COMPLEXITY THEORY

Lecture 5: Time Complexity and Polynomial Time

Markus Krötzsch
Knowledge-Based Systems

TU Dresden, 29th Oct 2019
Time Complexity
Measuring Complexity

Complexity Theory
Study the fine structure of decidable languages.

Goal
Classify languages by the amount of resources needed to solve them.

Resources
When dealing with Turing machines, we will primarily consider

- **time**: the running time of algorithms (steps on a Turing-machine)
- **space**: the amount of additional memory needed
  (cells on the Turing-tapes)
**Definition 5.1:** Consider a Turing machine $M$ and a function $f : \mathbb{N} \to \mathbb{R}^+$. 

1. $M$ is $f$-time bounded if it halts on every input $w \in \Sigma^*$ after $\leq f(|w|)$ steps.
2. $M$ is $f$-space bounded if it halts on every input $w \in \Sigma^*$ using $\leq f(|w|)$ cells on its tapes.

(Here we typically assume that Turing machines have a separate input tape that we do not count in measuring space complexity.)

**Notation 5.2:** Sometimes notations like “$f(n)$-time bounded” are used, assuming inputs to be of length $n$.

$\sim$ we use this when convenient, e.g., to write “$n^3$-bounded”
Big-O and Small-o

Algorithms are often judged by their asymptotic complexity, i.e., their behaviour in the limit.

We recall and extend the definition from Lecture 1:

**Definition 5.3:** The *Big-O notation* classifies functions using asymptotic upper bounds:

\[ f(n) = O(g(n)) \iff \exists c > 0 \exists n_0 \in \mathbb{N} \forall n > n_0 : f(n) \leq c \cdot g(n) \]

Then \( f \) is *asymptotically bounded* by \( g \) up to a constant factor.
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Then \( f \) is asymptotically bounded by \( g \) up to a constant factor.

**Definition 5.4:** The **small-o notation** classifies by a function that dominates them:

\[ f(n) = o(g(n)) \quad \text{iff} \quad \forall c > 0 \ \exists n_0 \in \mathbb{N} \ \forall n > n_0 : f(n) \leq c \cdot g(n) \]

Then \( f \) is asymptotically dominated by \( g \).
Relatives of the $O$ Notation

There are a number of further asymptotic notations besides Big-O and small-o. Their essence and underlying intuition is as follows:

<table>
<thead>
<tr>
<th>Notation</th>
<th>$C = \lim_{n \to \infty} \frac{f(n)}{g(n)}$</th>
<th>Intuition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f \in O(g)$</td>
<td>$C &lt; \infty$</td>
<td>“$f \leq g$”</td>
</tr>
<tr>
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**Note:** Both “$f \in O(g)$” and “$f = O(g)$” etc. are sometimes used in the literature, with the same intended meaning.
Relaxed Time and Space Bounds

We can use Big-O notation to generalise bounded TMs:

**Definition 5.5:** A Turing machine $\mathcal{M}$ is

1. $O(g(n))$-time bounded if it is $f$-time bounded for some $f$ with $f(n) = O(g(n))$
2. $O(g(n))$-space bounded if it is $f$-space bounded for some $f$ with $f(n) = O(g(n))$

**Notation 5.6:** We generally allow the use of $O(g(n))$ in place of a function $f(n)$ with analogous meaning.
Bounding TMs is the basis for both complexity theory and for studies of algorithmic complexity.

**Definition 5.7:** Let $f : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function.

(1) $\text{DTime}(f(n))$ is the class of all languages $L$ for which there is an $O(f(n))$-time bounded Turing machine deciding $L$.

(2) $\text{DSpace}(f(n))$ is the class of all languages $L$ for which there is an $O(f(n))$-space bounded Turing machine deciding $L$.

**Notation 5.8:** Sometimes $\text{Time}(f(n))$ is used instead of $\text{DTime}(f(n))$. 
The classes $\text{DTIME}(f)$ and $\text{DSPACE}(f)$ depend on

- details of the computational model
- details of the input encoding
- details of the implementation

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**Example 5.9:** A naive algorithm can perform matrix multiplication in $\text{DTIME}(n^3)$. Since many decades, researchers have been searching for better solutions: $\text{DTIME}(n^{2.808})$ [Strassen, 1969], $\text{DTIME}(n^{2.796})$ [Pan, 1978], $\text{DTIME}(n^{2.780})$ [Bini et al., 1979], $\text{DTIME}(n^{2.522})$ [Schönhage, 1981], $\text{DTIME}(n^{2.517})$ [Romani, 1982], $\text{DTIME}(n^{2.496})$ [Coppersmith & Winograd, 1981], $\text{DTIME}(n^{2.479})$ [Strassen, 1986], $\text{DTIME}(n^{2.376})$ [Coppersmith & Winograd, 1990], $\text{DTIME}(n^{2.374})$ [Stothers, 2010], and $\text{DTIME}(n^{2.373})$ [Williams, 2011].
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Defining Complexity Classes

Solution: Make complexity classes big enough to hide such details.

\[ P = \text{PTime} = \bigcup_{d \geq 1} \text{DTime}(n^d) \quad \text{polynomial time} \]

\[ \text{Exp} = \text{ExpTime} = \bigcup_{d \geq 1} \text{DTime}(2^{n^d}) \quad \text{exponential time} \]

\[ 2\text{Exp} = \text{2ExpTime} = \bigcup_{d \geq 1} \text{DTime}(2^{2^{n^d}}) \quad \text{double-exponential time} \]

\[ E = \text{ETime} = \bigcup_{d \geq 1} \text{DTime}(2^{d^n}) \quad \text{exp. time with linear exponent} \]

\[ L = \text{LogSpace} = \text{DSpace}(\log n) \quad \text{logarithmic space} \]

\[ \text{PSpace} = \bigcup_{d \geq 1} \text{DSpace}(n^d) \quad \text{polynomial space} \]

\[ \text{ExpSpace} = \bigcup_{d \geq 1} \text{DSpace}(2^{n^d}) \quad \text{exponential space} \]
Time Complexity Classes

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**Note:** Complexity classes are classes of languages.

**Observation:** The following relationships are clear from the definition:

\[ P \subseteq \text{ExpTime} \subseteq 2\text{ExpTime} \subseteq 3\text{ExpTime} \subseteq 4\text{ExpTime} \subseteq \ldots \]
A Hierarchy of Complexity Classes?

Many fundamental questions arise:

- Can we always solve more problems if we have more resources?

Discussing this further later on.

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A Hierarchy of Complexity Classes?

Many fundamental questions arise:

- Can we always solve more problems if we have more resources?
- If not, how much more resources do we need to be able to solve strictly more problems?

[...] discussed in future lectures
- How do we classify “efficient” in terms of complexity classes? [coming up next]
A Hierarchy of Complexity Classes?

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- If not, how much more resources do we need to be able to solve strictly more problems?
- How do the complexity classes relate to each other?
- Are there any tools by which we can show that a problem is in any of these classes but not in another?

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How is complexity affected by the chosen model of computation?

- Is $\text{DTime}(f)$ the same for multi-tape TMs?
- And how about non-deterministic TMs?
- Or TMs with a two-way infinite tape?
- Or random access machines?
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Many complexity classes are robust against many such variations

$\sim$ coming up next
Polynomial Time
An “intuitive” definition of “efficient”:

- Any linear time computation is “efficient”.
- Any program that
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This turns out to be equivalent to PTime.

\[
\text{PTime} := \bigcup_{d \geq 1} \text{DTime}(n^d)
\]

PTime serves as a mathematical model of “efficient” computation.
Robustness of the Definition

If PTime is to be the mathematical model of efficient computation, it should not depend on

- the exact computational model we are using,
- or how we encode the input (within reason).
Theorem 5.10 (Sipser, Theorem 7.8):
Consider a function \( f \) with \( f(n) \geq n \). Then, for every \( f(n) \)-time bounded \( k \)-tape Turing machine (\( k > 1 \)), there is an equivalent \( O(f^2(n)) \)-time bounded single-tape Turing machine.

Proof:
Simulate a multi-tape TM with a single-tape TM as shown in Lecture 2:
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Multi-Tape Turing Machines

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**Proof (cont.):** Then analyse how long this simulation really takes:

- Observation: the tapes can never have more than \( f(n) \) symbols on them.
- The simulation scans the whole tape once to find out what to do: \( O(f(n)) \) steps.
- Then it updates the whole tape in one pass: \( O(f(n)) \) steps.
- Sometimes the whole tape is shifted to make space: at most \( k \) times \( O(f(n)) \) steps.
- Overall: one step is simulated in \( O(f(n)) \) steps.
- Simulating \( f(n) \) such steps takes \( f(n) \cdot O(f(n)) = O(f^2(n)) \) steps.
- Tape initialisation takes another \( O(f(n)) \) (irrelevant).
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**Total simulation possible in $O(f^2(n))$.**
Let $\text{DTime}_k (f(n))$ denote “$\text{DTime}(f(n))$ for a $k$-tape TM”.

**Theorem 5.11:**

\[
\bigcup_{d \in \mathbb{N}} \text{DTime}(n^d) = \bigcup_{d \in \mathbb{N}} \text{DTime}_k (n^d) \text{ for every } k \geq 1
\]

**Proof:** The inclusion $\subseteq$ is clear.
The inclusion $\supseteq$ follows from the previous Theorem 5.10.

\[\square\]
P is robust against further models of computation:

1. We can simulate \( f(n) \) steps of a two-way infinite \( k \)-tape Turing-machine with an equivalent standard \( k \)-tape TM in \( O(f(n)) \) steps.

2. We can simulate \( f(n) \) steps of a RAM-machine with a 3-tape TM in \( O(f^3(n)) \) steps.

Vice-versa in \( O(f(n)) \) steps.

Consequences:

- \( \text{PTime} \) is the same for all these models (unlike linear time)
- The exponential time complexity classes are as robust as \( \text{P} \)

How about non-deterministic TMs?

It is unknown if \( \text{PTime} \) is robust against this, but most think it is not
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→ see next lectures
Linear Speed-Up

The Big-O notation in DTime hides arbitrary linear factors. Is it justified to rely on this for defining $P$?
Linear Speed-Up

The Big-O notation in DTime hides arbitrary linear factors. Is it justified to rely on this for defining P?

Yes, it turns out that we can make multi-tape TMs "arbitrarily fast":

**Theorem 5.12 (Linear Speed-Up Theorem):** Consider an $f(n)$-time bounded $k$-tape Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$ with $k > 1$.

Then, for every constant $c > 0$, there is a $(\frac{1}{c} \cdot f(n) + n + 2)$-time bounded $k$-tape TM $M' = (Q', \Sigma, \Gamma', \delta', q'_0, q'_{\text{accept}}, q'_{\text{reject}})$ that accepts the same language.
Proof (sketch): Let $\Gamma' := \Sigma \cup \Gamma^m$ where $m := \lceil 6c \rceil$. We construct $M'$ as follows:
**Linear Speed-Up (Proof)**

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**Step 1: Compress $M$’s input.**

Copy the input to tape 2, compressing $m$ symbols into one (i.e., each symbol corresponds to an $m$-tuple from $\Gamma^m$). This takes $n + 2$ steps.

For further details see Papadimitriou, Theorem 2.2. □
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**Step 2: Simulate $M$’s computation, $m$ steps at once.**

1. Read (in 4 steps) symbols to the left, right and the current position and “store” in $Q'$, using $|Q \times \{1, \ldots, m\}^k \times \Gamma^{3mk}|$ extra states.

2. Simulate (in 2 steps) the next $m$ steps of $M$ (as $M$ can only modify the current position and one of its neighbours)

3. $M'$ accepts (rejects) if $M$ accepts (rejects)

For further details see Papadimitriou, Theorem 2.2.

□
Different Encodings

Some simple observations:

(1) For any $n \in \mathbb{N}$, the length of the encoding of $n$ in base $b_1$ and base $b_2$ are related by a constant factor, for all $b_1, b_2 \geq 2$. 
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2. For any graph $G$, the length of its encoding as an
   - adjacency matrix
   - list of nodes + list of edges
   - adjacency list
   - ...

   are all polynomially related.
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Consequence:
PTime is the same for all these encodings (unlike linear time).
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However: This correspondence is not exact.

- When the degree of polynomials is very high, the time grows so quickly that in practice the problem is not solvable.
- The constants may also be very large
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And yet: For many concrete PTime-problems arising in practice, algorithms with moderate exponents and constants have been found.
Growth Rate of Some Functions

- $n!$
- $2^n$
- $n^3$
- $n^2$
- $n$

Time in $\mu s$:
- $10^0$
- $10^3$
- $10^6$
- $10^9$
- $10^{12}$
- $10^{15}$
- $10^{18}$
- $10^{21}$
- $10^{24}$

- 1 min.
- 1 day
- 1 year
- 1000 years
- Age of the universe

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Problems in P
Proving a Problem is in PTime

- The most direct way to show that a problem is in PTime is to exhibit a polynomial time algorithm that solves it.
- Even a naive polynomial-time algorithm often provides a good insight into how the problem can be solved efficiently.
- Because of robustness, we do not generally need to specify all the details of the machine model or the encoding.

\[ \leadsto \text{pseudo-code is sufficient} \]
Example: Satisfiability

Some of the most important problems concern logical formulae.

**Definition 5.13 (Propositional Logic Syntax):** Formulae of propositional logic are built up inductively

- (Propositional) Variables: $X_i$, $i \in \mathbb{N}$
- Boolean connectives: If $\varphi, \psi$ are propositional formulae then so are
  - $(\psi \lor \varphi)$
  - $(\psi \land \varphi)$
  - $\neg \varphi$
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- Boolean connectives: If \( \varphi, \psi \) are propositional formulae then so are
  - \( (\psi \lor \varphi) \)
  - \( (\psi \land \varphi) \)
  - \( \neg \varphi \)

**Example 5.14:** The following is a propositional logic formula:

\[
(X_1 \lor X_2 \lor \neg X_5) \land (\neg X_2 \lor X_4 \lor \neg X_5) \land (X_2 \lor X_3 \lor X_4)
\]
**Definition 5.15 (Conjunctive Normal Form):** A propositional logic formula $\varphi$ is in conjunctive normal form (CNF) if

$$\varphi = C_1 \land \cdots \land C_m$$

where each $C_i$ is a clause, that is, a disjunction of literals

$$C_i = (L_{i1} \lor \cdots \lor L_{ik})$$

and a literal is a variable $X_i$ or a negation $\neg X_i$ thereof.

A CNF $\varphi$ is in $k$-CNF is it has at most $k$ literals per clause.
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A CNF \( \varphi \) is in \( k \)-CNF if it has at most \( k \) literals per clause.

Example 5.16: The following formula is in 3-CNF:
\[
(X_1 \lor X_2 \lor \neg X_5) \land (\neg X_2 \lor \neg X_4 \lor \neg X_5) \land (X_2 \lor X_3 \lor X_4)
\]
Definition 5.17: A formula $\varphi$ is **satisfiable** if it is satisfied by an assignment that maps each variable in $\varphi$ to either 0 or 1 (and recursively defined for larger formulae as usual).

Specifically: A formula in CNF is satisfiable if there is an assignment $\beta$ for variables of $\varphi$ so that every clause contains at least

- one variable to which $\beta$ assigns 1, or
- one negated variable to which $\beta$ assigns 0.
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- one variable to which \( \beta \) assigns \( 1 \), or
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**Example 5.18:** The formula

\[
(X_1 \lor X_2 \lor \neg X_5) \land (\neg X_2 \lor \neg X_4 \lor \neg X_5) \land (X_2 \lor X_3 \lor X_4)
\]

is satisfied by \( \{X_1 \mapsto 1, \ X_2 \mapsto 0, \ X_3 \mapsto 1, \ X_4 \mapsto 0, \ X_5 \mapsto 1\} \).
The Satisfiability Problem

Related to propositional formulae, the following two problems are the most important:

\[ \text{\textbf{Sat}} \]
Input: Propositional formula \( \varphi \) in CNF
Problem: Is \( \varphi \) satisfiable?

\[ \text{\textbf{k-Sat}} \]
Input: Propositional formula \( \varphi \) in \( k \)-CNF
Problem: Is \( \varphi \) satisfiable?
2-Sat is Polynomial

**Theorem 5.19:** 2-Sat ∈ PTime.
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Proof: The following algorithm solves the problem in polynomial time.

**Main:** Input $\Gamma$ in CNF

$bcp(\Gamma)$

if conflict return UNSAT

while $\Gamma \neq \emptyset$ do

choose var. $X$ from $\Gamma$

set $\Gamma' := \Gamma$

assign($\Gamma$, $X$, 1)

$bcp(\Gamma)$

if conflict

$\Gamma := \Gamma'$

assign($\Gamma$, $X$, 0)

$bcp(\Gamma)$

if conflict

return UNSAT

While $\Gamma$ contains unit-clause $C$ do

if $C = \{X\}$ assign($\Gamma$, $X$, 1)

if $C = \{\neg X\}$ assign($\Gamma$, $X$, 0)

if $\Gamma$ contains empty clause return conflict

Assign($\Gamma$, $X$, $c$)

if $c = 1$

remove from $\Gamma$ all clauses $C$ with $X \in C$

remove $\neg X$ from all remaining clauses

if $c = 0$

remove from $\Gamma$ all clauses $C$ with $\neg X \in C$

remove $X$ from all remaining clauses

\[\square\]
Polynomial-Time Reductions

As for decidability we can use reductions to show membership in PTime.

**Definition 5.20:** A language $L_1 \subseteq \Sigma^*$ is polynomially many-one reducible to $L_2 \subseteq \Sigma^*$, denoted $L_1 \leq_p L_2$, if there is a polynomial-time computable function $f$ such that for all $w \in \Sigma^*$

$$w \in L_1 \text{ if and only if } f(w) \in L_2.$$
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**Theorem 5.21:** If $L_1 \leq_p L_2$ and $L_2 \in \text{PTime}$ then $L_1 \in \text{PTime}$.

**Proof:** The sum and composition of polynomials is a polynomial. □
Reductions in PTime

All non-trivial members of PTime can be reduced to each other:

**Theorem 5.22:** If $B$ is any language in $P$, $B \neq \emptyset$, and $B \neq \Sigma^*$, then $A \leq_p B$ for any $A \in P$. 
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**Proof:** Choose $w \in B$ and $w' \notin B$.

Define the function $f$ by setting

$$f(x) := \begin{cases} w & \text{if } x \in A \\ w' & \text{if } x \notin A \end{cases}$$

Since $A \in P$, this function $f$ is computable in polynomial time, and it is a reduction from $A$ to $B$. □
Definition 5.23 (Vertex Colouring): A vertex colouring of $G$ with $k$ colours is a function

$$c : V(G) \rightarrow \{1, \ldots, k\}$$

such that adjacent nodes have different colours, that is:

$$\{u, v\} \in E(G) \implies c(u) \neq c(v)$$

$k$-Colouring

Input: Graph $G$, $k \in \mathbb{N}$

Problem: Does $G$ have a vertex colouring with $k$ colours?

For $k = 2$ this is the same as Bipartite.
Reducing 2-Colourability to 2-Sat

**Theorem 5.24:** \( \text{2-COLOURABILITY} \leq_p \text{2-SAT} \), and therefore \( \text{2-COLOURABILITY} \in \text{P} \).

**Proof:** We define a reduction as follows: Given graph \( G \):

- For each vertex \( v \in V(G) \) of the graph, introduce new variable \( X_v \).
- For each \( \{u, v\} \in E(G) \), add clauses \( (X_u \lor X_v) \) and \( (\neg X_u \lor \neg X_v) \). This is obviously computable in polynomial time.

We check that it is a reduction:

- If \( G \) is 2-colourable, use colouring to assign truth values. (One colour is true, the other false)
- If the formula is satisfiable, the truth assignment defines valid 2-colouring. For every edge \( \{u, v\} \in E(G) \), one variable \( X_u, X_v \) must be set to true, the other to false.

\( \square \)
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□
A large class of languages is generally tractable:

**Theorem 5.25:** If \( L \) is a finite language, then it is decided by an \( O(1) \)-time bounded TM. In other words, all finite languages are decidable in constant time (and hence also in polynomial time).
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**Proof:**

- As \( L \) is finite, there is a maximum length \( m \) of words in \( L \).
- Read the input up to the first \( m \) letters.
- The state space contains a table containing the correct result for all such inputs.
- All other inputs are rejected.

□
The next result is an example of a theorem that proves the existence of a P algorithm in cases where we do not know what this algorithm is.

**Example 5.26:** Let $L$ be the language that contains all correct sentences from the following set:

\{
  \text{“P is the same as NP”, “P is not the same as NP”}
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Then $L$ is decidable in constant time.
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However, we don’t know which constant-time algorithm decides it.
A Note on Constructiveness

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**Example 5.26:** Let $L$ be the language that contains all correct sentences from the following set:

\{'P is the same as NP', 'P is not the same as NP'\}

Then $L$ is decidable in constant time. However, we don’t know which constant-time algorithm decides it.

Non-constructiveness:

- We can prove that there is a correct polynomial time algorithm.
- We cannot construct such an algorithm.

Such solutions are called non-constructive.
Theorem 5.27: It is decidable in polynomial-time ($O(n^3)$) if a graph can knotlessly be embedded into 3-dimensional space.

Proof (sketch):
• Robertson & Seymour proved a general result that implies the existence of a finite set of forbidden structures in knotlessly embeddable graphs.
• For each of these forbidden structures we can test whether a graph contains one of them in time $O(n^3)$.
• Hence, to decide if a graph is knotlessly embeddable, we only need to test for each of the finitely many forbidden structures, whether they occur in the graph. This yields a cubic time decision procedure. □

However: We do not currently know what these structures are.
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Summary and Outlook

Complexity classes are based on asymptotic resource estimates, further generalised by considering general classes of bounds (e.g., all polynomial functions).

Ignoring constant factors is justified due to Linear Speedup.

P is the most common approximation of “efficient”

Polynomial many-one reductions are used to show membership in P.

What’s next?

• NP
• Hardness and completeness
• More examples of problems