When Horn is All You Need

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1984 Congressional Voting Records

Implication Basis

- $h \to D$
- $m \to D$
- $D, R \to \bot$

Implications don't capture that

- everyone is a Democrat or a Republican;
- a Democrat would not vote against both h and m.

#Congressmen	*R*	*D*	*h*	*m*
123	×			
116		×	×	×
72		×		×
40		×	×	
...	...			

$A \to B$ holds if every object with A has B.

- R: Republican
- D: Democrat
- h: handicapped infants
- m: mx-missile

1984 Congressional Voting Records

Implication Basis

- $h \to D$
- $m \to D$
- $D, R \to \bot$

The Logic of the Context

$$\Phi(\mathbb{K}) = (\ \ R \land \neg D \land \neg h \land \neg m) \lor$$
$$(\neg R \land \ \ D \land \ \ h \land \ \ m) \lor$$
$$(\neg R \land \ \ D \land \neg h \land \ \ m) \lor$$
$$(\neg R \land \ \ D \land \ \ h \land \neg m)$$

#Congressmen	R	D	h	m
123	×			
116		×	×	×
72		×		×
40		×	×	

$A \to B$ holds if every object with A has B.

- R: Republican
- D: Democrat
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- m: mx-missile

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Implication Basis

$$\Psi(\mathbb{K}) = (h \rightarrow D) \wedge$$
$$(m \rightarrow D) \wedge$$
$$(D \wedge R \rightarrow \bot)$$

The Logic of the Context

$$\Phi(\mathbb{K}) = (\ R \wedge \neg D \wedge \neg h \wedge \neg m) \vee$$
$$(\neg R \wedge \ D \wedge \ h \wedge \ m) \vee$$
$$(\neg R \wedge \ D \wedge \neg h \wedge \ m) \vee$$
$$(\neg R \wedge \ D \wedge \ h \wedge \neg m)$$

#Congressmen	*R*	*D*	*h*	*m*
123	×			
116		×	×	×
72		×		×
40		×	×	

- R: Republican
- D: Democrat
- h: handicapped infants
- m: mx-missile

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Implication Basis

$$\Psi(\mathbb{K}) = (\neg h \vee D) \wedge (\neg m \vee D) \wedge (\neg D \vee \neg R)$$

Horn CNF

(at most one non-negated literal per clause)

$$\Phi(\mathbb{K}) \models \Psi(\mathbb{K})$$

The Logic of the Context

$$\Phi(\mathbb{K}) = (R \wedge \neg D \wedge \neg h \wedge \neg m) \vee (\neg R \wedge D \wedge h \wedge m) \vee (\neg R \wedge D \wedge \neg h \wedge m) \vee (\neg R \wedge D \wedge h \wedge \neg m)$$

Full DNF

$$\Psi(\mathbb{K}) \not\models \Phi(\mathbb{K})$$

#Congressmen	R	D	h	m
123	×			
116		×	×	×
72		×		×
40		×	×	

- R: Republican
- D: Democrat
- h: handicapped infants
- m: mx-missile

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Implication Basis

- $h \to \neg R$
- $m \to \neg R$
- $R \to \neg h, \neg m$
- $h, \neg h \to \bot$
- $\neg R, \neg h \to m$
- $\neg R, \neg m \to h$
- $\neg h, \neg m \to R$
- $m, \neg m \to \bot$

Adding negations

Literal context

#Congressmen	R	¬R	h	¬h	m	¬m
123	×			×		×
116		×	×		×	
72		×		×	×	
40		×	×			×

- R: Republican
- $\neg R$: Democrat
- h: handicapped infants
- m: mx-missile

1984 Congressional Voting Records

Implication Basis

- $h \to \neg R$
- $\neg R, \neg h \to m$
- $m \to \neg R$
- $\neg R, \neg m \to h$
- $R \to \neg h, \neg m$
- $\neg h, \neg m \to R$
- $h, \neg h \to \bot$
- $m, \neg m \to \bot$

An implication basis of a literal context can be *logically* redundant.

It fully captures the logic of the context: its models correspond precisely to the object intents of the context.

Adding negations

Literal context

#Congressmen	R	$\neg R$	h	$\neg h$	m	$\neg m$
123	×			×		×
116		×	×		×	
72		×		×	×	
40		×	×			×

- R: Republican
- $\neg R$: Democrat
- h: handicapped infants
- m: mx-missile

$$\Phi(\mathbb{K}) \equiv \Psi(\mathbb{K})$$

Literal Contexts

With atoms and their negations, we can express an arbitrary clause:

$$m_1 \vee \ldots \vee m_k \vee \neg m_{k+1} \vee \ldots \vee \neg m_n$$

$$\Updownarrow$$

$$\neg m_1, \ldots, \neg m_k, m_{k+1}, \ldots, m_n \quad \rightarrow \quad \bot$$

and, hence, an arbitrary CNF and an arbitrary Boolean function.

$\bot$ is a shorthand for $\{m_1, \neg m_1\}$

Disjunctive Contexts

With atoms and disjunctions over them, we can express an arbitrary clause:

$$m_1 \vee \ldots \vee m_k \vee \neg m_{k+1} \vee \ldots \vee \neg m_n$$

$$\Updownarrow$$

$$m_{k+1}, \ldots, m_n \quad \rightarrow \quad (m_1 \vee \ldots \vee m_k)$$

and, hence, an arbitrary CNF and an arbitrary Boolean function.

$\bigvee \varnothing = \bot$

In a disjunctive context, $M = \left\{ \bigvee Q \mid Q \subseteq P \right\}$ for a set P of propositional atoms.

When Horn Is All You Need

φ is valid in $\mathbb{K}$ if
$\Phi(\mathbb{K}) \models \varphi$

- If the attributes of a context $\mathbb{K}$ are
  - propositional atoms and their negations
  - or all possible disjunctions of propositional atoms,
- then the validity of any formula in $\mathbb{K}$ can be determined from its implication basis.

- We call attribute sets with this property implication-exact.

Coherent Attribute Sets

An attribute subset $C \subseteq M$ is coherent if $\left(\bigwedge_{m \in C} m\right) \wedge \left(\bigwedge_{m \in M \setminus C} \neg m\right)$ is satisfiable.

- $\{\neg R, h, h \vee m, m\}$ and $\{\neg R\}$ are coherent
- $\varnothing$, $\{R, \neg R\}$, and $\{R, h\}$ are not coherent
- If M is implication-exact, then M is not coherent and $\bigwedge M \equiv \bot$.

#Congressmen	R	¬R	h	h ∨ m	m
123	×				
116		×	×	×	×
72		×		×	×
40		×	×	×	

Coherent Attribute Sets

- Object intents must be coherent.
- For $C \subseteq M$, define

$$\varphi_{\overline{C}} := \Big(\bigvee_{m \in C} \neg m\Big) \vee \Big(\bigvee_{m \in M \setminus C} m\Big)$$

- For coherent $C_1 \neq C_2$:

$$\varphi_{\overline{C_1}} \not\equiv \varphi_{\overline{C_2}} \qquad C_1 \not\models \varphi_{\overline{C_1}} \qquad C_2 \models \varphi_{\overline{C_1}}$$

- Contexts with different sets of object intents validate different sets of formulae.

Implication-Exact Attribute Sets

Necessary Conditions

- Contexts with different sets of object intents validate different sets of formulae.
- Assume that the attribute set M is implication-exact.
- A context can have any collection of coherent sets as its object intents.
- Contexts with different sets of object intents have different implication sets.
- No coherent set is equal to the intersection of other coherent sets.
  - Removing any object intent must change the implication set.

Implication-Exact Attribute Sets

Necessary Conditions

- No coherent set is equal to the intersection of other coherent sets.
  - Removing any object intent must change the implication set.
- Every (object-clarified) context with an implication-exact attribute set is object-reduced.
- For coherent $C \subseteq M$, define

$$Exc(C) := C \to \underbrace{\bigcap \{D \mid C \subset D \text{ and } D \text{ is coherent}\}}_{\neq C}$$

$\bigcap \varnothing = M$

- Coherent set C is the only coherent set not respecting $Exc(C)$.

Implication-Exact Attribute Sets

Sufficient Conditions

- Assume that, for every coherent C, it is the only coherent set not respecting $Exc(C)$.
- For a context $\mathbb{K} = (G, M, I)$, define

$$\Sigma_{\mathbb{K}} = \{Exc(C) \mid C \text{ is coherent but } g' \neq C \text{ for all } g \in G\}$$

- We have $\Phi(\mathbb{K}) \equiv \Sigma_{\mathbb{K}} \equiv \Psi(\mathbb{K})$.
- M is implication-exact.

When Horn Is All You Need

The following conditions are equivalent:

- An attribute set M is implication-exact
- $\Phi(\mathbb{K}) \equiv \Psi(\mathbb{K})$ for every context $\mathbb{K}$ with attribute set M
- $C \neq \bigcap \{D \mid C \subset D \text{ and } D \text{ is coherent}\}$ for every coherent $C \subseteq M$
  - That is, the implication $Exc(C)$ is respected by all coherent sets except C

When Horn Is All You Need

Example

- Propositional atoms: $P \cup Q, \qquad P \neq \varnothing$
- Attribute set:

$$M = \{\underbrace{p_1, \neg p_1, \ldots, p_k, \neg p_k,}_{P \cup \{\neg p \mid p \in P\}} \quad \underbrace{q_1, q_2, (q_1 \vee q_2), \ldots, q_\ell, (q_1 \vee q_\ell), (q_2 \vee q_\ell), \ldots, (q_1 \vee \ldots \vee q_\ell)}_{\{\bigvee A \mid A \subseteq Q,\, A \neq \varnothing\}}\}$$

- Let $C \subseteq M$ be coherent.

C includes p_i or $\neg p_i$ for all i

- If $Q \subseteq C$, then C has no coherent proper supersets.
- Otherwise, every coherent proper superset of C includes $\bigvee(Q \backslash C)$.

$\bigvee(Q \backslash C) \notin C$

- $C \neq \bigcap\{D \mid C \subset D \text{ and } D \text{ is coherent}\}$, and M is implication-exact.
- It follows that $\{p_1, \neg p_1, \ldots, p_k, \neg p_k, \quad q\}$ is also implication-exact (for any q).

Implication-Exact Attribute Sets

Concept Lattices

- A finite lattice is isomorphic to the concept lattice of a literal context if and only if it is atomistic and $\sqcap$-complemented [1].
- The concept lattice of a disjunctive context is distributive. Every distributive lattice is isomorphic to the concept lattice of a disjunctive context.
- Can anything be said about an arbitrary implication-exact attribute set?

[1] Rodríguez-Jiménez, J.M., Cordero, P., Enciso, M., Rudolph, S.: Concept lattices with negative information: A characterization theorem. Inf. Sci. **369**, 51–62 (2016)

Logical Basis

Implication Basis

- $h \to \neg R$
- $m \to \neg R$
- $R \to \neg h, \neg m$
- $h, \neg h \to \bot$
- $\neg R, \neg h \to m$
- $\neg R, \neg m \to h$
- $\neg h, \neg m \to R$
- $m, \neg m \to \bot$

An implication basis of a logical context can be *logically* redundant.

How to (efficiently) compute a (compact) logically non-redundant basis?

Literal context

#Congressmen	R	¬R	h	¬h	m	¬m
123	×			×		×
116		×	×		×	
72		×		×	×	
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