if $A(e)@[a\colon b] \models A(e)@[c\colon d]$ for some arbitrary $A \in \mathsf{N_C}$ and $e \in \mathsf{N_I}$. Based on a given $\mathsf{N_I}$, this implication relationship is computable in polynomial time. We then extend $\mathcal{O}^\dagger$ with all inclusions $A_S \sqsubseteq A_T$ and $r_S \sqsubseteq r_T$, where A_S, A_T and r_S, r_T are concept and role names occurring in $\mathcal{O}^\dagger$, including those introduced in (4), such that for each temporal attribute-value pair $c\colon d$ in T there is a temporal attribute-value pair $a\colon b$ in S such that $a\colon b$ implies $c\colon d$ and:

- T is an open specifier and the set of non-temporal attribute-value pairs in S is a superset of the set of non-temporal attribute-value pairs in T; or
- S, T are closed specifiers and the set of non-temporal attribute-value pairs in S is equal to the set of non-temporal attribute-value pairs in T.

This finishes the construction of $\mathcal{O}^\dagger$. As shown in the appendix, $\mathcal{O}$ is satisfiable iff $\mathcal{O}^\dagger$ is satisfiable.

While ground $\mathcal{ALCH}^{\mathbb{T}}_{@}$ can already be used for some interesting conclusions, it is still rather limited. However, satisfiability of (non-ground) $\mathcal{ALCH}_{@}$ ontologies is also decidable [14], and indeed we can regain decidability in $\mathcal{ALCH}^{\mathbb{T}}_{@}$ by disallowing expressions of the form $X.a$ to be used with temporal attributes a. Indeed, using a similar reasoning as in the case of $\mathcal{ALCH}_{@}$, we obtain a 2ExpTime upper bound by constructing an equisatisfiable (exponentially larger) ground $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontology.

Theorem 4. *Satisfiability of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies without expressions of the form $X.a$ for temporal attributes a is* 2ExpTime*-complete.*

Another way for regaining decidability is by limiting the temporal attributes that make reference to time points in the future. Using this assumption, we can translate any $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontology into a ground $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontology. In this case, however, the resulting ground ontology is double-exponentially larger if we assume that the size of the temporal domain has been encoded in binary. We therefore obtain a 3ExpTime upper bound (using Theorem 3).

Theorem 5. *Satisfiability of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies with only the temporal attributes* during, time, before *and* until *is in* 3ExpTime.

Our result in our next Theorem 6 below is that this upper bound is tight. The proof is by reduction from the word problem for double-exponentially space-bounded alternating Turing machines (ATMs) [8] to the entailment problem for $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies. The main challenge in this reduction is that we need a mechanism that allows us to transfer the information of a double-exponentially space bounded tape, so that each configuration following a given configuration is actually a successor configuration (i.e., tape cells are changed according to the transition relation). We encode our tape using time: we can have exponentially many time points in an interval with end points encoded in binary. So considering each time point as a bit position, we construct a counter with *exponentially many bits*, encoding the position of double-exponentially many tape cells.

Theorem 6. *Satisfiability of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies with only* time *and* before *is* 3ExpTime*-hard.*

5 Lightweight Temporal Attributed DLs

In this section we investigate $\mathcal{ELH}^{\mathbb{T}}_{@}$, the fragment of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ which uses only $\exists$, $\sqcap$, $\top$ and $\bot$ in concept expressions. Though, even for ground ontologies, the satisfiability problem for $\mathcal{ELH}^{\mathbb{T}}_{@}$ is not tractable.

Theorem 7. *Satisfiability of ground $\mathcal{ELH}^{\mathbb{T}}_{@}$ ontologies is* ExpTime*-complete.*

Proof. The upper bound follows from Theorem 3. For the lower bound, we show how one can encode disjunctions (i.e., inclusions of the form $\top \sqsubseteq B \sqcup C$), which allow us to reduce satisfiability of ground $\mathcal{ALCH}^{\mathbb{T}}_{@}$ to satisfiability of ground $\mathcal{ELH}^{\mathbb{T}}_{@}$ ontologies. In fact, several combinations of the temporal attributes time, between, before and after suffice to encode $\top \sqsubseteq B \sqcup C$. As an example, see the following inclusions using the temporal attributes time and $\mathsf{between}$: $\top \sqsubseteq A@\lfloor \mathsf{between} : [1,2] \rfloor$, $A@\lfloor \mathsf{time} : 1 \rfloor \sqsubseteq B$, $A@\lfloor \mathsf{time} : 2 \rfloor \sqsubseteq C$. One can also obtain the same type of encoding with before and after: $\top \sqsubseteq A@\lfloor\ \rfloor$, $A@\lfloor \mathsf{before} : 1 \rfloor \sqsubseteq B$, $A@\lfloor \mathsf{after} : 0 \rfloor \sqsubseteq C$.

It is known that the entailment problem for $\mathcal{EL}$ ontologies with concept and role names annotated with time intervals over finite models is in PTime [17]. Indeed, our temporal attribute during can be seen as a syntactic variant of the time intervals in the mentioned work and, if we restrict to the temporal attributes time, during, since and until, the complexity of the satisfiability problem for ground $\mathcal{ELH}^{\mathbb{T}}_{@}$ ontologies is in PTime. Our proof here (for ground $\mathcal{ELH}^{\mathbb{T}}_{@}$ over $\mathbb{N}$ or over a finite interval in $\mathbb{N}$) is based on a polynomial translation to $\mathcal{ELH}$ extended with role conjunction, where satisfiability is PTime-complete [24].

Theorem 8. *Satisfiability of ground $\mathcal{ELH}^{\mathbb{T}}_{@}$ ontologies without the temporal attributes* $\mathsf{between}$, before *and* after *is* PTime*-complete.*

Proof. Hardness follows from the PTime-hardness of $\mathcal{EL}$ [4]. For membership, note that the translation in Theorem 3 for the temporal attributes during, since and until does not introduce disjunctions or negations. So the result of translating a ground $\mathcal{ELH}^{\mathbb{T}}_{@}$ ontology belongs to $\mathcal{ELH}$ extended with role conjunction.

6 Discussion and Conclusion

We investigated decidability and complexities of attributed description logics enriched with special attributes whose values are interpreted over a temporal dimension. We discussed several ways of restricting the general, undecidable setting in order to regain decidability. Our complexity results range from PTime to 3ExpTime. Some of the statements used in our examples can also be naturally expressed in temporal DLs. For instance, the first statement of Example 3 is expressible by $\mathcal{ALC}$ extended with Linear Temporal Logic [20, 26] with:

$$\mathsf{Lived} \sqcap \Diamond \exists \mathsf{bornIn}.\top \sqsubseteq \bot.$$

Other authors have also considered extending $\mathcal{ALC}$ with Metric Temporal Logic (MTL) [11, 3], where the last statement of Example 1 can be expressed with:

$$\Diamond_{[1394,1404]} \mathsf{bornIn}(\mathsf{Gutenberg}, \mathsf{Mainz}).$$

However, the statement in Example 2 cannot be naturally expressed by temporal DLs. The complexity results can also be very different, for instance, the complexity of propositional MTL is already undecidable over the reals and ExpSpace-complete over the naturals [1], whereas in Theorem 3 of this paper we show that we can enhance $\mathcal{ALC}$ with many types of time related annotations with time points encoded in *binary* while keeping the same ExpTime complexity of $\mathcal{ALC}$. As future work, we plan to study forms of generalising our logic to capture the semantics of other standard types of annotations in knowledge graphs, such as provenance and spatial information.

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A Proofs for Section: Reasoning in $\mathcal{ALCH}^{\mathbb{T}}_{@}$

Theorem 2. *Satisfiability of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies with the temporal attributes* time, after *and* before *but* after *only in assertions is Σ^0_1-complete (recall Σ^0_1 stands for* RE*). The problem is Σ^0_1-hard even with at most one set variable per inclusion.*

Proof. We first show hardness. We reduce the word problem for deterministic Turing machines (DTM) to satisfiability of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies with the temporal attribute after occurring only in assertions. A DTM is a tuple $(Q, \Sigma, \Theta, q_0, q_f)$, where:

- Q is a finite set of states,
- Σ is a finite alphabet containing the *blank symbol* $\sqcup$,
- $\{q_0, q_f\} \subseteq Q$ are the *initial* and the *final* states, resp., and
- $\Theta : Q \times \Sigma \to Q \times \Sigma \times \{l, r\}$ is the *transition function*.

A *configuration* of $\mathcal{M}$ is a word wqw' with $w, w' \in \Sigma^*$ and q in Q. The meaning is that the (one-sided infinite) tape contains the word ww' with only blanks behind it, the machine is in state q and the head is on the left-most symbol of w'. The notion of a *successive configuration* is defined in the usual way, in terms of the transition relation Θ. A *computation* of $\mathcal{M}$ on a word w is a sequence of successive configurations $\alpha_0, \alpha_1, \ldots$, where $\alpha_0 = q_0 w$ is the *initial configuration* for the input w. Let $\mathcal{M}$ be a DTM and $w = \sigma_1\sigma_2 \cdots \sigma_n$ an input word. Assume w.l.o.g. that $\mathcal{M}$ never attempts to move to the left when its head is in the left-most tape position and that q_0 occurs only in the domain of Θ (but not in the range).

We construct an $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontology $\mathcal{O}_{\mathcal{M},w}$ with after occurring only in assertions that is satisfiable iff $\mathcal{M}$ accepts w. Models of $\mathcal{O}_{\mathcal{M},w}$ have a similar structure as in the proof of Theorem 1. We create a vertical chain with:

$$I(a), \quad I \sqsubseteq B \quad \text{and} \quad B \sqsubseteq \exists s.B$$

and ensure that horizontally the set of time points is the same:

$$\exists r.A@X \sqsubseteq \forall s.\exists r.A@\lfloor \mathsf{time}\colon X.\mathsf{time} \rfloor, \tag{5}$$

$$\exists s.\exists r.A@X \sqsubseteq \exists r.A@\lfloor \mathsf{time}\colon X.\mathsf{time} \rfloor. \tag{6}$$

Every element representing a tape cell is marked with A in at most one time point (in fact, it will be exactly one):

$$A@X \sqsubseteq \neg A@\lfloor \mathsf{before}\colon X.\mathsf{time} \rfloor$$

The main difference is that horizontally we do not have infinitely many sibling nodes. That is, over the naturals, adding the inclusion $\exists r.A@X \sqsubseteq \exists r.A@\lfloor \mathsf{before}\colon X.\mathsf{time} \rfloor$ would make $\mathcal{O}_{\mathcal{M},w}$ unsatisfiable and here we cannot use after in inclusions. Instead, for each $q \neq q_f$ in Q, we add to $\mathcal{O}_{\mathcal{M},w}$ the inclusions:

$$S_q \sqcap A@X \sqsubseteq S_q@\lfloor \mathsf{time}\colon X.\mathsf{time} \rfloor, \tag{7}$$

$$\exists r.S_q@X \sqsubseteq \exists r.A@\lfloor \mathsf{before}\colon X.\mathsf{time} \rfloor \tag{8}$$

where S_q is a concept name representing a state. Intuitively, each vertically aligned set of elements (w.r.t. time) represents a configuration and a sequence of configurations going backwards in time represents a computation of $\mathcal{M}$ with input w. The goal is to ensure that $\mathcal{O}_{\mathcal{M},w}$ is satisfiable iff we reach the final state, that is, w is accepted by $\mathcal{M}$.

We now add to $\mathcal{O}_{\mathcal{M},w}$ assertions to trigger the inclusions in Equations 5, 6, 7 and 8:

$$r(a,b), \quad S_{q_0}(b), \quad A(b)@[\mathsf{after}\colon 0].$$

We also use in our encoding concepts C_σ for each symbol $\sigma \in \Sigma$. To encode the input word $w = \sigma_1\sigma_2\cdots\sigma_n$, we add:

$$C_\sigma \sqcap A@X \sqsubseteq C_\sigma@\lfloor\mathsf{time}\colon X.\mathsf{time}\rfloor \text{ for each } \sigma \in \Sigma,$$
$$C_{\sigma_1}(b), \quad \exists r.S_{q_0}@X \sqsubseteq \forall s^i.\exists r.C_{\sigma_{i+1}}@\lfloor\mathsf{time}\colon X.\mathsf{time}\rfloor$$

for $1 \leq i < n$. It is straightforward to add inclusions encoding that (i) the rest of the tape in the initial configuration is filled with the blank symbol, (ii) each node representing a tape cell in a configuration is associated with only one C_σ with $\sigma \in \Sigma$ and (iii) at most one S_q with $q \in Q$ (exactly the node representing the head position). Also, for each element, the time point associated with A is the same for the concepts of the form C_σ and S_q (if true in the node).

To access the 'next' configuration, we use an auxiliary concept F that keeps time points in the future. Recall that since a computation here goes backwards in time, these time points are associated with previous configurations:

$$\exists r.A@X \sqsubseteq \forall r.(\neg A@\lfloor\mathsf{before}\colon X.\mathsf{time}\rfloor \sqcup F@\lfloor\mathsf{time}\colon X.\mathsf{time}\rfloor).$$

We now ensure that tape contents are transferred to the 'next' configuration, except for the tape cell at the head position:

$$\exists r.(C_\sigma@X \sqcap S_{\overline{q}}) \sqsubseteq \forall r.(F@\lfloor\mathsf{before}\colon X.\mathsf{time}\rfloor \sqcup \neg A@\lfloor\mathsf{before}\colon X.\mathsf{time}\rfloor \sqcup C_\sigma)$$

for each $\sigma \in \Sigma$, where $S_{\overline{q}}$ is a shorthand for $\neg \bigsqcup_{q\in Q} S_q$. Finally we encode the transition function. We explain for $\Theta(q,\sigma) = (q', \tau, D)$ with $D = r$ (the case with $D = l$ can be handled analogously). We encode that the 'next' state is q':

$$\exists r.(S_q@X \sqcap C_\sigma) \sqsubseteq \forall s.\forall r.(F@\lfloor\mathsf{before}\colon X.\mathsf{time}\rfloor \sqcup \neg A@\lfloor\mathsf{before}\colon X.\mathsf{time}\rfloor \sqcup S_{q'}) \quad (9)$$

and change to τ the tape cell at the (previous) head position:

$$\exists r.(S_q@X \sqcap C_\sigma) \sqsubseteq \forall r.(F@\lfloor\mathsf{before}\colon X.\mathsf{time}\rfloor \sqcup \neg A@\lfloor\mathsf{before}\colon X.\mathsf{time}\rfloor \sqcup C_\tau).$$

Equation 9 also increments the head position.

This finishes our reduction.

For the upper bound, we point out that if an $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontology $\mathcal{O}$ with after only in assertions is satisfiable then there is a satisfiable ontology $\mathcal{O}'$ that is the result of replacing each occurrence of after : k in $\mathcal{O}$ by some time : l with $k < l \in \mathbb{N}$. By Theorem 5, one can decide satisfiability of $\mathcal{O}'$ (that is, satisfiability of ontologies with only the temporal attributes time and before). As the replacements of after : k by time : l in assertions can be enumerated, it follows that satisfiability of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies is in Σ^0_1.

B Proofs for Section: Regaining decidability

Theorem 3. *Satisfiability of ground $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies is* ExpTime*-complete.*

Proof. We show that $\mathcal{O}$ is satisfiable iff $\mathcal{O}^{\dagger}$ is satisfiable. Given a model $\mathcal{I}$ of $\mathcal{O}$, we directly obtain an $\mathcal{ALCH}b$ interpretation $\mathcal{J}$ over $\Delta^{\mathcal{I}}$ by undoing the renaming and applying $\mathcal{I}$, i.e., by mapping $A_S \in \mathsf{N_C}$ to $A@S^{\mathcal{I}}$, $r_S \in \mathsf{N_R}$ to $r@S^{\mathcal{I}}$, and $a \in \mathsf{N_I}$ to $a^{\mathcal{I}}$. By the semantics of $\mathcal{ALCH}^{\mathbb{T}}_{@}$, $\mathcal{J} \models \mathcal{O}^{\dagger}$. Conversely, given an $\mathcal{ALCH}b$ model $\mathcal{J}$ of $\mathcal{O}^{\dagger}$, we construct an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ with $\Delta^{\mathcal{I}}_{T} = [\max(0, k_{\mathsf{min}} - 2), k_{\mathsf{max}} + 2]$ and $\Delta^{\mathcal{I}}_{I} = \Delta^{\mathcal{J}} \cup \{\star\} \cup \mathbb{T}$, where $\mathbb{T}$ is the set of temporal attributes and $\star$ is a fresh individual name. We define $a^{\mathcal{I}} := a^{\mathcal{J}}$ for all $a \in \mathsf{N_I} \cup \mathsf{N_T} \cup \mathsf{N}^2_{\mathsf{T}}$.

For a ground closed specifier S with $a_1 : b_1, \ldots, a_n : b_n$ as non-temporal attributes, we define:

$$F_S := \{(a_1^{\mathcal{I}}, b_1^{\mathcal{I}}), \ldots, (a_n^{\mathcal{I}}, b_n^{\mathcal{I}})\}.$$

Similarly, for a ground open specifier S with $a_1 : b_1, \ldots, a_n : b_n$ as non-temporal attribute-value pairs, we define:

$$F_S := \{(a_1^{\mathcal{I}}, b_1^{\mathcal{I}}), \ldots, (a_n^{\mathcal{I}}, b_n^{\mathcal{I}}), (\star, \star)\}.$$

To simplify the presentation, we write $a : b \in S$ if $a : b$ occurs in S. Furthermore, let $A^{\mathcal{I}_i}$ be the set of all tuples (δ, F_S) such that one of the following holds:

- $\delta \in A_S^{\mathcal{J}}$, $\mathsf{during} : v \in S$ and $i \in v$;
- $\delta \in A_S^{\mathcal{J}}$, $\mathsf{after} : k_{\mathsf{max}} \in S$ and $i = k_{\mathsf{max}} + 1$;
- $\delta \in A_S^{\mathcal{J}}$, $\mathsf{since} : k_{\mathsf{max}} \in S$ and $k_{\mathsf{max}} + 1 \leq i \leq k_{\mathsf{max}} + 2$;
- $\delta \in A_S^{\mathcal{J}}$, $\mathsf{before} : k_{\mathsf{min}} \in S, i = k_{\mathsf{min}} - 1$ and $k_{\mathsf{min}} > 0$;
- $\delta \in A_S^{\mathcal{J}}$, $\mathsf{until} : k_{\mathsf{min}} \in S, \max(k_{\mathsf{min}} - 2, 0) \leq i \leq k_{\mathsf{min}} - 1$ and $k_{\mathsf{min}} > 0$.

We define $r^{\mathcal{I}_i}$ analogously. Given the definitions of $A^{\mathcal{I}_i}$ and $r^{\mathcal{I}_i}$, for all $i \in \mathbb{N}$, $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$, we define $\cdot^{\mathcal{I}}$ as in Definition 1.

Claim. For all A_S, r_S occurring in $\mathcal{O}^{\dagger}$: (1) $A_S^{\mathcal{J}} = A@S^{\mathcal{I}}$ and (2) $r_S^{\mathcal{J}} = r@S^{\mathcal{I}}$.

Proof of the Claim. If no temporal attribute occurs in S then by definition of $\mathcal{I}$ (in particular, F_S), we clearly have that $\delta \in A_S^{\mathcal{J}}$ iff $\delta \in A@S^{\mathcal{I}}$. Also, by semantics of $\mathcal{ALCH}^{\mathbb{T}}_{@}$, for a ground specifier S with a non-empty set $S_{\mathbb{T}}$ of temporal attributes the following holds for any $\mathcal{I}$ and concept $A@S$:

$$A@S^{\mathcal{I}} = \bigcap_{a : b \in S_{\mathbb{T}}} A@S(a : b)^{\mathcal{I}}$$

So we can consider $A@S$ with S containing only one temporal attribute. We argue for during and $\mathsf{between}$ (one can give a similar argument for the other temporal attributes):

- if the temporal attribute-value pair $\mathsf{during} : v$ is in S then, by definition of $\mathcal{I}$ (and F_S), $\delta \in A_S^{\mathcal{J}}$ iff $\delta \in A@S^{\mathcal{I}}$;

- if the temporal attribute-value pair $\mathsf{between}\colon v$ is in S then, by Equation 4, $\delta \in A_S^{\mathcal{J}}$ iff $\delta \in \bigcup_{k \in v \cap \mathbb{N}_{\mathcal{O}}} A^{\mathcal{J}}_{S(\mathsf{during}\colon[k,k])}$. By definition of $\mathcal{I}$, $\delta \in A^{\mathcal{J}}_{S(\mathsf{during}\colon[k,k])}$ iff $\delta \in A@S(\mathsf{during}\colon[k,k])^{\mathcal{I}}$, for $k \in v \cap \mathbb{N}_{\mathcal{O}}$. Then,
$$\delta \in A_S^{\mathcal{J}} \text{ iff } \delta \in \bigcup_{k \in v \cap \mathbb{N}_{\mathcal{O}}} A@S(\mathsf{during}\colon[k,k])^{\mathcal{I}};$$
so $\delta \in A@S^{\mathcal{I}}$.

In the definition of $\mathbb{N}_{\mathcal{O}}$, we add $k_i + 1$ for each k_i occurring in $\mathcal{O}$, to ensure that axioms such as $\top \sqsubseteq A@\lfloor\mathsf{between}\colon[k,l]\rfloor \sqcap \neg A@\lfloor\mathsf{time}\colon k\rfloor \sqcap \neg A@\lfloor\mathsf{time}\colon l\rfloor$ with $l - k \geq 2$ remain satisfiable. Also, in the definition of $\mathcal{I}$ we use the interval $\Delta_T^{\mathcal{I}} = [\max(0, k_{\mathsf{min}} - 2), k_{\mathsf{max}} + 2]$, and so, we give a margin of two 'additional' points in each side of the interval $[k_{\mathsf{min}}, k_{\mathsf{max}}]$ used in the translation. This is to ensure that axioms such as $\top \sqsubseteq A@\lfloor\mathsf{before}\colon k_{\mathsf{min}}\rfloor \sqcap \neg A@\lfloor\mathsf{until}\colon k_{\mathsf{min}}\rfloor$ with $k_{\mathsf{min}} \geq 2$ remain satisfiable. Point (2) can be proven with an easy adaptation of Point (1).

The Claim directly implies that $\mathcal{I} \models \mathcal{O}$. Note that $\star$ ensures that axioms such as $\top \sqsubseteq A@\lfloor a\colon b\rfloor \sqcap \neg A@[a\colon b]$ remain satisfiable.

Theorem 4. *Satisfiability of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies without expressions of the form $X.a$ for temporal attributes a is* 2ExpTime*-complete.*

Proof. The 2ExpTime lower bound follows from the fact that satisfiability of $\mathcal{ALCH}_{@}$ (so without temporal attributes) is already 2ExpTime-hard [14]. Our proof strategy for the upper bound consists on defining an ontology with grounded versions of inclusion axioms. Let $\mathcal{O}$ be an $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontology and let $\mathsf{N} := \mathsf{N}_{\mathsf{I}}^{\mathcal{O}} \cup \mathsf{N}_{\mathsf{T}}^{\mathcal{O}} \cup \mathsf{N}_{\mathsf{T}}^{2\mathcal{O}}$ be the union of the sets of individual names, time points, and intervals, occurring in $\mathcal{O}$, respectively. Let $\mathcal{I}$ be an interpretation of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ over the domain $\Delta^{\mathcal{I}} = \mathsf{N} \cup \{x\}$, where x is a fresh individual name, satisfying $a^{\mathcal{I}} = a$ for all $a \in \mathsf{N}$. Let $\mathcal{Z} : \mathsf{N}_{\mathsf{V}} \to \Phi^{\mathcal{I}}_{\mathcal{O}}$ be a variable assignment, where $\Phi^{\mathcal{I}}_{\mathcal{O}} := \mathcal{P}_{\mathsf{fin}}\left(\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}\right)$. Consider a concept inclusion α of the form $X_1 : S_1, \ldots, X_n : S_n\ (C \sqsubseteq D)$. We say that $\mathcal{Z}$ is *compatible with* α if $\mathcal{Z}(X_i) \in S_i^{\mathcal{I},\mathcal{Z}}$ for all $1 \leq i \leq n$. In this case, the *$\mathcal{Z}$-instance* $\alpha_{\mathcal{Z}}$ of α is the concept inclusion $C' \sqsubseteq D'$ obtained by

- replacing each X_i by $[a\colon b \mid (a,b) \in \mathcal{Z}(X_i)]$; and
- replacing every $a\colon X_i.b$ occurring in some specifier (with a, b *non-temporal* attributes) by all $a\colon c$ such that $(b,c) \in \mathcal{Z}(X_i)$.

Then, the grounding $\mathcal{O}_g$ of $\mathcal{O}$ contains all $\mathcal{Z}$-instances $\alpha_{\mathcal{Z}}$ for all concept inclusions α in $\mathcal{O}$ and all compatible variable assignments $\mathcal{Z}$; and analogous axioms for role inclusions.

There may be (at most) exponentially many different instances for each terminological axiom in $\mathcal{O}$, thus $\mathcal{O}_g$ is of exponential size. We show that $\mathcal{O}$ is satisfiable iff $\mathcal{O}_g$ is satisfiable. By construction, we have $\mathcal{O} \models \mathcal{O}_g$, i.e., any model of $\mathcal{O}$ is also a model of $\mathcal{O}_g$. Conversely, let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a model of $\mathcal{O}_g$. W.l.o.g., assume that there is $x \in \Delta^{\mathcal{I}}$ such that $x \neq a^{\mathcal{I}}$ for all $a \in \mathsf{N}_{\mathsf{I}}^{\mathcal{O}} \setminus \{x\}$. For an annotation set $F \in \mathcal{P}_{\mathsf{fin}}\left(\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}\right)$, we define $\mathrm{rep}_x(F)$ to be the annotation set obtained from F by replacing any individual $\delta \notin \mathcal{I}(\mathsf{N}_{\mathsf{I}}^{\mathcal{O}})$ in F by x.

Let $\sim$ be the equivalence relation induced by $\text{rep}_x(F) = \text{rep}_x(G)$ and define an interpretation $\mathcal{J}$ of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ over the domain $\Delta^{\mathcal{J}} := \Delta^{\mathcal{I}}$, where $A^{\mathcal{J}} := \{(\delta, F) \mid (\delta, G) \in A^{\mathcal{I}} \text{ and } F \sim G\}$ for all $A \in \mathsf{N_C}$, $r^{\mathcal{J}} := \{(\delta, \epsilon, F) \mid (\delta, \epsilon, G) \in r^{\mathcal{I}} \text{ and } F \sim G\}$ for all $r \in \mathsf{N_R}$, and $a^{\mathcal{J}} := a^{\mathcal{I}}$ for all $a \in \mathsf{N_I} \cup \mathsf{N_T} \cup \mathsf{N_T^2}$. It remains to show that $\mathcal{J}$ is indeed a model of $\mathcal{O}$. Suppose for a contradiction that there is a concept inclusion α in $\mathcal{O}$ that is not satisfied by $\mathcal{J}$ (the case for role inclusions is analogous). Then we have some compatible variable assignment $\mathcal{Z}$ that leaves α unsatisfied. Let $\mathcal{Z}_x$ be the variable assignment $X \mapsto \text{rep}_x(\mathcal{Z}(X))$ for all $X \in \mathsf{N_V}$. Clearly, as expressions of the form $a\colon X_i.b$ are only allowed for a, b being *non-temporal* attributes, $\mathcal{Z}_x$ is also compatible with α. But now we have $C^{\mathcal{J},\mathcal{Z}} = C^{\mathcal{I},\mathcal{Z}_x}$ for all $\mathcal{ALCH}^{\mathbb{T}}_{@}$ concepts C, yielding the contradiction $\mathcal{I} \not\models \alpha_{\mathcal{Z}_x}$. Thus, $\mathcal{O}$ is satisfiable iff $\mathcal{O}_g$ is satisfiable. The result then follows from Theorem 3.

Theorem 5. *Satisfiability of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies with only the temporal attributes* during, time, before *and* until *is in* 3ExpTime.

Proof. The difference w.r.t. the proof of Theorem 4 is that here expressions of the form $X.a$ may occur in front of the temporal attributes during, before, time and until and the other temporal attributes are not allowed (not even in assertions). Let v be the internal $[0, k]$, where k is the largest number occurring in $\mathcal{O}$ (or 0 if no number occurs). To define our ground translation, we consider variable assignments $\mathcal{Z} : \mathsf{N_V} \to \Phi^{\mathcal{I}}_{\mathcal{O},v}$, where $\Phi^{\mathcal{I}}_{\mathcal{O},v} := \mathcal{P}_{\text{fin}}\left(\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}\right)$ and $\Delta^{\mathcal{I}}$ is the set of all individual names in $\mathcal{O}$ plus a fresh individual name x, all time points in v and all intervals contained in v. This gives us a ground ontology $\mathcal{O}_g$ with size double-exponential in the size of $\mathcal{O}$. Clearly, $\mathcal{O}$ is satisfiable iff $\mathcal{O}_g$ is satisfiable.

Theorem 6. *Satisfiability of $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies with only* time *and* before *is* 3ExpTime-*hard.*

Proof. We reduce the word problem for double-exponentially space-bounded alternating Turing machines (ATMs) to the entailment problem for $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontologies. We consider w.l.o.g. ATMs with only finite computations on any input. As usual, an ATM is a tuple $\mathcal{M} = (Q, \Sigma, \Theta, q_0)$, where:

- $Q = Q_\exists \uplus Q_\forall$ is a finite set of states, partitioned into *existential states* $Q_\exists$ and *universal states* $Q_\forall$,
- Σ is a finite alphabet containing the *blank symbol* $\sqcup$,
- $q_0 \in Q$ is the *initial state*, and
- $\Theta \subseteq Q \times \Sigma \times Q \times \Sigma \times \{l, r\}$ is the *transition relation*.

We use the same notions of configuration, computation and initial configuration given in the proof of Theorem 2. We recall the acceptance condition of an ATM. A configuration $\alpha = wqw'$ is *accepting* iff

- α is a universal configuration and all its successor configurations are accepting, or
- α is an existential configuration and at least one of its successor configurations is accepting.

Note that, by the definition above, universal configurations without any successors are accepting. We assume w.l.o.g. that all configurations wqw of a computation of $\mathcal{M}$ satisfy $|ww'| \leq 2^{2^n}$. $\mathcal{M}$ *accepts* a word in $(\Sigma \setminus \{\llcorner\!\lrcorner\})^*$ (in space double-exponential in the size of the input) iff the initial configuration is accepting.

There exists a double-exponentially space bounded ATM $\mathcal{M} = (Q, \Sigma, q_0, \Theta)$ whose word problem is 3ExpTime-hard [8]. Let $\mathcal{M}$ be such a double-exponentially space bounded ATM and $w = \sigma_1\sigma_2 \cdots \sigma_n$ an input word. W.l.o.g., we assume that $\mathcal{M}$ never attempts to move to the left (right) when the head is on the left-most (right-most) tape cell.

We construct an $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontology $\mathcal{O}_{\mathcal{M},w}$ that entails $A(a)$ iff $\mathcal{M}$ accepts w. We represent configurations using individuals in $\mathcal{O}_{\mathcal{M},w}$, which are connected to the corresponding successor configurations by roles encoding the transition. W.l.o.g., we assume that these individuals form a tree, which we call the *configuration tree*. Furthermore, each node of this tree, i.e., each configuration, is connected to 2^{2^n} individuals representing the tape cells. The main ingredients of our construction are:

- an individual a denoting the root of the configuration tree;
- an attribute bit, with values in $\{0, 1\}$, used to encode double-exponentially many tape positions;
- an attribute flip which has value 1 at a (unique) time point where bit has value 0 and bit has value 1 in all subsequent time points;
- a concept A marking accepting configurations;
- a concept H marking the head position;
- a concept T marking tape cells;
- a concept I marking the initial configuration;
- concepts S_q for each state $q \in Q$;
- concepts C_σ for each symbol $\sigma \in \Sigma$;
- roles r_θ for all transitions $\theta \in \Theta$;
- a role tape connecting configurations to tape cells; and
- attributes $a_0, \ldots, a_n$ to encode the binary representation of time values.

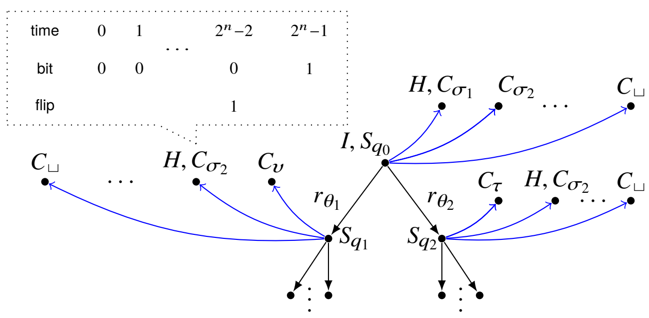

Fig. 1. A model of $\mathcal{O}_{\mathcal{M},w}$ encoding the computation tree of an ATM; blue edges (potentially grey) represent the tape role (we omit for brevity T in nodes representing tape cells)

To encode the binary representation of time values we first state that for $\mathsf{time}\colon 2^n - 1$ we have all a_i set to 1:

$$T \sqsubseteq T@\lfloor \mathsf{time}\colon 2^n - 1, a_n\colon 1, \dots, a_0\colon 1\rfloor.$$

We now use the following intuition: if the a_i attributes represent a pattern $s \cdot 1000$, where s is a binary sequence and $\cdot$ means concatenation, then $s \cdot 0111$ should occur *before* that pattern in the time line. To ensure this, we add concept inclusions of the form, for all $0 \le i \le n$:

$$X : S\ (T@X \sqsubseteq T@\lfloor \mathsf{before}\colon X.\mathsf{time}, P^X_{a_i}\rfloor)$$

where S is $\lfloor a_i\colon 1, a_{i-1}\colon 0, \dots, a_0\colon 0\rfloor$ and $P^X_{a_i}$ abbreviates

$$a_n\colon X.a_n, \dots, a_{i+1}\colon X.a_{i+1}, a_i\colon 0, a_{i-1}\colon 1, \dots, a_0\colon 1.$$

By further adding a concept inclusion encoding that a_i can only be one of $1, 0$ at the same time point we have that, in any model, the a_i attributes encode the binary representation of the corresponding time value, for time points in $[0, 2^n - 1]$. This means that, for time points in $[0, 2^n - 1]$, we can simulate the temporal attribute after by using variables and specifiers of the form $X : \lfloor a_i\colon 1\rfloor$ and $\lfloor a_n\colon X.a_n, \dots, a_{i-1}\colon X.a_{i-1} a_i\colon 0\rfloor$, for all $0 \le i \le n$.

Remark 1. To simplify the presentation, in the following, we use the temporal attributes after and during (the latter is used to encode the initial configuration). Given the construction above it is straightforward to replace the inclusions using after and during with inclusions using the attributes a_i.

We encode the meaning of the attribute flip (i.e., it has value 1 at the time point from which bits should be flipped to increment a tape position) with the following concept inclusions:

$$T@\lfloor \mathsf{bit}\colon 0\rfloor \sqsubseteq T@\lfloor \mathsf{flip}\colon 1\rfloor \tag{10}$$

$$X : \lfloor \mathsf{flip}\colon 1\rfloor (T@X \sqsubseteq \neg T@\lfloor \mathsf{bit}\colon 0, \mathsf{after}\colon X.\mathsf{time}\rfloor) \tag{11}$$

$$X : \lfloor \mathsf{flip}\colon 1\rfloor (T@X \sqsubseteq T@\lfloor \mathsf{bit}\colon 0, \mathsf{time}\colon X.\mathsf{time}\rfloor) \tag{12}$$

Intuitively, in Equation 10 we say that if there is a time point where we have bit with value 0 then there is a time point where we should flip some bit to increment the tape position, i.e., where flip is 1. In Equation 11 we state that there is no bit with value 0 after a time point marked with flip set to 1. Finally, in Equation 12, we state that bit has value 0 where flip has value 1. Thus, Equations 11 and 12 ensure that there is at most one time point where flip has value 1.

Let $\mathbf{\Omega}$ be a sequence with the following variables X^j_i, with $1 \le i \le n$ and $1 \le j \le 5$, and their respective specifiers:

- $X^1_i : \lfloor \mathsf{flip}\colon 1\rfloor$, we look at our auxiliary attribute that indicates from which time point we should flip our bits to obtain the next tape position (this will be a time point with bit value 0);

– we also define X_i^2 **:** $\lfloor$before: X_i^1.time, bit: 0$\rfloor$ and X_i^3 **:** $\lfloor$before: X_i^1.time, bit: 1$\rfloor$, to filter time points with bit values 0 and 1, respectively, before the time point with flip : 1, related to X_i^1;
– we use X_i^4 **:** $\lfloor$bit : 0$\rfloor$ and X_i^5 **:** $\lfloor$bit : 1$\rfloor$ to filter time points bit values 0 and 1, respectively.

Basically, the first three variables are related to specifiers that filter the information needed to increment the tape position encoded with the bit attribute. The last two variables X_i^j are related to specifiers that filter the information needed to copy the tape position. We now define specifiers S_i^j, S_i, for $1 \leq i \leq n$ and $1 \leq j \leq 5$. Intuitively, the next four specifiers are used to increment the tape position, using the information given by the X_i^j variables. The last two specifiers copy the tape position, again using the information given by the X_i^j variables:

– the negation of a concept expression associated with $S_i = \lfloor$after : X_i^1.time, bit : 1$\rfloor$ ensures that we have bit : 0 in all time points after the time point marked with flip : 1 in the previous position;
– we use $S_i^1 = \lfloor$time : X_i^1.time, bit : 1$\rfloor$ to flip to 1 the bit marked with flip : 1 in the previous position;
– in addition, we define $S_i^2 = \lfloor$time : X_i^2.time, bit : 0$\rfloor$ and $S_i^3 = \lfloor$time : X_i^3.time, bit : 1$\rfloor$ to transfer to the next tape position bit values which should not be flipped (i.e., those that are before the time point with flip : 1);
– finally, we define $S_i^4 = \lfloor$time : X_i^4.time, bit : 0$\rfloor$ and $S_i^5 = \lfloor$time : X_i^5.time, bit : 1$\rfloor$, to receive a copy of the bit values.

To simplify the presentation, we define the abbreviations $P_i, P_i^+, P_i^=$ for the following concepts, respectively, to be used in concept inclusions with $\mathbf{\Omega}$:

– $\bigsqcap_{1 \leq j \leq 5} T@X_i^j$, we filter the bits encoding a tape position and the information of which bits should be flipped in order to increment it;
– $\bigsqcap_{1 \leq j \leq 3} T@S_i^j \sqcap \neg T@S_i$, we increment the tape position,
– $\bigsqcap_{4 \leq j \leq 5} T@S_i^j$, we copy the tape position.

We may also write $P, P^+, P^=$ if $i = 1$.

Encoding the initial configuration We add assertions to $\mathcal{O}_{\mathcal{M},w}$ that encode the initial configuration of $\mathcal{M}$. We mark the root of the configuration tree with the initial state by adding $S_{q_0}(a)$ and initialise the tape cells with the input word by adding $I(a)$ and the concept inclusions:

$$\mathbf{\Omega}\ (I \sqsubseteq \exists \mathsf{tape}.(H \sqcap C_{\sigma_1} \sqcap T@\lfloor \mathsf{during} : [0, 2^n - 1], \mathsf{bit}\colon 0\rfloor))$$
$$\mathbf{\Omega}\ (I \sqcap \exists \mathsf{tape}.P_i \sqsubseteq \exists \mathsf{tape}.(C_{\sigma_{i+1}} \sqcap P_i^+))\ \text{for}\ 1 \leq i < n$$
$$\mathbf{\Omega}\ (I \sqcap \exists \mathsf{tape}.P_n \sqsubseteq \exists \mathsf{tape}.(C_{\sqcup} \sqcap P_n^+))$$

The intuition is as follows. In the first inclusion, we place the head, represented by the concept H, in the first position of the tape and fill the tape cell with the first symbol of the input word, represented by the concept C_{σ_1}. We then add the remaining symbols of the input word in their corresponding tape positions. In the last inclusion we add a blank symbol after the input word. We now add the following concept inclusion fill the remaining tape cells with blank in the initial configuration marked with the concept I:

$$\mathbf{\Omega}\ (I \sqcap \exists \mathsf{tape}.(C_{\sqcup} \sqcap P) \sqsubseteq \exists \mathsf{tape}.(C_{\sqcup} \sqcap P^{+}))$$

Synchronising configurations For each transition $\theta \in \Theta$, we make sure that tape contents are transferred to successor configurations, except for the tape cell at the head position:

$$\mathbf{\Omega}\ (\exists \mathsf{tape}.(P \sqcap \neg H \sqcap C_{\sigma}) \sqsubseteq \forall r_{\theta}.\exists \mathsf{tape}.(P^{=} \sqcap C_{\sigma}))$$

We now encode our transitions $\theta = (q, \sigma, q', \tau, D) \in \Theta$ with concept inclusions of the form (we explain for $D = r$, the case $D = l$ is analogous):

$$\mathbf{\Omega}\Big(S_q \sqcap \exists \mathsf{tape}.(H \sqcap P \sqcap C_{\sigma}) \sqcap \exists \mathsf{tape}.(P^{+} \sqcap C_{\upsilon}) \sqsubseteq \exists r_{\theta}.(S_{q'} \sqcap \exists \mathsf{tape}.(H \sqcap P^{+} \sqcap C_{\upsilon}) \sqcap \exists \mathsf{tape}.(P^{=} \sqcap C_{\tau}))\Big)$$

Essentially, if the head is at position P then, to move it to the right, we increment the head position using P^{+} in the successor configuration. We use the specifiers in $\mathbf{\Omega}$ to modify the tape cell with C_{σ} in the head position to C_{τ} in the successor configuration.

Acceptance Condition Finally, we add concept inclusions that propagate acceptance from the leaf nodes of the configuration tree backwards to the root of the tree. For existential configurations, we add $S_q \sqcap \exists r_{\theta}.A \sqsubseteq A$ for each $q \in Q_{\exists}$, whereas to handle universal configurations, we add, for each $q \in Q_{\forall}$, the concept inclusion

$$S_q \sqcap \exists \mathsf{tape}.(C_{\sigma} \sqcap H) \sqcap \bigsqcap_{\substack{\theta \in \Theta \\ \theta = (q, \sigma, q', \tau, D)}} \exists r_{\theta}.A \sqsubseteq A$$

where the conjunction may be empty if there are no suitable $\theta \in \Theta$.

With an inductive argument along the recursive definition of acceptance, we show that $\mathcal{O}_{\mathcal{M},w} \models A(a)$ iff $\mathcal{M}$ accepts w.

Given a natural number $i < 2^{2^n}$, we write $i_{\mathbf{b}}[j]$ for the value of the j-th bit of the binary representation of i using 2^n bits, where $i_{\mathbf{b}}[0]$ is the value the most significant bit. In the following, we write B_i as a shorthand for the concept:

$$\bigsqcap_{0 \leq y < 2^n} T @ \lfloor \mathsf{bit} : i_{\mathbf{b}}[y], \mathsf{time} : y \rfloor.$$

Following the terminology provided in [16], given an interpretation $\mathcal{I}$ of $\mathcal{ALCH}^{\mathbb{T}}_{@}$, we say that an element $\delta \in \Delta^{\mathcal{I}}_{I}$ *represents* a configuration $\tau_1 \ldots \tau_{i-1} q \tau_i \ldots \tau_m$ if $(\delta, F) \in S^{\mathcal{I}}_{q}$, for some $F \in \Phi^{\mathcal{I}}$, $\delta \in (\exists \mathsf{tape}.(B_i \sqcap H))^{\mathcal{I}}$ and $\delta \in (\exists \mathsf{tape}.(B_j \sqcap C_{\tau_j}))^{\mathcal{I}}$, for all $0 \leq j < 2^{2^n}$. We are now ready to show Claims 1 and 2.

Claim 1 If $\delta \in \Delta^{\mathcal{I}}_{I}$ represents a configuration α and some transition $\theta \in \Theta$ is applicable to α then δ has an r_{θ}-successor that represents the result of applying θ to α.

Proof of Claim 1. Let $\delta \in \Delta_I^{\mathcal{I}}$ be an element representing a configuration α and assume $\theta \in \Theta$ is applicable to α. To synchronise configurations, we added to $\mathcal{O}_{\mathcal{M},w}$ concept inclusions that (1) ensure that tape contents other than the content at the head position are copied to all r_θ-successors of δ; and (2) create an r_θ-successor that represents the correct state, position of the head and corresponding symbols at the previous and current position of the head. Then our concept inclusions ensure that δ has an r_θ-successor that represents the result of applying θ to α.

Claim 2 w is accepted by $\mathcal{M}$ iff $\mathcal{O}_{\mathcal{M},w} \models A(a)$.

Proof of Claim 2. Consider an arbitrary interpretation $\mathcal{I}$ of $\mathcal{ALCH}_{@}^{\mathbb{T}}$ that satisfies $\mathcal{O}_{\mathcal{M},w}$. First we show that if any element $\delta \in \Delta_I^{\mathcal{I}}$ represents an accepting configuration then $(\delta, F) \in A^{\mathcal{I}}$, for some $F \in \Phi^{\mathcal{I}}$. We make a case distinction.

- If α is a universal configuration, then all successor configurations of α must be accepting. By Claim 1, for any θ-successor configuration α' of α there is a corresponding r_θ-successor δ' of δ. By induction hypothesis for α', (δ', F') is in $A^{\mathcal{I}}$, for some $F' \in \Phi^{\mathcal{I}}$. Since this holds for all θ-successor configurations of α, our concept inclusion encoding acceptance of universal configurations implies that $(\delta, F) \in A^{\mathcal{I}}$, for some $F \in \Phi^{\mathcal{I}}$, as required. This argument covers the base case where α has no successors.
- If α is an existential configuration, then there is some accepting θ-successor configuration α' of α. By Claim 1, there is an r_θ-successor δ' of δ that represents α' and, by induction hypothesis, $(\delta', F') \in A^{\mathcal{I}}$, for some $F' \in \Phi^{\mathcal{I}}$. Then, our concept inclusion encoding acceptance of existential configurations applies and so, we conclude that $(\delta, F) \in A^{\mathcal{I}}$, for some $F \in \Phi^{\mathcal{I}}$.

Since elements in $I^{\mathcal{I}}$ represent the initial configuration of $\mathcal{M}$, this shows that $I^{\mathcal{I}} \subseteq A^{\mathcal{I}}$ when the initial configuration is accepting. As $I(a)$ is an assertion in $\mathcal{O}_{\mathcal{M},w}$, we have that $(a^{\mathcal{I}}, G) \in A^{\mathcal{I}}$, for some $G \in \Phi^{\mathcal{I}}$.

We now show that if the initial configuration is not accepting, then there is some interpretation $\mathcal{I}$ of $\mathcal{ALCH}_{@}^{\mathbb{T}}$ such that $I^{\mathcal{I}} \nsubseteq A^{\mathcal{I}}$, in particular, $(a^{\mathcal{I}}, G) \notin A^{\mathcal{I}}$, for all $G \in \Phi^{\mathcal{I}}$. To show this we construct a canonical interpretation $\mathcal{J}$ of $\mathcal{O}_{\mathcal{M},w}$ as follows. Let $\mathsf{Con}_{\mathcal{M}} := \{wqw' \mid |ww'| \leq 2^{2^n}, q \in Q, \{w, w'\} \subseteq \Sigma^*\}$ be the set of all possible $\mathcal{M}$ configurations with size bounded by 2^{2^n}. Also, we define a set $\mathsf{Tp}_{\mathcal{M}} := \{\alpha \cdot c_\sigma^i \mid \alpha \in \mathsf{Con}_{\mathcal{M}}, 0 \leq i < 2^{2^n}, \sigma \in \Sigma\}$, containing individuals that represent tape cells, related to each possible configuration of a computation of $\mathcal{M}$. The domain $\Delta^{\mathcal{J}}$ is a disjoint union of $\Delta_I^{\mathcal{J}} \cup \Delta_T^{\mathcal{J}} \cup \Delta_{2T}^{\mathcal{J}}$, where:

- $\Delta_I^{\mathcal{J}} = \mathsf{Con}_{\mathcal{M}} \cup \mathsf{Tp}_{\mathcal{M}} \cup \mathbb{T}$, where $\mathbb{T} \subseteq \mathsf{N_I}$ is either time or before;
- $\Delta_T^{\mathcal{J}} = \{0^{\mathcal{J}}, \ldots, (2^n - 1)^{\mathcal{J}}\}$; and $\Delta_{2T}^{\mathcal{J}} = \Delta_T^{\mathcal{J}} \times \Delta_T^{\mathcal{J}}$.

The extension of the concepts C_σ, H and B_j in the interpretation is defined as expected so that every element $\alpha \cdot c_\sigma^i \in \mathsf{Tp}_{\mathcal{M}}$ is in C_σ and B_i and no other C_τ or B_j, with $\tau \neq \sigma$ or $i \neq j$. Also, $\alpha \cdot c_\sigma^i$ is in H iff α is of the form wqw' and $|w| = i - 1$. We connect α to $\alpha \cdot c_\sigma^i$ using the role tape iff α has σ at position i. Moreover, α is in S_q iff α is of the form wqw'. We then have that every configuration $\alpha \in \mathsf{Con}_{\mathcal{M}}$ represents itself and no other configuration. $I^{\mathcal{J}}$ is the singleton set containing the initial configuration $a^{\mathcal{J}}$. Given two configurations α and α' and a transition $\theta \in \Theta$, we connect α to α' using the role r_θ iff there is a transition θ from α to α'. Finally, $A^{\mathcal{J}}$ is defined to be the set of tuples (α, F), for some $F \in \Phi^{\mathcal{J}}$, where α is an accepting configuration.

Now, if the initial configuration $a^{\mathcal{J}}$ is not accepting then, by construction, $(a, G) \notin A^{\mathcal{J}}$, for all $G \in \Phi^{\mathcal{J}}$. By checking the concept inclusions in $\mathcal{O}_{\mathcal{M},w}$, we can see that $\mathcal{J}$ satisfies $\mathcal{O}_{\mathcal{M},w}$. Then, $\mathcal{J}$ is a counterexample for $\mathcal{O}_{\mathcal{M},w} \models A(a)$, and so $\mathcal{O}_{\mathcal{M},w} \not\models A(a)$.

Theorem. *In $\mathcal{ALCH}^{\mathbb{T}}_{@}$, any combination of temporal attributes containing* {time, after} *is undecidable. Moreover, the combination* {time, before} *is* 3ExpTime*-complete, and the combination* {time, during, since, until} *and every subset of it are* 2ExpTime*-complete.*

Proof. The proof of Theorem 1 uses only the temporal attributes time and after. Thus, any combination containing these attributes is Σ^1_1-hard. By Theorems 5 and 6 the combination {time, before} is 3ExpTime-complete. Here we show that the combination {time, during, since, until} is in 2ExpTime (since 2ExpTime hardness is already for $\mathcal{ALCH}_{@}$ [14]).

Our proof strategy consists on showing that given an $\mathcal{ALCH}^{\mathbb{T}}_{@}$ interpretation and an $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontology containing only the temporal attributes in {time, during, since, until}, one can always transform this interpretation so that only time points explicitly mentioned in the ontology are relevant to determine if the interpretation is a model of the ontology. Then, one can check satisfiability by grounding the ontology using only those time points explicitly mentioned. We start by providing some notation.

Given an $\mathcal{ALCH}^{\mathbb{T}}_{@}$ ontology $\mathcal{O}$, we define the set $\mathbb{N}_{\mathcal{O}}$ in a similar way as in Theorem 3, except that here we do not need $k_i + 1$. Let $k_0 < \ldots < k_n$ be the ascending sequence of all numbers mentioned in time points or in time intervals (as end-points) in $\mathcal{O}$. We define $\mathbb{N}_{\mathcal{O}}$ as $\{k_i \mid 0 \leq i \leq n\}$, and let $k_{\mathsf{min}} := \min(\mathbb{N}_{\mathcal{O}})$ and $k_{\mathsf{max}} = \max(\mathbb{N}_{\mathcal{O}})$, where we assume $k_{\mathsf{min}} = k_{\mathsf{max}} = 0$ if $\mathbb{N}_{\mathcal{O}} = \emptyset$.

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an $\mathcal{ALCH}^{\mathbb{T}}_{@}$ interpretation. By Definition 1, $\mathcal{I}$ is a global interpretation of a sequence $(\mathcal{I}_i)_{i \in \Delta^{\mathcal{I}}_T}$ of $\mathcal{ALCH}_{@}$ interpretations with domain $\Delta^{\mathcal{I}}_I$. We now define a sequence $(\mathcal{J}_i)_{i \in \Delta^{\mathcal{J}}_T}$ of $\mathcal{ALCH}_{@}$ interpretations as follows. Let $\Delta^{\mathcal{J}}_I = \Delta^{\mathcal{I}}_I$ and let $\Delta^{\mathcal{J}}_T = \{k^{\mathcal{J}}_{\mathsf{min}}, \ldots, k^{\mathcal{J}}_{\mathsf{max}}\}$. For all $A \in \mathsf{N_C}$, all $F \in \Phi^{\mathcal{I}}$ with $F \setminus F_I \neq \emptyset$ and $k \in [k_{\mathsf{min}}, k_{\mathsf{max}}]$:

$$(\delta, F_I) \in A^{\mathcal{J}_k} \text{ iff } (\delta, F_I) \in A^{\mathcal{I}_k}$$

and either:

1. $k \in \mathbb{N}_{\mathcal{O}}$; or
2. there is $k_i < k$ such that $k_i \in \mathbb{N}_{\mathcal{O}}$ and $(\delta, F_I) \in A^{\mathcal{I}_j}$ for all $k_i \leq j \leq k_{\mathsf{max}}$; or
3. there is $k_i > k$ such that $k_i \in \mathbb{N}_{\mathcal{O}}$ and $(\delta, F_I) \in A^{\mathcal{I}_j}$ for all $k_{\mathsf{min}} \leq j \leq k_i$.

We analogously apply the definition above for all role names $r \in \mathsf{N_R}$. We define $\mathcal{I}_{\mathcal{O}}$ as a global interpretation of the sequence $(\mathcal{J}_i)_{i \in \Delta^{\mathcal{J}}_T}$ and set $(\delta, F) \in A^{\mathcal{I}_{\mathcal{O}}}$ iff $(\delta, F) \in A^{\mathcal{I}}$ for all $A \in \mathsf{N_C}$ with $F = F_I$, and similarly for all role names $r \in \mathsf{N_R}$. Let $\mathcal{O}_{\mathsf{g}}$ be the result of grounding $\mathcal{O}$ in the same way as in the proof of Theorem 4 using time points in $\mathbb{N}_{\mathcal{O}}$ (here $\mathcal{O}$ may have expressions of the form $X.a$, with $a \in$ {time, during, since, until}).

Claim. For all $A@S, r@S$ occurring in $\mathcal{O}_{\mathsf{g}}$: $A@S^{\mathcal{I}_{\mathcal{O}}} = A@S^{\mathcal{I}}$ and $r@S^{\mathcal{I}_{\mathcal{O}}} = r@S^{\mathcal{I}}$.

Proof of the Claim. This claim follows by definition of $(\mathcal{J}_i)_{i\in\Delta_T^{\mathcal{J}}}$ and the fact that only the temporal attributes {time, during, since, until} are allowed. Point 1 covers concept expressions with the temporal attributes time and during, whereas Points 2 and 3 cover concept expressions with the temporal attributes since and until, respectively.

By definition of $\mathcal{O}_\mathsf{g}$, we know that $\mathcal{O} \models \mathcal{O}_\mathsf{g}$. So if $\mathcal{O}$ is satisfiable then $\mathcal{O}_\mathsf{g}$ is satisfiable. Conversely, by the Claim, one can show with an inductive argument that $C^{\mathcal{I}_\mathcal{O}} = C^{\mathcal{I}}$ for all $\mathcal{ALCH}^{\mathbb{T}}_{@}$ concepts C occurring in $\mathcal{O}_\mathsf{g}$. So, if an $\mathcal{ALCH}^{\mathbb{T}}_{@}$ interpretation $\mathcal{I}$ satisfies $\mathcal{O}_\mathsf{g}$ then $\mathcal{I}_\mathcal{O}$ satisfies $\mathcal{O}$. Since $\mathcal{O}_g$ is at most exponentially larger than $\mathcal{O}$, it follows that satisfiability in this fragment is in 2ExpTime.