Exercise 2.1. Using an oracle that decides the halting problem, construct a decider for the language \( \{ \langle M, w \rangle \mid M \text{ is a TM that accepts } w \} \).

Exercise 2.2. A useless state in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a Turing machine has any useless states. Show that this language is undecidable.

Exercise 2.3. Show the following: “If a language \( L \) is Turing-recognisable and \( \overline{L} \) is many-one reducible to \( L \), then \( L \) is decidable.”

Exercise 2.4. Let
\[
L = \{ \langle M \rangle \mid M \text{ a TM that accepts } w^r \text{ whenever it accepts } w \},
\]
where \( w^r \) is the word \( w \) reversed. Show that \( L \) is undecidable.

Exercise 2.5. Consider the following languages \( L \) and \( L' \):
\[
L = \{ \langle M, w \rangle \mid M \text{ is a TM that accepts } w \}
\]
\[
L' = \{ \langle M \rangle \mid M \text{ is a TM that does not accept any word} \}
\]
Show that there cannot exist a reduction from \( L \) to \( L' \).

Exercise 2.6. Show that every Turing-recognisable language can be mapping-reduced to the following language.
\[
\{ \langle M, w \rangle \mid M \text{ is a TM that accepts the word } w \} \]