TECHNISCHE UNIVERSITÄT DRESDEN

International Center for Computational Logic

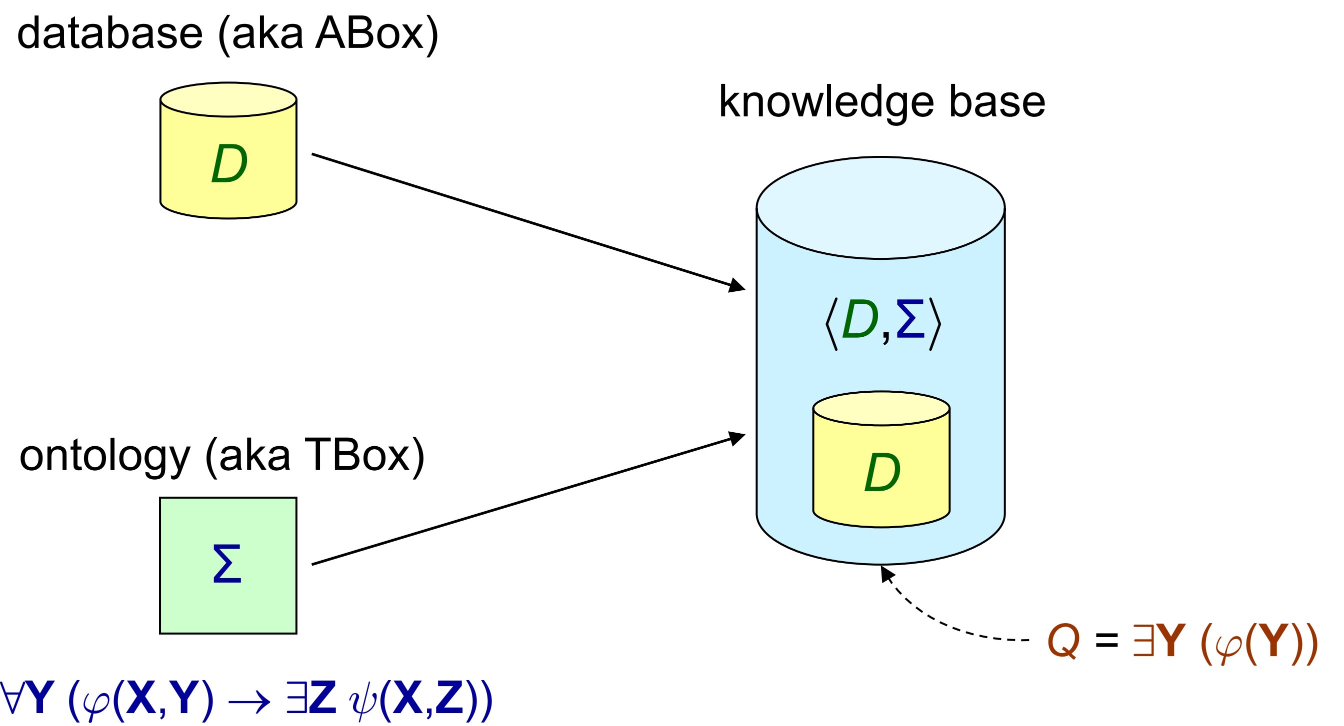
Computational Logic ∴ Group

Sebastian Rudolph
International Center for Computational Logic
TU Dresden

Existential Rules – Lecture 8

Adapted from slides by Andreas Pieris and Michaël Thomazo
Winter Term 2025/26

BCQ-Answering: Our Main Decision Problem



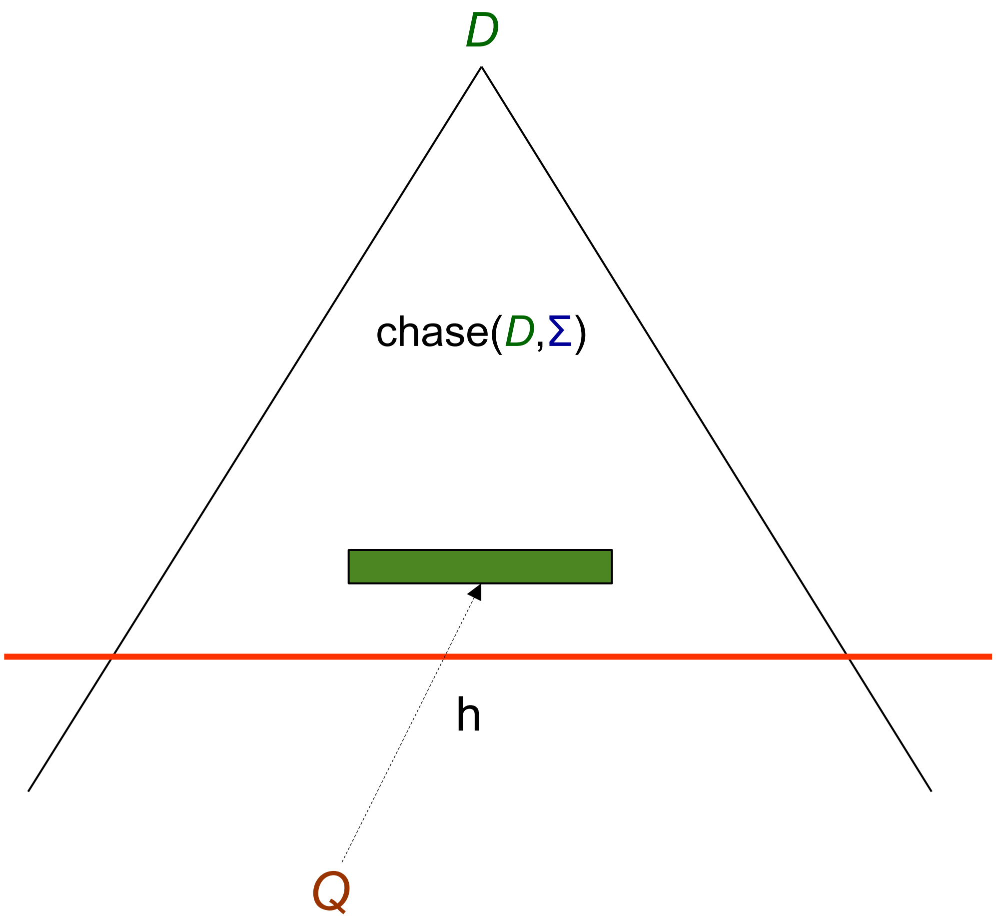
decide whether $D \wedge \Sigma \models Q$

Sum Up

	Data Complexity	
FULL	PTIME-c	Naïve algorithm
		Reduction from Monotone Circuit Value problem
ACYCLIC	in LOGSPACE	**Second part of our course**
LINEAR		

	Combined Complexity	
FULL	EXPTIME-c	Naïve algorithm
		Simulation of a deterministic exponential time TM
ACYCLIC	NEXPTIME-c	Small witness property
		Reduction from Tiling problem
LINEAR	PSPACE-c	Level-by-level non-deterministic algorithm
		Simulation of a deterministic polynomial space TM

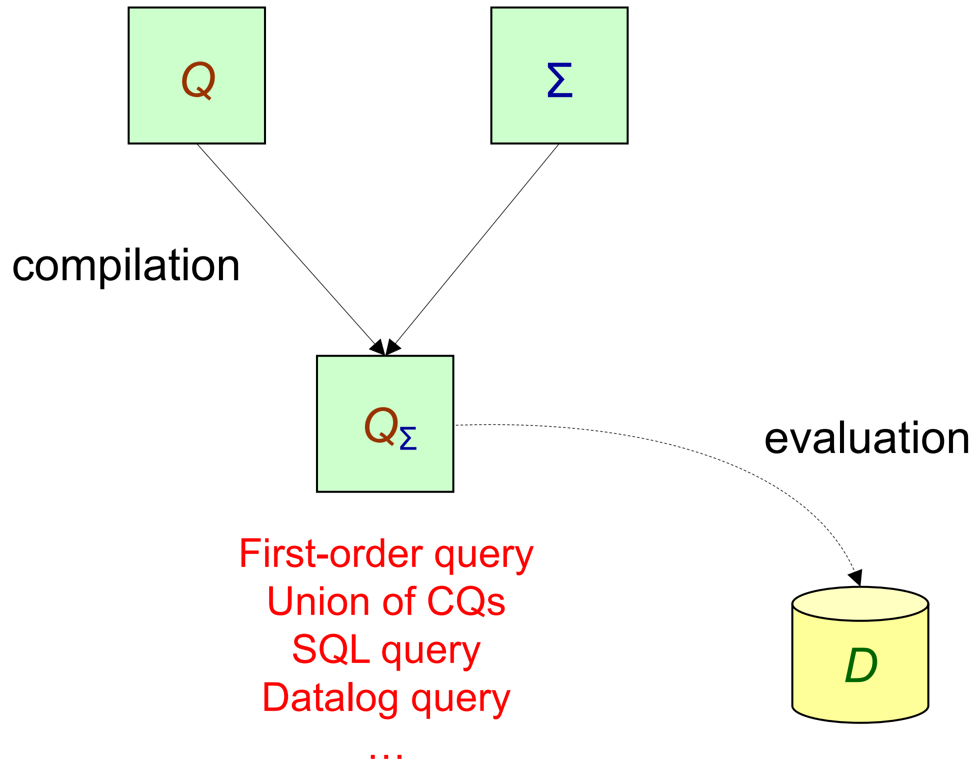
Forward Chaining Techniques



Useful techniques for establishing optimal upper bounds

…but not practical - we need to store instances of very large size

Query Rewriting



$\forall D \; : \; D \wedge \Sigma \models Q \quad \Leftrightarrow \quad D \models Q_{\Sigma}$

evaluated and optimized by

exploiting existing technology

Query Rewriting: Formal Definition

Consider a class of existential rules $\mathcal{L}$, and a query language $\mathcal{Q}$.

BCQ-Answering under $\mathcal{L}$ is $\mathcal{Q}$-rewritable if, for every $\Sigma \in \mathcal{L}$ and BCQ Q,

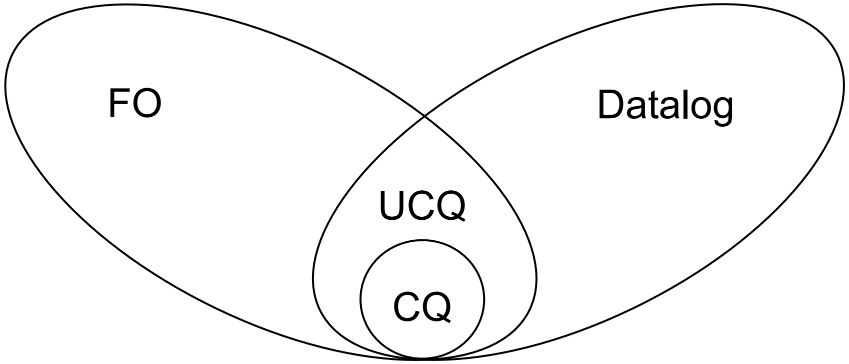
we can construct a query $Q_\Sigma \in \mathcal{Q}$ such that,

for every database D, $D \wedge \Sigma \models Q$ iff $D \models Q_\Sigma$

NOTE: The construction of Q_Σ is database-independent – the pure approach to query rewriting

Target Query Language

we target the weakest query language



	CQ	UCQ	FO	Datalog
FULL	✗	✗	✗	✓
ACYCLIC	✗	✓	✓	✓
LINEAR	✗	✓	✓	✓

UCQ-Rewritings

- The standard algorithm for computing UCQ-rewritings performs an exhaustive application of the following two steps:
    1. Rewriting
    2. Minimization

- The standard algorithm is designed for normalized existential rules, where only one atom appears in the head

Normalization Procedure

$$\forall X \forall Y\ (\varphi(X,Y) \rightarrow \exists Z\ (P_1(X,Z) \wedge \ldots \wedge P_n(X,Z)))$$

↓

$$\forall X \forall Y\ (\varphi(X,Y) \rightarrow \exists Z\ \mathit{Auxiliary}(X,Z))$$

$$\forall X \forall Z\ (\mathit{Auxiliary}(X,Z) \rightarrow P_1(X,Z))$$

$$\forall X \forall Z\ (\mathit{Auxiliary}(X,Z) \rightarrow P_2(X,Z))$$

…

$$\forall X \forall Z\ (\mathit{Auxiliary}(X,Z) \rightarrow P_n(X,Z))$$

NOTE 1: Acyclicity and linearity are preserved

NOTE 2: We obtain an equivalent set w.r.t. query answering (not logically equivalent)

UCQ-Rewritings

- The standard algorithm for computing UCQ-rewritings performs an exhaustive application of the following two steps:
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- The standard algorithm is designed for normalized existential rules, where only one atom appears in the head

Rewriting Step

Σ = {∀X∀Y (*project*(X) ∧ *inArea*(X,Y) → ∃Z *hasCollaborator*(Z,Y,X))}

Q = ∃A∃B *hasCollaborator*(A,db,B)

g = {X→ B, Y→ db, Z→ A}

hasCollaborator(A,db,B)

Thus, we can simulate a “backward chase step” by a resolution step

Q_Σ = ∃A∃B *hasCollaborator*(A,db,B)

∨

∃B (*project*(B) ∧ *inArea*(B,db))

Unsound Rewritings

$\Sigma = \{\forall X \forall Y\ (project(X) \land inArea(X,Y) \rightarrow \exists Z\ hasCollaborator(Z,Y,X))\}$

$Q = \exists B\ hasCollaborator(c,db,B)$

$g = \{X \rightarrow B,\ Y \rightarrow db,\ Z \rightarrow c\}$

$hasCollaborator(c,db,B)$

After applying the rewriting step we obtain the following UCQ

$$Q_\Sigma = \exists B\ hasCollaborator(c,db,B)$$

$$\lor$$

$$\exists B\ (project(B) \land inArea(B,db))$$

Unsound Rewritings

$\Sigma = \{\forall X \forall Y\, (\mathit{project}(X) \land \mathit{inArea}(X,Y) \rightarrow \exists Z\, \mathit{hasCollaborator}(Z,Y,X))\}$

$Q = \exists B\, \mathit{hasCollaborator}(c,db,B)$

$$Q_\Sigma = \exists B\, \mathit{hasCollaborator}(c,db,B) \lor \exists B\, (\mathit{project}(B) \land \mathit{inArea}(B,db))$$

- Consider the database $D = \{\mathit{project}(a), \mathit{inArea}(a,db)\}$
- Clearly, $D \models Q_\Sigma$
- However, $D \land \Sigma$ does not entail Q since there is no way to obtain an atom of the form *hasCollaborator*(c,db,_) during the chase

Unsound Rewritings

$\Sigma = \{\forall X \forall Y\ (\mathit{project}(X) \wedge \mathit{inArea}(X,Y) \rightarrow \exists Z\ \mathit{hasCollaborator}(Z,Y,X))\}$

$Q = \exists B\ \mathit{hasCollaborator}(c,db,B)$

$$Q_\Sigma = \exists B\ \mathit{hasCollaborator}(c,db,B) \vee \exists B\ (\mathit{project}(B) \wedge \mathit{inArea}(B,db))$$

the information about the constant c in the original query is lost after the application of the rewriting step since c is unified with an ∃-variable

Unsound Rewritings

Σ = {∀X∀Y (*project*(X) ∧ *inArea*(X,Y) → ∃Z *hasCollaborator*(Z,Y,X))}

Q = ∃B *hasCollaborator*(B,db,B)

g = {X→ B, Y→ db, Z→ B}

hasCollaborator(B,db,B)

After applying the rewriting step we obtain the following UCQ

Q_Σ = ∃B *hasCollaborator*(B,db,B)

∨

∃B (*project*(B) ∧ *inArea*(B,db))

Unsound Rewritings

$\Sigma = \{\forall X \forall Y\ (\mathit{project}(X) \wedge \mathit{inArea}(X,Y) \rightarrow \exists Z\ \mathit{hasCollaborator}(Z,Y,X))\}$

$Q = \exists B\ \mathit{hasCollaborator}(B,db,B)$

$$Q_\Sigma = \exists B\ \mathit{hasCollaborator}(c,db,B) \vee \exists B\ (\mathit{project}(B) \wedge \mathit{inArea}(B,db))$$

- Consider the database $D = \{\mathit{project}(a), \mathit{inArea}(a,db)\}$
- Clearly, $D \vDash Q_\Sigma$
- However, $D \wedge \Sigma$ does not entail Q since there is no way to obtain an atom of the form $\mathit{hasCollaborator}(t,db,t)$ during the chase

Unsound Rewritings

$\Sigma = \{\forall X \forall Y\ (\mathit{project}(X) \land \mathit{inArea}(X,Y) \rightarrow \exists Z\ \mathit{hasCollaborator}(Z,Y,X))\}$

$Q = \exists B\ \mathit{hasCollaborator}(B,db,B)$

$$Q_\Sigma = \exists B\ \mathit{hasCollaborator}(c,db,B) \lor \exists B\ (\mathit{project}(B) \land \mathit{inArea}(B,db))$$

the fact that B in the original query participates in a join is lost after the application of the rewriting step since B is unified with an ∃-variable

Applicability Condition

Consider a BCQ Q, an atom α in Q, and a (normalized) rule σ.

We say that σ is applicable to α if the following conditions hold:

1. head(σ) and α unify via h : terms(head(σ)) $\rightarrow$ terms(α)

2. For every variable X in head(σ), if h(X) is a constant, then X is a $\forall$-variable

3. For every variable X in head(σ), if h(X) = h(Y), where Y is a shared variable of α, then X is a $\forall$-variable

4. If X is an $\exists$-variable of head(σ), and Y is a variable in head(σ) such that X $\neq$ Y, then h(X) $\neq$ h(Y)

...but, although this is crucial for soundness, it may destroy completeness

Incomplete Rewritings

$\Sigma = \{\forall X \forall Y\ (\mathit{project}(X) \wedge \mathit{inArea}(X,Y) \rightarrow \exists Z\ \mathit{hasCollaborator}(Z,Y,X)),$
$\forall X \forall Y \forall Z\ (\mathit{hasCollaborator}(X,Y,Z) \rightarrow \mathit{collaborator}(X))\}$

$Q = \exists A \exists B \exists C\ (\mathit{hasCollaborator}(A,B,C) \wedge \mathit{collaborator}(A))$

$$Q_\Sigma = \exists A \exists B \exists C\ (\mathit{hasCollaborator}(A,B,C) \wedge \mathit{collaborator}(A))$$
$$\vee$$
$$\exists A \exists B \exists C \exists E \exists F\ (\mathit{hasCollaborator}(A,B,C) \wedge \mathit{hasCollaborator}(A,E,F))$$

- Consider the database $D = \{\mathit{project}(a), \mathit{inArea}(a,db)\}$
- Clearly, chase(D,Σ) = $D \cup \{\mathit{hasCollaborator}(z,db,a), \mathit{collaborator}(z)\} \models Q_\Sigma$
- However, D does not entail Q_Σ

Incomplete Rewritings

$\Sigma = \{\forall X \forall Y\ (\mathit{project}(X) \wedge \mathit{inArea}(X,Y) \rightarrow \exists Z\ \mathit{hasCollaborator}(Z,Y,X)),$

$\forall X \forall Y \forall Z\ (\mathit{hasCollaborator}(X,Y,Z) \rightarrow \mathit{collaborator}(X))\}$

$Q = \exists A \exists B \exists C\ (\mathit{hasCollaborator}(A,B,C) \wedge \mathit{collaborator}(A))$

$$Q_\Sigma = \exists A \exists B \exists C\ (\mathit{hasCollaborator}(A,B,C) \wedge \mathit{collaborator}(A))$$

$$\vee$$

$$\exists A \exists B \exists C \exists E \exists F\ (\mathit{hasCollaborator}(A,B,C) \wedge \mathit{hasCollaborator}(A,E,F))$$

$$\vee$$

$$\exists B \exists C\ (\mathit{project}(C) \wedge \mathit{inArea}(C,B))$$

...but, we cannot obtain the last query due to the applicablity condition

Minimization Step

$\Sigma = \{\forall X \forall Y\, (\mathit{project}(X) \land \mathit{inArea}(X,Y) \rightarrow \exists Z\, \mathit{hasCollaborator}(Z,Y,X)),$

$\forall X \forall Y \forall Z\, (\mathit{hasCollaborator}(X,Y,Z) \rightarrow \mathit{collaborator}(X))\}$

$Q = \exists A \exists B \exists C\, (\mathit{hasCollaborator}(A,B,C) \land \mathit{collaborator}(A))$

$Q_\Sigma = \exists A \exists B \exists C\, (\mathit{hasCollaborator}(A,B,C) \land \mathit{collaborator}(A))$

$\lor$

$\exists A \exists B \exists C \exists E \exists F\, (\mathit{hasCollaborator}(A,B,C) \land \mathit{hasCollaborator}(A,E,F))$

$\mathit{hasCollaborator}(A,B,C)$

Minimization Step

Σ = {∀X∀Y (*project*(X) ∧ *inArea*(X,Y) → ∃Z *hasCollaborator*(Z,Y,X)),

∀X∀Y∀Z (*hasCollaborator*(X,Y,Z) → *collaborator*(X))}

Q = ∃A∃B∃C (*hasCollaborator*(A,B,C) ∧ *collaborator*(A))

Q_Σ = ∃A∃B∃C (*hasCollaborator*(A,B,C) ∧ *collaborator*(A))

∨

∃A∃B∃C∃E∃F (*hasCollaborator*(A,B,C) ∧ *hasCollaborator*(A,E,F))

∨

∃A∃B∃C (*hasCollaborator*(A,B,C)) - by minimization

Minimization Step

$\Sigma = \{\forall X \forall Y\ (\mathit{project}(X) \wedge \mathit{inArea}(X,Y) \rightarrow \exists Z\ \mathit{hasCollaborator}(Z,Y,X)),$
$\forall X \forall Y \forall Z\ (\mathit{hasCollaborator}(X,Y,Z) \rightarrow \mathit{collaborator}(X))\}$

$Q = \exists A \exists B \exists C\ (\mathit{hasCollaborator}(A,B,C) \wedge \mathit{collaborator}(A))$

$Q_\Sigma = \exists A \exists B \exists C\ (\mathit{hasCollaborator}(A,B,C) \wedge \mathit{collaborator}(A))$

$\vee$

$\exists A \exists B \exists C \exists E \exists F\ (\mathit{hasCollaborator}(A,B,C) \wedge \mathit{hasCollaborator}(A,E,F))$

$\vee$

$\exists A \exists B \exists C\ (\mathit{hasCollaborator}(A,B,C))$ - by minimization

$\vee$

$\exists B \exists C\ (\mathit{project}(C) \wedge \mathit{inArea}(C,B))$ - by rewriting

UCQ-Rewritings

- The standard algorithm for computing UCQ-rewritings performs an exhaustive application of the following two steps:
    1. Rewriting
    2. Minimization

- The standard algorithm is designed for normalized existential rules, where only one atom appears in the head

The Rewriting Algorithm

Q_Σ := Q;
repeat
    Q_{aux} := Q_Σ;
    foreach disjunct q of Q_{aux} do
    //Rewriting Step
        foreach atom α in q do
            foreach rule σ in Σ do
                if σ is applicable to α then
                    q_{rew} := *rewrite*(q,α,σ);   // resolve α using σ
                    if q_{rew} does not appear in Q_Σ (modulo variable renaming) then
                        $Q_\Sigma := Q_\Sigma \vee q_{rew}$;
    //Minimization Step
        foreach pair of atoms α,β in q that unify do
            q_{min} := *minimize*(q,α,β);   // apply most general unifier of α and β on q
            if q_{min} does not appear in Q_Σ (modulo variable renaming) then
                        $Q_\Sigma := Q_\Sigma \vee q_{min}$;
until $Q_{aux} = Q_\Sigma$;
return Q_Σ;

Termination

Theorem: The rewriting algorithm terminates under ACYCLIC and LINEAR

Proof (ACYCLIC):

- Key observation: after arranging the disjuncts of the rewriting in a tree T, the branching of T is finite, and the depth of T is at most the number of predicates occurring in the rule set
- Therefore, only finitely many partial rewritings can be constructed - in general, exponentially many

Termination

Theorem: The rewriting algorithm terminates under ACYCLIC and LINEAR

Proof (LINEAR):

- Key observation: the size of each partial rewriting is at most the size of the given CQ Q
- Thus, each partial rewriting can be transformed into an equivalent query that contains at most $|Q| \cdot$ maxarity variables
- The number of queries that can be constructed using a finite number of predicates and a finite number of variables is finite
- Therefore, only finitely many partial rewritings can be constructed - in general, exponentially many

Complexity of BCQ-Answering

		Data Complexity
FULL	PTIME-c	Naïve algorithm
		Reduction from Monotone Circuit Value problem
ACYCLIC	in LOGSPACE	**UCQ-rewriting**
LINEAR		

		Combined Complexity
FULL	EXPTIME-c	Naïve algorithm
		Simulation of a deterministic exponential time TM
ACYCLIC	NEXPTIME-c	Small witness property
		Reduction from Tiling problem
LINEAR	PSPACE-c	Level-by-level non-deterministic algorithm
		Simulation of a deterministic polynomial space TM

Size of the Rewriting

- Ideally, we would like to construct UCQ-rewritings of polynomial size
- But, the standard rewriting algorithm produces rewritings of exponential size
- Can we do better? NO!!!

$$\Sigma = \{\forall X\, (R_k(X) \rightarrow P_k(X))\}_{k \in \{1,\ldots,n\}} \qquad Q = \exists X\, (P_1(X) \wedge \ldots \wedge P_n(X))$$

$$\exists X\, (P_1(X) \wedge \ldots \wedge P_n(X))$$

$$P_1(X) \vee R_1(X) \qquad P_n(X) \vee R_n(X)$$

thus, we need to consider 2^n disjuncts

Size of the Rewriting

- Ideally, we would like to construct UCQ-rewritings of polynomial size
- But, the standard rewriting algorithm produces rewritings of exponential size
- Can we do better? NO!!!

- **Although the standard rewriting algorithm is worst-case optimal, it can be significantly improved**
- **Optimization techniques can be applied in order to compute efficiently small rewritings - field of intense research**

Minimization Step Revisited

$\Sigma = \{\forall X\, (P(X) \rightarrow \exists Y\, R(X,Y))\}$

$Q = \exists A_1 \ldots \exists A_n \exists B\, (S_1(A_1) \wedge R(A_1,B) \wedge \ldots \wedge S_n(A_n) \wedge R(A_n,B))$

exponentially many minimization steps must be applied in order to get the query

$$\exists A \exists B\, (S_1(A) \wedge \ldots \wedge S_n(A) \wedge R(A,B))$$

and then apply the rewriting step, which will lead to the query

$$\exists A\, (S_1(A) \wedge \ldots \wedge S_n(A) \wedge P(A))$$

Minimization Step Revisited

$\Sigma = \{\forall X\ (P(X) \rightarrow \exists Y\ R(X,Y))\}$

$Q = \exists A_1 \ldots \exists A_n \exists B\ (S_1(A_1) \wedge R(A_1,B) \wedge \ldots \wedge S_n(A_n) \wedge R(A_n,B))$

Piece-based Rewriting

- Instead of rewriting a single atom
- Rewrite a set of atoms that have to be rewritten together

Computing the Piece

Input: CQ q, atom $\alpha = R(t_1,\dots,t_n)$ in q, rule σ
Output: piece of α in q w.r.t. σ

Piece := $\{R(t_1,\dots,t_n)\}$;
while TRUE do
    if *Piece* and head(σ) do not unify then
        return ∅;
    h := most general unifier of *Piece* and head(σ);
    if h violates points 2 or 4 of the applicability condition then
        return ∅;
    if h violates point 3 of the applicability condition then
        Piece := *Piece* ∪ {atoms containing a variable that unifies with an ∃-variable};
    else
        return *Piece*;

The Piece-based Rewriting Algorithm

Q_Σ := Q;
repeat
    Q_{aux} := Q_Σ;
    foreach disjunct q of Q_{aux} do
        foreach atom α in q do
            foreach rule σ in Σ do
                //Rewriting Step
                if σ is applicable to α then
                    q_{rew} := *rewrite*(q,α,σ);   // resolve α using σ
                    if q_{rew} does not appear in Q_Σ (modulo variable renaming) then
                        $Q_\Sigma := Q_\Sigma \vee q_{rew}$;
                //Minimization Step
                P := piece of α in q w.r.t. σ;
                q_{min} := *minimize*(q,P);   // apply the most general unifier of P on q
                if q_{min} does not appear in Q_Σ (modulo variable renaming) then
                    $Q_\Sigma := Q_\Sigma \vee q_{min}$;
until $Q_{aux} = Q_\Sigma$;
return Q_Σ;

Termination

$\Sigma = \{\forall X \forall Y\, (R(X,Y) \wedge P(Y) \rightarrow P(X))\}$

$Q = \exists X\, P(X)$

$$Q_\Sigma = \exists X\, P(X)$$
$$\vee$$
$$\exists X \exists Y_1\, (R(c,Y_1) \wedge P(Y_1))$$
$$\vee$$
$$\exists X \exists Y_1 \exists Y_2\, (R(c,Y_1) \wedge R(Y_1,Y_2) \wedge P(Y_2))$$
$$\vee$$
$$\exists X \exists Y_1 \exists Y_2 \exists Y_3\, (R(c,Y_1) \wedge R(Y_1,Y_2) \wedge R(Y_2,Y_3) \wedge P(Y_3))$$
$$\vee$$
$$\dots$$

- The piece-based rewriting algorithm does not terminate
- However, there exists a finite UCQ-rewritings, that is, $\exists X\, P(X)$

...careful application of the homomorphism check

Limitations of UCQ-Rewritability

$\forall D : D \land \Sigma \models Q \quad \Leftrightarrow \quad D \models Q_\Sigma$

evaluated and optimized by exploiting existing technology

- What about the size of Q_Σ? - very large, no rewritings of polynomial size
- What kind of ontology languages can be used for Σ? - below PTIME