Undecidability so far

We have seen several undecidable problems for TMs:

- The **Halting Problem**: recognise TM-word pairs where the TM halts
- The **Non-Halting Problem**: recognise TM-word pairs where the TM does not halt
- The **$\varepsilon$-Halting Problem**: recognise TMs that halt on the empty input

Many further TM-related problems are undecidable . . .
We have seen several undecidable problems for TMs:

- The **Halting Problem**: recognise TM-word pairs where the TM halts
- The **Non-Halting Problem**: recognise TM-word pairs where the TM does not halt
- The **ε-Halting Problem**: recognise TMs that halt on the empty input

Many further TM-related problems are undecidable . . .

. . . but we can use a shortcut to proving many of them:

**Theorem 4.1 (Rice’s Theorem, informal)**: Any interesting property related to the language recognised by a given TM is undecidable.
We can make this formal as follows:

**Definition 4.2:** Let $\mathcal{P}$ be a set of languages. A language $L$ has the property $\mathcal{P}$ if $L \in \mathcal{P}$. Property $\mathcal{P}$ is a non-trivial property of recognisable languages if there are TM-recognisable languages that have it and others that do not have it.

**Theorem 4.1 (Rice’s Theorem):** If $\mathcal{P}$ is a non-trivial property of recognisable languages, then the following problem is undecidable:

$$\mathcal{P}\text{-ness} = \{ \langle M \rangle \mid L(M) \in \mathcal{P} \}$$
Proof of Rice’s Theorem

**Theorem 4.1 (Rice’s Theorem):** If $\mathcal{P}$ is a non-trivial property of recognisable languages, then the following problem is undecidable:

$$\mathcal{P} \text{-ness} = \{ \langle M \rangle \mid L(M) \in \mathcal{P} \}$$

Proof:

We reduce $\epsilon$-Halting to $\mathcal{P}$-ness.

- Assume w.l.o.g. that $\emptyset < \mathcal{P}$ (otherwise do the proof for $\mathcal{P}$).
- Let $M_L$ be some TM that recognises a language $L \in \mathcal{P}$.
- Given any TM $M$, compute a TM $M^*$ that behaves as follows:
  - On input $w \in \Sigma^*$: (1) Simulate $M$ on input $\epsilon$.
  - If $M$ halts, simulate $M_L$ on $w$.

Then $L(M^*) = L \in \mathcal{P}$ if $M$ halts on $\epsilon$, and $L(M^*) = \emptyset < \mathcal{P}$ if $M$ does not halt on $\epsilon$.

For the required Turing reduction, we construct a TM that:

1. checks if the input is a TM encoding $\langle M \rangle$ and rejects otherwise,
2. returns the result of the check $\langle M^* \rangle \in \mathcal{P}$. This would decide $\epsilon$-Halting.

□
Proof of Rice’s Theorem

**Theorem 4.1 (Rice’s Theorem):** If $\mathcal{P}$ is a non-trivial property of recognisable languages, then the following problem is undecidable:

$$\mathcal{P}\text{-ness} = \{\langle M \rangle \mid L(M) \in \mathcal{P}\}$$

**Proof:** We reduce $\varepsilon$-Halting to $\mathcal{P}$-ness.
Proof of Rice’s Theorem

**Theorem 4.1 (Rice’s Theorem):** If $\mathcal{P}$ is a non-trivial property of recognisable languages, then the following problem is undecidable:

$$\mathcal{P}\text{-ness} = \{\langle M \rangle \mid L(M) \in \mathcal{P}\}$$

**Proof:** We reduce $\varepsilon$-Halting to $\mathcal{P}$-ness.

- Assume w.l.o.g. that $\emptyset \notin \mathcal{P}$ (otherwise do the proof for $\overline{\mathcal{P}}$)
Proof of Rice’s Theorem

**Theorem 4.1 (Rice’s Theorem):** If \( \mathcal{P} \) is a non-trivial property of recognisable languages, then the following problem is undecidable:

\[
\mathcal{P}\text{-ness} = \{ \langle M \rangle | L(M) \in \mathcal{P} \}
\]

**Proof:** We reduce \( \varepsilon \)-Halting to \( \mathcal{P}\text{-ness} \).

- Assume w.l.o.g. that \( \emptyset \notin \mathcal{P} \) (otherwise do the proof for \( \overline{\mathcal{P}} \))
- Let \( M_L \) be some TM that recognises a language \( L \in \mathcal{P} \)
Proof of Rice’s Theorem

**Theorem 4.1 (Rice’s Theorem):** If \( \mathcal{P} \) is a non-trivial property of recognisable languages, then the following problem is undecidable:

\[
\mathcal{P}\text{-ness} = \{\langle M \rangle \mid L(M) \in \mathcal{P}\}
\]

**Proof:** We reduce \( \varepsilon \)-Halting to \( \mathcal{P}\text{-ness} \).

- Assume w.l.o.g. that \( \emptyset \notin \mathcal{P} \) (otherwise do the proof for \( \overline{\mathcal{P}} \))
- Let \( M_L \) be some TM that recognises a language \( L \in \mathcal{P} \)
- Given any TM \( M \), compute a TM \( M^* \) that behaves as follows:
  - On input \( w \in \Sigma^* \):
    1. (1) Simulate \( M \) on input \( \varepsilon \)
    2. (2) If \( M \) halts, simulate \( M_L \) on \( w \)
Proof of Rice’s Theorem

**Theorem 4.1 (Rice’s Theorem):** If $\mathcal{P}$ is a non-trivial property of recognisable languages, then the following problem is undecidable:

$$\mathcal{P}\text{-ness} = \{\langle M \rangle \mid L(M) \in \mathcal{P}\}$$

**Proof:** We reduce $\varepsilon$-Halting to $\mathcal{P}$-ness.

- Assume w.l.o.g. that $\emptyset \notin \mathcal{P}$ (otherwise do the proof for $\overline{\mathcal{P}}$)
- Let $M_L$ be some TM that recognises a language $L \in \mathcal{P}$
- Given any TM $M$, compute a TM $M^*$ that behaves as follows:
  
  On input $w \in \Sigma^*$:
  1. Simulate $M$ on input $\varepsilon$
  2. If $M$ halts, simulate $M_L$ on $w$

- Then $L(M^*) = L \in \mathcal{P}$ if $M$ halts on $\varepsilon$, and $L(M^*) = \emptyset \notin \mathcal{P}$ if $M$ does not halt on $\varepsilon$
**Proof of Rice’s Theorem**

**Theorem 4.1 (Rice’s Theorem):** If \( \mathcal{P} \) is a non-trivial property of recognisable languages, then the following problem is undecidable:

\[
\mathcal{P}\text{-ness} = \{\langle M \rangle \mid L(M) \in \mathcal{P}\}
\]

**Proof:** We reduce \( \varepsilon \)-Halting to \( \mathcal{P} \)-ness.

- Assume w.l.o.g. that \( \emptyset \notin \mathcal{P} \) (otherwise do the proof for \( \overline{\mathcal{P}} \))
- Let \( M_L \) be some TM that recognises a language \( L \in \mathcal{P} \)
- Given any TM \( M \), compute a TM \( M^* \) that behaves as follows:
  
  On input \( w \in \Sigma^* \):
  1. Simulate \( M \) on input \( \varepsilon \)
  2. If \( M \) halts, simulate \( M_L \) on \( w \)

- Then \( L(M^*) = L \in \mathcal{P} \) if \( M \) halts on \( \varepsilon \), and
  - \( L(M^*) = \emptyset \notin \mathcal{P} \) if \( M \) does not halt on \( \varepsilon \)

For the required Turing reduction, we construct a TM that:

(Step 1) checks if the input is a TM encoding \( \langle M \rangle \) and rejects otherwise,

(Step 2) returns the result of the check \( \langle M^* \rangle \in \mathcal{P} \). This would decide \( \varepsilon \)-Halting.

\( \square \)
Here are some simple results that Rice gives us:

**Corollary 4.3:** Given an arbitrary TM $M$, it is undecidable whether the language recognised by $M$ has any of the following properties:

- emptiness
- finiteness
- decidability
- regularity
- context-freedom
- contains any given word $w$ (word problem for TMs)
Using Rice’s Theorem

Here are some simple results that Rice gives us:

**Corollary 4.3:** Given an arbitrary TM $M$, it is undecidable whether the language recognised by $M$ has any of the following properties:

- emptiness
- finiteness
- decidability
- regularity
- context-freedom
- contains any given word $w$ (word problem for TMs)

**Attention:** There are of course many non-trivial properties of TMs that can be decided, and which do not relate to their language:

**Example 4.4:** It is decidable if a TM has at least three states.
Semi-decidability and Co-semi-decidability

We can distinguish the following two cases:

1. $L$ is Turing-recognisable: $L$ is semi-decidable
2. $\overline{L}$ is Turing-recognisable: $L$ is co-semi-decidable
Semi-decidability and Co-semi-decidability

We can distinguish the following two cases:

1. $L$ is Turing-recognisable: $L$ is semi-decidable
2. $\overline{L}$ is Turing-recognisable: $L$ is co-semi-decidable

We have seen examples for both:

**Theorem 4.5:** The Halting Problem is semi-decidable.

**Proof:** Use the universal TM to simulate an input TM, and accept if it halts. □
Semi-decidability and Co-semi-decidability

We can distinguish the following two cases:

(1) $L$ is Turing-recognisable: $L$ is semi-decidable

(2) $\overline{L}$ is Turing-recognisable: $L$ is co-semi-decidable

We have seen examples for both:

**Theorem 4.5:** The Halting Problem is semi-decidable.

**Proof:** Use the universal TM to simulate an input TM, and accept if it halts. □

**Corollary 4.6:** The Non-Halting Problem is co-semi-decidable.
Semi-decidable + Co-semi-decidable = Decidable

An easy but important observation:

**Theorem 4.7**: If \( L \) is semi-decidable and co-semi-decidable, then \( L \) is decidable.

Proof: On input \( w \), simulate, in parallel, a recogniser for \( L \) and a recogniser for the complement of \( L \). At least one of them eventually must halt, so we can decide if \( w \in L \).

\( \square \)

We thus obtain an example of a problem that is not Turing-recognisable.

**Corollary 4.8**: The Non-Halting Problem is not Turing-recognisable.
Semi-decidable + Co-semi-decidable = Decidable

An easy but important observation:

**Theorem 4.7:** If $L$ is semi-decidable and co-semi-decidable, then $L$ is decidable.

**Proof:** On input $w$, simulate, in parallel, a recogniser for $L$ and a recogniser for $\overline{L}$. At least one of them eventually must halt, so we can decide if $w \in L$. \hfill \Box
An easy but important observation:

**Theorem 4.7:** If $L$ is semi-decidable and co-semi-decidable, then $L$ is decidable.

**Proof:** On input $w$, simulate, in parallel, a recogniser for $L$ and a recogniser for $\overline{L}$. At least one of them eventually must halt, so we can decide if $w \in L$.

We thus obtain an example of a problem that is not Turing-recognisable.

**Corollary 4.8:** The Non-Halting Problem is not Turing-recognisable.
Turing reductions and semi-decidability

**Observation:**
- If \( Q \) is decidable and \( P \leq_T Q \), then \( P \) is decidable (Theorem 3.17)
- **But:** if \( Q \) is semi-decidable and \( P \leq_T Q \), then \( P \) may or may not be semi-decidable

**Reason:** An oracle for Halting is as good as an oracle for Non-Halting, since we are free to complement the answer in an oracle machine.

This is a general insight: complementing oracles has no effect
Turing reductions and semi-decidability

Observation:

- If $Q$ is decidable and $P \leq_T Q$, then $P$ is decidable (Theorem 3.17)
- But: if $Q$ is semi-decidable and $P \leq_T Q$, then $P$ may or may not be semi-decidable

Reason: An oracle for Halting is as good as an oracle for Non-Halting, since we are free to complement the answer in an oracle machine.

This is a general insight: complementing oracles has no effect

To preserve (co-)semi-decidability, one needs a more restricted form of reduction:

**Definition 4.9:** A language $P$ is many-one reducible to a language $Q$, written $P \leq_m Q$ if there exists a total computable function $f : \Sigma^* \to \Sigma^*$ such that, for all $w \in \Sigma^*$:

$$w \in P \text{ if and only if } f(w) \in Q.$$

This is sometimes called a mapping-reduction or an $m$-reduction.
Properties of Many-One-Reductions

Many-one reductions are special kinds of Turing reductions:

**Theorem 4.10:** If $P \leq_m Q$ then $P \leq_T Q$.

**Proof:**
We obtain an OTM with oracle $Q$ that recognises $P$ as follows:

- On input $w$, compute $f(w)$
- Call the oracle and return its result (yes = accept; no = reject)

□

An easy consequence of Theorem 3.17 therefore is:

**Corollary 4.11:** If $P \leq_m Q$ and $Q$ is decidable, then $P$ is decidable.

However, now we also have the following:

**Theorem 4.12:** If $P \leq_m Q$ and $Q$ is semi-decidable, then $P$ is semi-decidable.

**Proof:**
Given a TM that recognises $Q$, we obtain a TM that recognises $P$ as follows:

- On input $w$, compute $f(w)$
- Simulate the TM for $Q$ and return the result (if any)

□
Many-one reductions are special kinds of Turing reductions:

**Theorem 4.10:** If $P \leq_m Q$ then $P \leq_T Q$.

**Proof:** We obtain an OTM with oracle $Q$ that recognises $P$ as follows:

- On input $w$, compute $f(w)$
- Call the oracle and return its result (yes = accept; no = reject)
Properties of Many-One-Reductions

Many-one reductions are special kinds of Turing reductions:

**Theorem 4.10:** If \( P \leq_m Q \) then \( P \leq_T Q \).

**Proof:** We obtain an OTM with oracle \( Q \) that recognises \( P \) as follows:

- On input \( w \), compute \( f(w) \)
- Call the oracle and return its result (yes = accept; no = reject)

An easy consequence of Theorem 3.17 therefore is:

**Corollary 4.11:** If \( P \leq_m Q \) and \( Q \) is decidable, then \( P \) is decidable.
Properties of Many-One-Reductions

Many-one reductions are special kinds of Turing reductions:

**Theorem 4.10:** If $P \leq_m Q$ then $P \leq_T Q$.

**Proof:** We obtain an OTM with oracle $Q$ that recognises $P$ as follows:

- On input $w$, compute $f(w)$
- Call the oracle and return its result (yes = accept; no = reject)

An easy consequence of Theorem 3.17 therefore is:

**Corollary 4.11:** If $P \leq_m Q$ and $Q$ is decidable, then $P$ is decidable.

However, now we also have the following:

**Theorem 4.12:** If $P \leq_m Q$ and $Q$ is semi-decidable, then $P$ is semi-decidable.

**Proof:** Given a TM that recognises $Q$, we obtain a TM that recognises $P$ as follows:

- On input $w$, compute $f(w)$
- Simulate the TM for $Q$ and return the result (if any)
Example: Many-one Reduction

Some of our previous Turing-reductions can easily be described as many-one, e.g., Halting can be many-one reduced to $\varepsilon$-Halting. Here is another example:

Definition 4.13: Two TMs $M$ and $N$ are equivalent if $L(M) = L(N)$.

Theorem 4.14: Equivalence of Turing machines is undecidable.

(Note that we could also get this from Rice’s Theorem, but we want to try out our new machinery.)

Proof: We define $f$ such that $w \in \varepsilon$-Halting iff $f(w) \in$ Equivalence.
Example: Many-one Reduction

Some of our previous Turing-reductions can easily be described as many-one, e.g., Halting can be many-one reduced to $\varepsilon$-Halting. Here is another example:

**Definition 4.13:** Two TMs $M$ and $N$ are **equivalent** if $L(M) = L(N)$.

**Theorem 4.14:** Equivalence of Turing machines is undecidable.

(Note that we could also get this from Rice’s Theorem, but we want to try out our new machinery.)

**Proof:** We define $f$ such that $w \in \varepsilon$-Halting iff $f(w) \in$ Equivalence.

Let $M_a$ be a TM that accepts all inputs.
Example: Many-one Reduction

Some of our previous Turing-reductions can easily be described as many-one, e.g., Halting can be many-one reduced to $\varepsilon$-Halting. Here is another example:

**Definition 4.13:** Two TMs $M$ and $N$ are **equivalent** if $L(M) = L(N)$.

**Theorem 4.14:** Equivalence of Turing machines is undecidable.

(Note that we could also get this from Rice’s Theorem, but we want to try out our new machinery.)

**Proof:** We define $f$ such that $w \in \varepsilon$-Halting iff $f(w) \in$ Equivalence.

Let $M_a$ be a TM that accepts all inputs.

For a TM $M$, we define the following TM $M^*$:

- Simulate $M$ on the empty input.
- If $M$ halts, accept.
Example: Many-one Reduction

Some of our previous Turing-reductions can easily be described as many-one, e.g., Halting can be many-one reduced to $\varepsilon$-Halting. Here is another example:

**Definition 4.13:** Two TMs $M$ and $N$ are equivalent if $L(M) = L(N)$.

**Theorem 4.14:** Equivalence of Turing machines is undecidable.

(Note that we could also get this from Rice's Theorem, but we want to try out our new machinery.)

**Proof:** We define $f$ such that $w \in \varepsilon$-Halting iff $f(w) \in$ Equivalence.

Let $M_a$ be a TM that accepts all inputs.

For a TM $M$, we define the following TM $M^*$:

- Simulate $M$ on the empty input.
- If $M$ halts, accept.

Then $M^*$ is equivalent to $M_a$ iff $M$ halts on the empty input. We define $f$:

$$f(w) = \begin{cases} 
\langle M^*, M_a \rangle & \text{if } w = \langle M \rangle \\
\varepsilon & \text{if } w \text{ is no encoded TM}
\end{cases}$$
Equivalence is Hard

We can show a somewhat stronger result:

**Theorem 4.15:** Equivalence of Turing machines is neither semi-decidable nor co-semi-decidable.

**Proof:** We have already shown $\varepsilon$-Halting $\leq_m$ Equivalence. Since we know that $\varepsilon$-Halting is not co-semi-decidable (similar to Halting), we conclude that Equivalence is neither.

However, we can also show that $\overline{\varepsilon}$-Halting $\leq_m$ Equivalence.

- Note that the TM $M^*$ defined on the previous slide either accepts all inputs (if $M$ halts on $\varepsilon$) or none (if it doesn’t)
- Equivalence to $M_a$ corresponds to $\varepsilon$-Halting
- On the other hand, equivalence to a TM $M_\emptyset$, which rejects all inputs, corresponds to $\varepsilon$-non-Halting

We can therefore use the reduction $f$:

$$f(w) = \begin{cases} 
\langle M^*, M_\emptyset \rangle & \text{if } w = \langle M \rangle \\
\langle M_\emptyset, M_\emptyset \rangle & \text{(an invalid input) if } w \text{ is no encoded TM}
\end{cases}$$
Recursion
A Paradox in the Study of Life:

(1) Living things are machines.
(2) Living things can reproduce.
(3) Machines cannot reproduce.
A Paradox

A Paradox in the Study of Life:

(1) Living things are machines.
(2) Living things can reproduce.
(3) Machines cannot reproduce.

Rationale:
A Paradox

A Paradox in the Study of Life:

(1) Living things are machines.
(2) Living things can reproduce.
(3) Machines cannot reproduce.

Rationale:

(1) Viewpoint of modern biology.
A Paradox

A Paradox in the Study of Life:

(1) Living things are machines.
(2) Living things can reproduce.
(3) Machines cannot reproduce.

Rationale:

(1) Viewpoint of modern biology.
(2) Clear.
A Paradox

A Paradox in the Study of Life:

(1) Living things are machines.
(2) Living things can reproduce.
(3) Machines cannot reproduce.

Rationale:

(1) Viewpoint of modern biology.
(2) Clear.
(3) If a machine $A$ produces a machine $B$, then $A$ must be more complex than $B$. For example, a car-producing factory is more complex than the cars it produces, as it contains the design of the cars and, in addition, the design of all manufacturing robots, among others. Since no machine is more complex than itself, a machine cannot reproduce itself.
Resolving the Paradox

A Paradox in the Study of Life:

1. Living things are machines.
2. Living things can reproduce.
3. Machines cannot reproduce.

Question: How to resolve this paradox?

Answer: Assertion (3) is wrong. In particular, the underlying argument of "more information" and "greater complexity" needed by the producing machine is flawed: there are TMs that reproduce themselves.
Resolving the Paradox

A Paradox in the Study of Life:
(1) Living things are machines.
(2) Living things can reproduce.
(3) Machines cannot reproduce.

Question: How to resolve this paradox?

Answer: Assertion (3) is wrong.

In particular, the underlying argument of “more information” and “greater complexity” needed by the producing machine is flawed: there are TMs that reproduce themselves.
Quines

Reproduction of TMs is closely related to the task of creating a program that prints its own source code:

**Definition 4.16:** A quine is a program that, when started without any input, will print out its own source code, and then stop.

Can Quines be created? How?
Quines

Reproduction of TMs is closely related to the task of creating a program that prints its own source code:

**Definition 4.16:** A quine is a program that, when started without any input, will print out its own source code, and then stop.

Can Quines be created? How?

**Example 4.17 (A quine in English):** Print this sentence.
Quines

Reproduction of TMs is closely related to the task of creating a program that prints its own source code:

**Definition 4.16:** A quine is a program that, when started without any input, will print out its own source code, and then stop.

Can Quines be created? How?

**Example 4.17 (A quine in English):** Print this sentence.

However, we cannot turn this into a program, since “this sentence” does often not correspond to available programming constructs.
Quines

Reproduction of TMs is closely related to the task of creating a program that prints its own source code:

**Definition 4.16:** A quine is a program that, when started without any input, will print out its own source code, and then stop.

Can Quines be created? How?

**Example 4.17 (A quine in English):** Print this sentence.

However, we cannot turn this into a program, since “this sentence” does often not correspond to available programming constructs.

**Example 4.18 (Another quine in English):** Print the following sentence twice, the second times in quotes. "Print the following sentence twice, the second times in quotes."
Some Real Quines

Example 4.19 (A classic C quine):
```c
main() char *c="main() char *c=%c\%s%c; printf(c,34,c,34);"; printf(c,34,c,34);
```

Example 4.20 (The shortest C quine, by Szymon Rusinkiewicz):

Example 4.21 (A Python quine by Frank Stajano):

```python
l='l=%s; print l%%'l''; print l%'l'
```

Note: A variation are ouroboros quines that print out another program that prints out the original again. More steps are possible. See, e.g., https://github.com/mame/quine-relay for one with 100 steps. Other variations exist (see Wikipedia).
Some Real Quines

Example 4.19 (A classic C quine):
```c
main()char *c="main()char *c=%c%s%c;printf(c,34,c,34);";printf(c,3 4,c,34);
```

Example 4.20 (The shortest C quine, by Szymon Rusinkiewicz):
```
l='l=%s;print l%%'l'';print l%'l'
```

Note:
A variation are ouroboros quines that print out another program that prints out the original again. More steps are possible. See, e.g., [https://github.com/mame/quine-relay](https://github.com/mame/quine-relay) for one with 100 steps. Other variations exist (see Wikipedia).
Some Real Quines

Example 4.19 (A classic C quine):  
```
main()char *c="main()char
  *c=%c%s%c;printf(c,34,c,34);";printf(c,3 4,c,34);
```

Example 4.20 (The shortest C quine, by Szymon Rusinkiewicz):  
```
```

Example 4.21 (A Python quine by Frank Stajano):  
```
l='l=%s;print l%%l';print l'l'
```
Some Real Quines

**Example 4.19 (A classic C quine):**

```
main()char *c="main()char *
*c=%c%s%c;printf(c,34,c,34);";printf(c,3 4,c,34);
```

**Example 4.20 (The shortest C quine, by Szymon Rusinkiewicz):**

**Example 4.21 (A Python quine by Frank Stajano):**

```
l='l=%s;print l%%'l'';print l%'l'
```

**Note:** A variation are **ouroboros quines** that print out another program that prints out the original again. More steps are possible. See, e.g., [https://github.com/mame/quine-relay](https://github.com/mame/quine-relay) for one with 100 steps.

Other variations exist (see Wikipedia).
Towards a TM Quine

We define a TM SELF that ignores its input and prints out a description of itself.
(A TM quine, where “source code” is interpreted as “encoding of the TM”)

The following small result is helpful:

**Lemma 4.22:** There is a computable function $q : \Sigma^* \rightarrow \Sigma^*$ such that, for each $w \in \Sigma^*$, the word $q(w)$ is (the encoding of) a TM that prints $w$ and halts.

Proof:
For any word $w$, let $P_w$ be a TM that replaces the tape contents with the word $w$ (clearly, this can easily be found for any $w$).

Now $q$ is simply computed by a TM that, given $w$ as input, constructs $P_w$ and then computes and outputs $\langle P_w \rangle$.

□

Intuition:
If we were using another programming language, the TM $P_w$ might be, e.g., `print(w)`, and the function we seek would simply turn input string $w$ into output string “print($w$)”. 
Towards a TM Quine

We define a TM SELF that ignores its input and prints out a description of itself.
(A TM quine, where “source code” is interpreted as “encoding of the TM”)

The following small result is helpful:

**Lemma 4.22:** There is a computable function $q : \Sigma^* \to \Sigma^*$ such that, for each $w \in \Sigma^*$, the word $q(w)$ is (the encoding of) a TM that prints $w$ and halts.

**Proof:** For any word $w$, let $P_w$ be a TM that replaces the tape contents with the word $w$ (clearly, this can easily be found for any $w$).

Now $q$ is simply computed by a TM that, given $w$ as input, constructs $P_w$ and then computes and outputs $\langle P_w \rangle$.

**Intuition:** If we were using another programming language, the TM $P_w$ might be, e.g., `print(w)`, and the function we seek would simply turn input string $w$ into output string "print($w$)".
Defining the TM SELF

Like other quines, SELF consists of two parts:

A  Compute the “source code” $\langle B \rangle$ of a suitable program $B$

B  Use $\langle B \rangle$ to print out:

  (1) source code $\langle A \rangle$ that computes $\langle B \rangle$ and (2) the source code $\langle B \rangle$ itself
Defining the TM SELF

Like other quines, SELF consists of two parts:

A  Compute the “source code” \( \langle B \rangle \) of a suitable program \( B \)

B  Use \( \langle B \rangle \) to print out:
   
   (1) source code \( \langle A \rangle \) that computes \( \langle B \rangle \) and (2) the source code \( \langle B \rangle \) itself

We know how to implement part A: use the TM \( P_{\langle B \rangle} \)

(However, to actually do this, we need to know \( B \) first)
Defining the TM SELF

Like other quines, SELF consists of two parts:

A Compute the “source code” $\langle B \rangle$ of a suitable program $B$

B Use $\langle B \rangle$ to print out:
   (1) source code $\langle A \rangle$ that computes $\langle B \rangle$ and (2) the source code $\langle B \rangle$ itself

We know how to implement part A: use the TM $P_{\langle B \rangle}$

(However, to actually do this, we need to know $B$ first)

$B$ in turn can work as follows:

Given some input string $\langle M \rangle$:

- compute $q(\langle M \rangle)$
- concatenate the TMs given by $q(\langle M \rangle)$ and $\langle M \rangle$
  
  (take a disjoint union of states where any halting state of $\langle M \rangle$ gets a transition to the starting state of $q(\langle M \rangle)$)
- output the encoding of the resulting machine

Then part $B$ does not depend on $A$, so we can really define $A$ as $P_{\langle B \rangle}$
Summary: SELF TM

So how did we construct our TM quine now?

**Step 1:** We define some TM $B$ that behaves as follows:

Given some input string $\langle M \rangle$:

- compute $q(\langle M \rangle)$
- concatenate the TMs given by $q(\langle M \rangle)$ and $\langle M \rangle$
  (take a disjoint union of states where any halting state of $\langle M \rangle$ gets a transition to the starting state of $q(\langle M \rangle)$)
- output the encoding of the resulting machine

**Step 2:** We define SELF to be the TM constructed by $B$ on input $\langle B \rangle$
Summary: SELF TM

So how did we construct our TM quine now?

**Step 1:** We define some TM $B$ that behaves as follows:

Given some input string $⟨M⟩$:

- compute $q(⟨M⟩)$
- concatenate the TMs given by $q(⟨M⟩)$ and $⟨M⟩$
  (take a disjoint union of states where any halting state of $⟨M⟩$ gets a transition to the starting state of $q(⟨M⟩)$)
- output the encoding of the resulting machine

**Step 2:** We define SELF to be the TM constructed by $B$ on input $⟨B⟩$

**Exercise:** Use this recipe to create a quine in your favourite programming language (or just use Python). What is the equivalent of “TM concatenation” here? Also note that the function $q$ is often more complicated than one might think, due to character escaping.
Going further, we can allow any TM to access its own description during the computation:

**Theorem 4.23 (Recursion Theorem):** Let $t : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ be a function computed by some TM $T$ (assuming a suitable encoding of pairs of words over $\Sigma^*$). Then there is a TM $R$ that computes a function $r : \Sigma^* \rightarrow \Sigma^*$ such that

$$r(w) = t(\langle R \rangle, w)$$

for every $w \in \Sigma^*$.
The Recursion Theorem

Going further, we can allow any TM to access its own description during the computation:

**Theorem 4.23 (Recursion Theorem):** Let $t : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ be a function computed by some TM $T$ (assuming a suitable encoding of pairs of words over $\Sigma^*$). Then there is a TM $R$ that computes a function $r : \Sigma^* \rightarrow \Sigma^*$ such that

$$r(w) = t(\langle R \rangle, w)$$

for every $w \in \Sigma^*$.

**Intuition:** To make a TM that can use its own description, we first devise a TM $T$ (to compute $t$) that receives the description of a machine as extra input. The theorem yields a TM $R$ that operates like $R$ does but with $R$’s description filled in automatically.
The Recursion Theorem: Proof

**Theorem 4.23 (Recursion Theorem):** Let \( t : \Sigma^* \times \Sigma^* \rightarrow \Sigma^* \) be a function computed by some TM \( T \) (assuming a suitable encoding of pairs of words over \( \Sigma^* \)). Then there is a TM \( R \) that computes a function \( r : \Sigma^* \rightarrow \Sigma^* \) such that \( r(w) = t(\langle R \rangle, w) \) for every \( w \in \Sigma^* \).
The Recursion Theorem: Proof

**Theorem 4.23 (Recursion Theorem):** Let \( t : \Sigma^* \times \Sigma^* \rightarrow \Sigma^* \) be a function computed by some TM \( T \) (assuming a suitable encoding of pairs of words over \( \Sigma^* \)). Then there is a TM \( R \) that computes a function \( r : \Sigma^* \rightarrow \Sigma^* \) such that \( r(w) = t(\langle R \rangle, w) \) for every \( w \in \Sigma^* \).

**Proof:** The proof is similar to the construction of SELF, using a TM with three parts \( A, B \) and \( T \):
The Recursion Theorem: Proof

**Theorem 4.23 (Recursion Theorem):** Let \( t : \Sigma^* \times \Sigma^* \rightarrow \Sigma^* \) be a function computed by some TM \( T \) (assuming a suitable encoding of pairs of words over \( \Sigma^* \)). Then there is a TM \( R \) that computes a function \( r : \Sigma^* \rightarrow \Sigma^* \) such that \( r(w) = t(\langle R \rangle, w) \) for every \( w \in \Sigma^* \).

**Proof:** The proof is similar to the construction of SELF, using a TM with three parts \( A, B \) and \( T \):

- **A:** print \( \langle BT \rangle \) (like \( P_{\langle BT \rangle} \), but without deleting the input)

  we use \( BT \) to denote the concatenation of the TM parts \( B \) and \( T \) in one TM
The Recursion Theorem: Proof

Theorem 4.23 (Recursion Theorem): Let \( t : \Sigma^* \times \Sigma^* \rightarrow \Sigma^* \) be a function computed by some TM \( T \) (assuming a suitable encoding of pairs of words over \( \Sigma^* \)). Then there is a TM \( R \) that computes a function \( r : \Sigma^* \rightarrow \Sigma^* \) such that \( r(w) = t(\langle R \rangle, w) \) for every \( w \in \Sigma^* \).

Proof: The proof is similar to the construction of SELF, using a TM with three parts \( A, B \) and \( T \):

- **A:** print \( \langle BT \rangle \) (like \( P_{\langle BT \rangle} \), but without deleting the input)
  
  we use \( BT \) to denote the concatenation of the TM parts \( B \) and \( T \) in one TM

- **B:** on an input of form \( w\langle M \rangle \), replace \( \langle M \rangle \) by an encoding of the concatenation of \( q'(\langle M \rangle) \) and \( \langle M \rangle \)
  
  where \( q'(v) \) is like \( q \) but returns a TM that adds \( v \) at the end of the tape
The Recursion Theorem: Proof

**Theorem 4.23 (Recursion Theorem):** Let \( t : \Sigma^* \times \Sigma^* \rightarrow \Sigma^* \) be a function computed by some TM \( T \) (assuming a suitable encoding of pairs of words over \( \Sigma^* \)). Then there is a TM \( R \) that computes a function \( r : \Sigma^* \rightarrow \Sigma^* \) such that \( r(w) = t(\langle R \rangle, w) \) for every \( w \in \Sigma^* \).

**Proof:** The proof is similar to the construction of SELF, using a TM with three parts \( A, B \) and \( T \):

- **A:** print \( \langle BT \rangle \) (like \( P_{\langle BT \rangle} \) but without deleting the input)

  we use \( BT \) to denote the concatenation of the TM parts \( B \) and \( T \) in one TM

- **B:** on an input of form \( w\langle M \rangle \), replace \( \langle M \rangle \) by an encoding of the concatenation of \( q'(<M>) \) and \( <M> \)

  where \( q'(v) \) is like \( q \) but returns a TM that adds \( v \) at the end of the tape

- **T:** run \( T \) on an input of form \( w\langle N \rangle \)

We assume here that our TM encoding can be written next to the input \( w \) without risk of confusion.
The Recursion Theorem: Proof

**Theorem 4.23 (Recursion Theorem):** Let \( t : \Sigma^* \times \Sigma^* \rightarrow \Sigma^* \) be a function computed by some TM \( T \) (assuming a suitable encoding of pairs of words over \( \Sigma^* \)). Then there is a TM \( R \) that computes a function \( r : \Sigma^* \rightarrow \Sigma^* \) such that \( r(w) = t(\langle R \rangle, w) \) for every \( w \in \Sigma^* \).

**Proof:** The proof is similar to the construction of SELF, using a TM with three parts \( A, B \) and \( T \):

- **A:** print \( \langle BT \rangle \) (like \( P_{\langle BT \rangle} \), but without deleting the input)
  
  we use \( BT \) to denote the concatenation of the TM parts \( B \) and \( T \) in one TM

- **B:** on an input of form \( w\langle M \rangle \), replace \( \langle M \rangle \) by an encoding of the concatenation of \( q'(\langle M \rangle) \) and \( \langle M \rangle \)
  
  where \( q'(v) \) is like \( q \) but returns a TM that adds \( v \) at the end of the tape

- **T:** run \( T \) on an input of form \( w\langle N \rangle \)

We assume here that our TM encoding can be written next to the input \( w \) without risk of confusion. Then \( R \) is the TM obtained as the concatenation of \( A, B, \) and \( T \).

This is the TM whose encoding \( B \) would write on some input \( w\langle BT \rangle \)
Using the Recursion Theorem

By the Recursion Theorem, we can now use instructions like “obtain own description \( \langle M \rangle \)” in our informal descriptions of TMs.

**Example 4.24:** We can describe a TM quine in the style of our previous SELF as follows:

On any input:

- Obtain own description \( \langle M \rangle \)
- Print \( \langle M \rangle \)

We can construct such a TM by applying the Recursion Theorem to the TM \( T \) described as follows:

On input \( \langle w, M \rangle \), print \( \langle M \rangle \)

The Recursion Theorem turns this into a TM \( R \) that is a quine.
Halting is Undecidable: Proof by Introspection

We can also use the Recursion Theorem for alternative proofs:
We can also use the Recursion Theorem for alternative proofs:

**Theorem 3.11** The Halting Problem $P_{\text{Halt}}$ is undecidable.

**Proof**: By contradiction: Suppose there is a decider $H$ for the Halting Problem

We construct a TM $M$ that, on input $w$, acts as follows:

1. Obtain own description $\langle M \rangle$
2. Simulate $H$ on input $\langle M \rangle \# \langle w \rangle$, that is, check if $M$ halts on $w$
3. If yes, enter an infinite loop;
   if no, halt and accept
Halting is Undecidable: Proof by Introspection

We can also use the Recursion Theorem for alternative proofs:

**Theorem 3.11** The Halting Problem $P_{\text{Halt}}$ is undecidable.

**Proof:** By contradiction: Suppose there is a decider $H$ for the Halting Problem.

We construct a TM $M$ that, on input $w$, acts as follows:

1. Obtain own description $\langle M \rangle$
2. Simulate $H$ on input $\langle M \rangle \# \# \langle w \rangle$, that is, check if $M$ halts on $w$
3. If yes, enter an infinite loop;
   if no, halt and accept

Then $M$ halts on $w$ if and only if it doesn’t – contradiction. □
Minimal TMs

**Definition 4.25:** A TM $M$ is called **minimal** if there is no TM equivalent to $M$ that has a shorter description. The problem of deciding if a TM is minimal is:

$$\text{MIN}_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a minimal TM} \}$$

**Theorem 4.26:** $\text{MIN}_{\text{TM}}$ is not Turing-recognisable.

**Proof:** Assume there is some TM $E$ enumerating $\text{MIN}_{\text{TM}}$.

We define a TM $C$ that processes an input $w$ as follows:

1. Obtain own description $\langle C \rangle$
2. Simulate $E$ until some TM $D$ is printed such that $\langle D \rangle$ is longer than $\langle C \rangle$
3. Simulate $D$ on $w$

Then $C$ is equivalent to $D$, but it has a shorter description, contradicting the assumption that $D$ is minimal.

□
Definition 4.25: A TM $M$ is called minimal if there is no TM equivalent to $M$ that has a shorter description. The problem of deciding if a TM is minimal is:

$$\text{MIN}_T = \{\langle M \rangle | M \text{ is a minimal } TM\}$$

Theorem 4.26: $\text{MIN}_T$ is not Turing-recognisable.
Definition 4.25: A TM $M$ is called minimal if there is no TM equivalent to $M$ that has a shorter description. The problem of deciding if a TM is minimal is:

$$\text{MIN}_\text{TM} = \{ \langle M \rangle \mid M \text{ is a minimal TM} \}$$

Theorem 4.26: $\text{MIN}_\text{TM}$ is not Turing-recognisable.

Proof: Assume there is some TM $E$ enumerating $\text{MIN}_{\text{TM}}$.

We define a TM $C$ that processes an input $w$ as follows:

1. Obtain own description $\langle C \rangle$
2. Simulate $E$ until some TM $D$ is printed such that $\langle D \rangle$ is longer than $\langle C \rangle$
3. Simulate $D$ on $w$
Definition 4.25: A TM $M$ is called minimal if there is no TM equivalent to $M$ that has a shorter description. The problem of deciding if a TM is minimal is:

$$\text{MIN}_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a minimal } TM\}$$

Theorem 4.26: $\text{MIN}_{\text{TM}}$ is not Turing-recognisable.

Proof: Assume there is some TM $E$ enumerating $\text{MIN}_{\text{TM}}$.

We define a TM $C$ that processes an input $w$ as follows:

1. Obtain own description $\langle C \rangle$
2. Simulate $E$ until some TM $D$ is printed such that $\langle D \rangle$ is longer than $\langle C \rangle$
3. Simulate $D$ on $w$

Then $C$ is equivalent to $D$, but it has a shorter description, contradicting the assumption that $D$ is minimal. $\square$
Summary and Outlook

Most properties related to the computation of TMs are undecidable

Many-one reductions establish a closer relationship between two problems than Turing reductions

There are non-semi-decidable problems

Turing machines can work with their own description

What’s next?

- Defining complexity classes
- Time complexity
- Non-deterministic time