Decomposing Finite Closure Operators by Attribute Exploration

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Motivation

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Implications and formal contexts

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NextClosure	input: M, $X \mapsto X''$
Close-by-One	input: (G, M, I)

Can Close-by-One be applied to an arbitrary closure operator c?

Decomposing Closure Operators

Definition

Let M be a finite set and let $c : \mathfrak{P}(M) \to \mathfrak{P}(M)$. Then c is a *closure operator on M* if and only if

- c is monotone: $\forall A, B \subseteq M : A \subseteq B \implies c(A) \subseteq c(B)$,
- c is extending: $\forall A \subseteq M : A \subseteq c(A)$,
- c is idempotent: $\forall A \subseteq M : c(c(A)) = c(A)$.

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Definition

A formal context $\mathbb{K} = (G, M, I)$ is a *decomposition of* c if and only if

$$\operatorname{Int}(\mathbb{K}) = c[\mathfrak{P}(M)]$$

i.e., the intents of $\mathbb{K}$ are precisely the closed sets of c.

Trivial Decomposition

Lemma

The formal context

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Definition

The formal context $\mathbb{K}_c$ is called the *trivial decomposition of c*.

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But what about the smallest possible decomposition?

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Can we compute the canonical decomposition without computing the trivial one?

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So can we do attribute exploration?

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  Therefore provide $c(A)$ as a counterexample.

Now attribute exploration can be used to compute a decomposition of c!

But this will not always yield the canonical decomposition of c.

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Lemma

Let $N \in c[\mathfrak{P}(M)]$. Then N is infimum-irreducible in $(c[\mathfrak{P}(M)], \subseteq)$ if and only if there exists an $n \in M \setminus N$ such that N is maximal in $(c[\mathfrak{P}(M)], \subseteq)$ with respect to not containing n.

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Idea

If $B \not\subseteq c(A)$, then choose $x \in B \setminus c(A)$ and maximize $N \supseteq c(A)$ with respect to $x \notin N$. Then call N a *maximal counterexample* for $A \to B$.

Decomposition by Attribute Exploration

Corollary

Attribute exploration using maximal counterexamples yields as the final context of the exploration the canonical decomposition of c.

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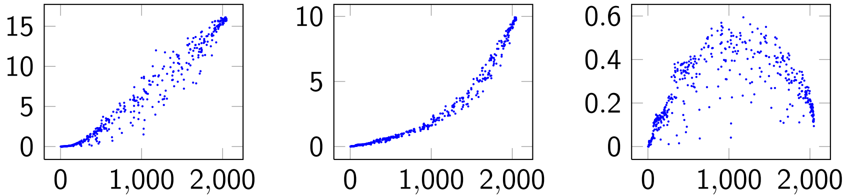
- Compute the canonical decomposition of $X \mapsto X''$ using
  - the naive algorithm
  - simple attribute exploration
  - attribute exploration with maximal counterexamples

Experimental Results

Number of intents vs. Runtime.



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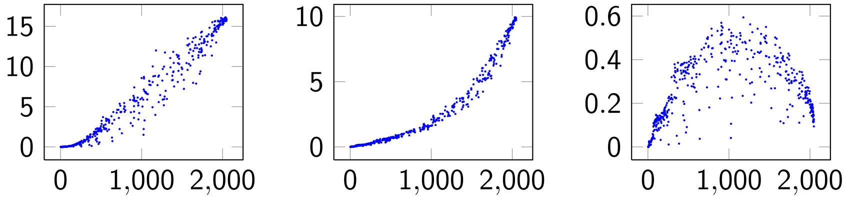
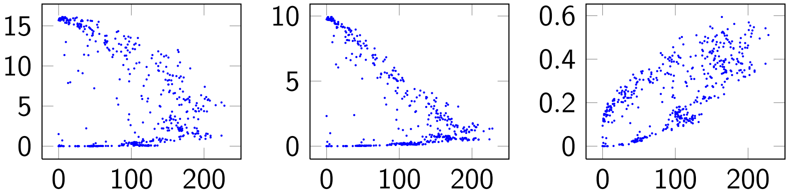
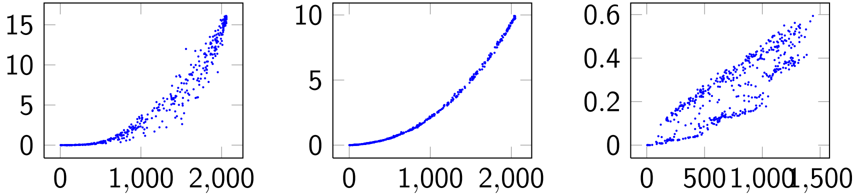


Number of pseudo-intents vs. Runtime.



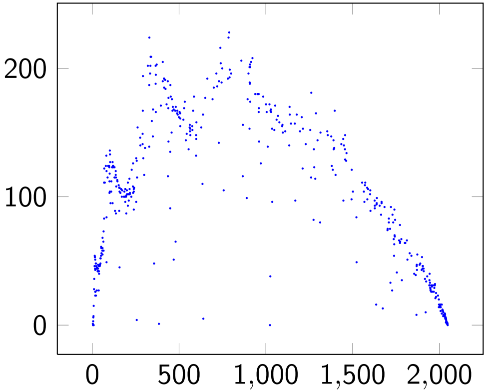
Experimental Results (cont.)

Calls of c vs. Runtime.



An Unexpected Observation

Number of intents vs. Number of pseudo-intents.



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  - Canonical decomposition might be exponentially large in $|M|$
  - How to represent c?
- Correlation between number of intents and number of pseudo-intents?

Thank You.