Nested Sequents for Quantified Modal Logics

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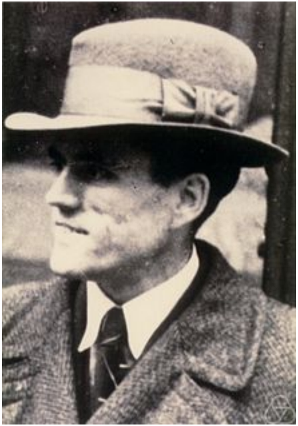
What are sequents? How are they useful?

- **Sequent:** $\Gamma \Rightarrow \Delta$
- Proofs **formalized** as objects in their own right
- Offers **constructive** and **syntactic** approach to studying properties of logics; e.g.
  - Consistency
  - Decidability
  - Interpolation
- **Fruitful** approach to **automated reasoning**; e.g.
  - Complexity optimal decision algorithms with witnesses



Gerhard Gentzen (1945)

A Prominent Desideratum: Analyticity

"A proof is **analytic** if it does not go beyond its **subject matter**."

Bernard Bolzano

Our Interpretation: A proof is **analytic** if it only contains **subformulae** of the **conclusion**.

A Jungle of Sequent Formalisms

$$Rxy, Rxz, x : A \Rightarrow y : B, y : C$$

$$\Gamma \Longrightarrow \Delta \| \Gamma_1 \Longrightarrow \Delta_1 / \ldots / \Gamma_n \Longrightarrow \Delta_n$$

$$A, B \vdash C, D, E$$

$$A \Rightarrow G, [\ \Rightarrow B, [C \Rightarrow D], [E \Rightarrow F]]$$

$$A \vdash B \mid C, D \vdash E \mid \ \vdash F$$

$$\Gamma_1 \Longrightarrow \Delta_1 \| \ldots \| \Gamma_n \Longrightarrow \Delta_n$$

$$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$$

$$A, \overset{1}{[}B, C\overset{1}{]}, \overset{3}{[}D, \overset{2}{[}E\overset{2}{]}, \overset{1}{[}F\overset{1}{]}\overset{3}{]}$$

Et cetera . . .

The Hierarchy of Sequents

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Structure

Gentzen Sequents $A, B \vdash C, D, E$

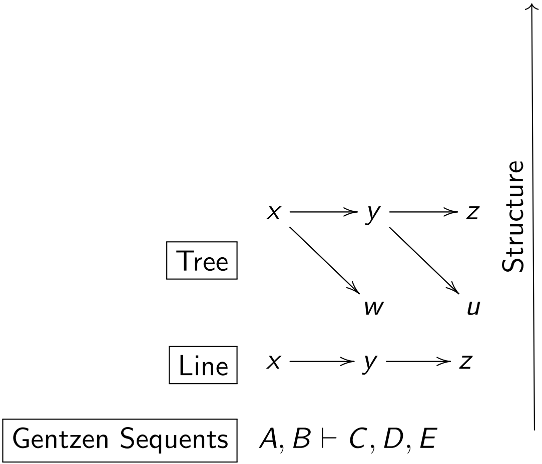
The Hierarchy of Sequents

Structure

Line $x \longrightarrow y \longrightarrow z$

Gentzen Sequents $A, B \vdash C, D, E$

The Hierarchy of Sequents



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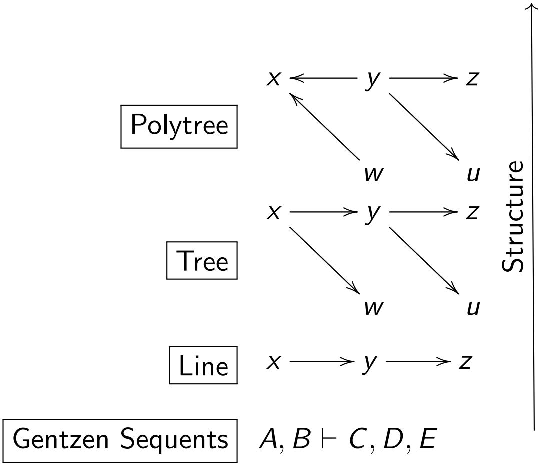

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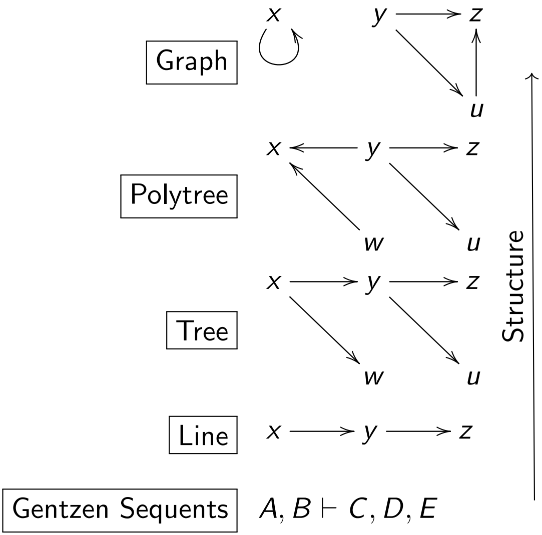

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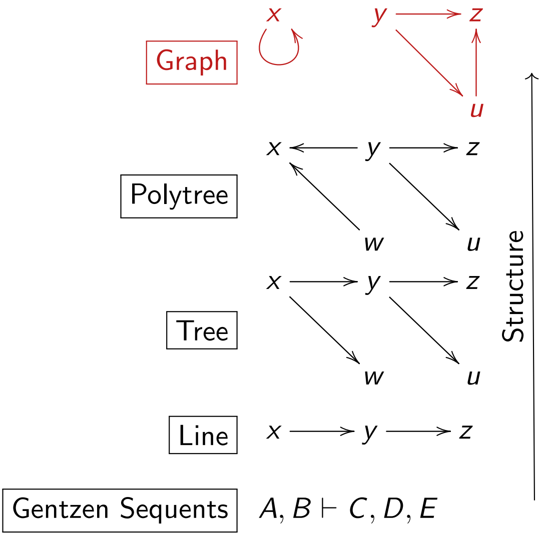

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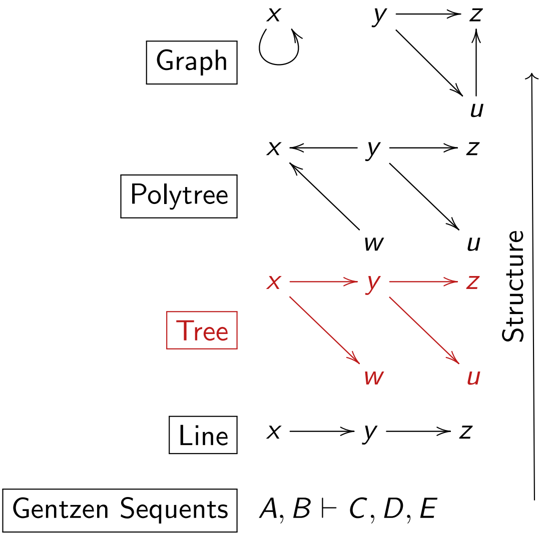
Q1 Reduce Sequent Structure?

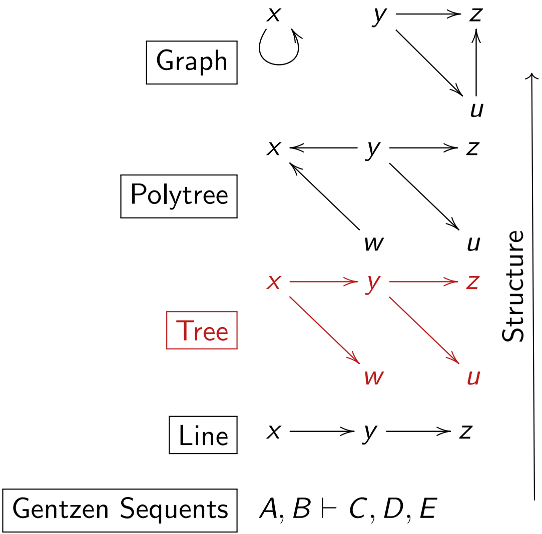


The Hierarchy of Sequents

Q1 Reduce Sequent Structure?

Q2 Retain ‘Nice’ Properties?

- Invertible Rules
- Admissible Rules
- Syntactic Cut-Elimination



Quantified Modal Logics

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Language:

$$A ::= R(x_1, \ldots, x_n) \mid x = y \mid \bot \mid A \supset A \mid \forall x A \mid \Box A$$

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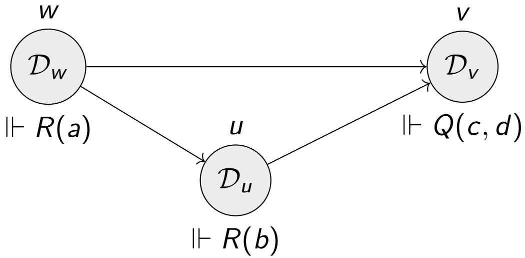

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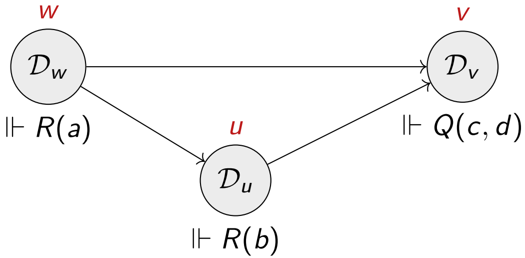

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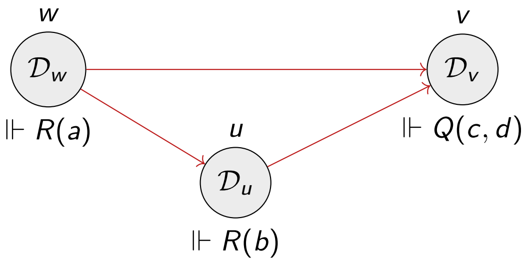

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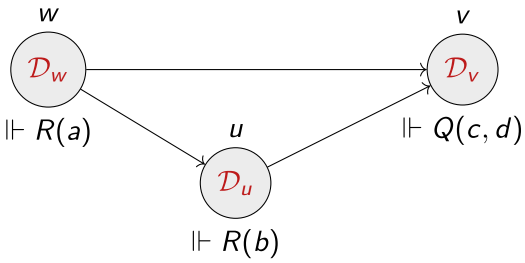

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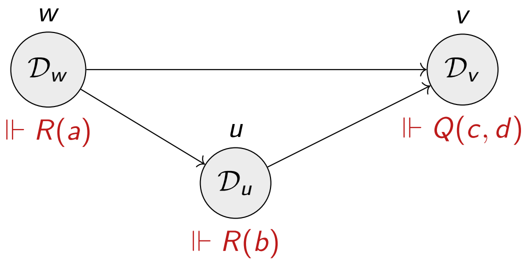

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Base Logic: QK = Set of Valid QML Formulae

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Base Logic: QK = Set of Valid QML Formulae

Extensions: QK +

1. Relational Properties
2. Domain Properties

Relational and Domain Conditions

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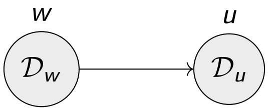

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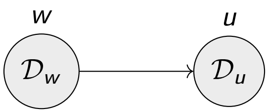

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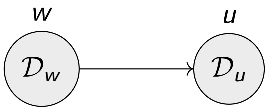

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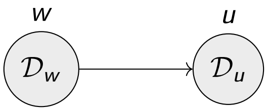
Varying Domains: No condition imposed.

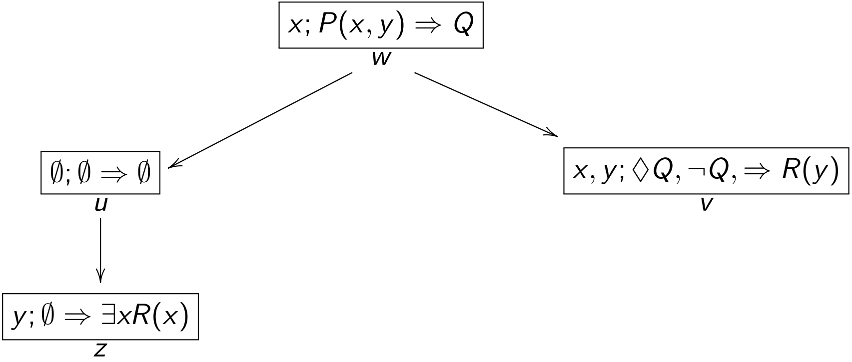


Nested Sequents

Nested Sequents: $x_1, \ldots, x_\ell; A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k, [\mathcal{S}_1], \ldots, [\mathcal{S}_m]$

Example:

$$x; P(x,y) \Rightarrow Q, [\emptyset; \emptyset \Rightarrow \emptyset, [y; \emptyset \Rightarrow \exists x R(x)]], [x, y; \Diamond Q, \neg Q, \Rightarrow R(y)]$$



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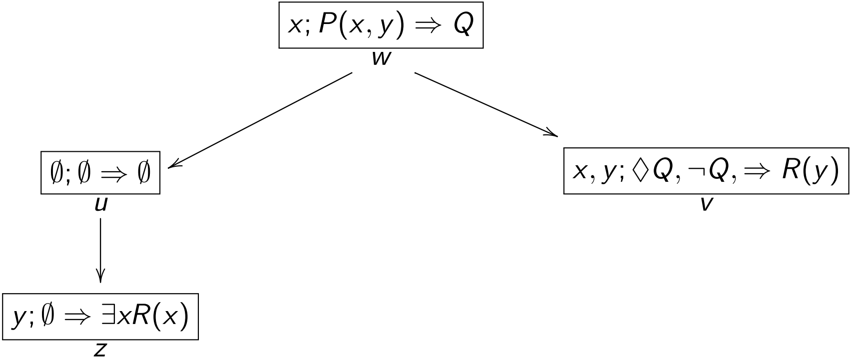

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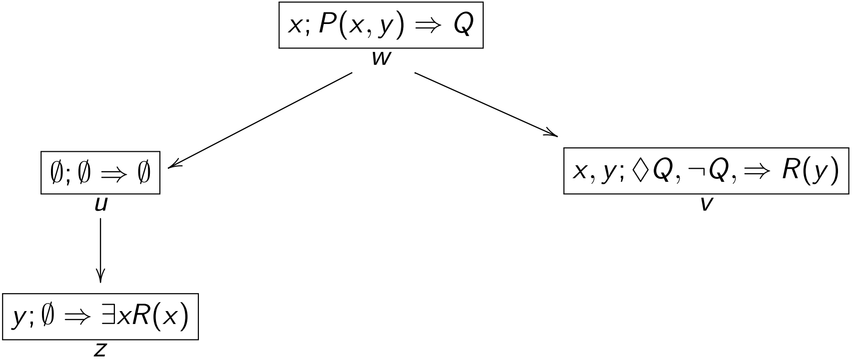

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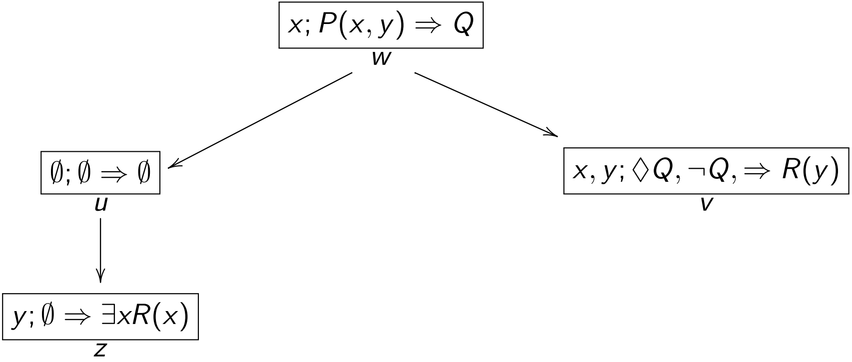

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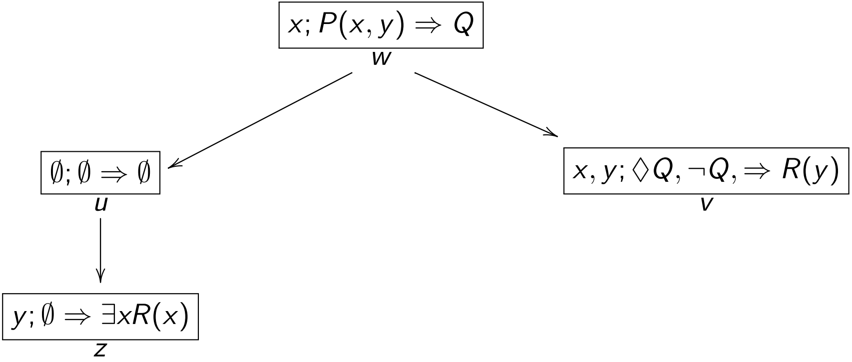

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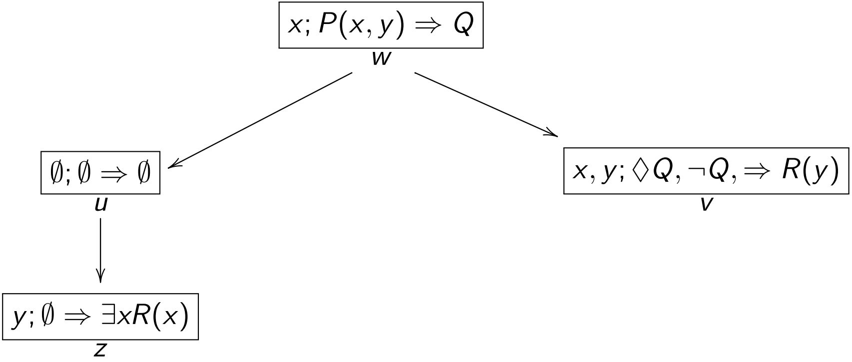

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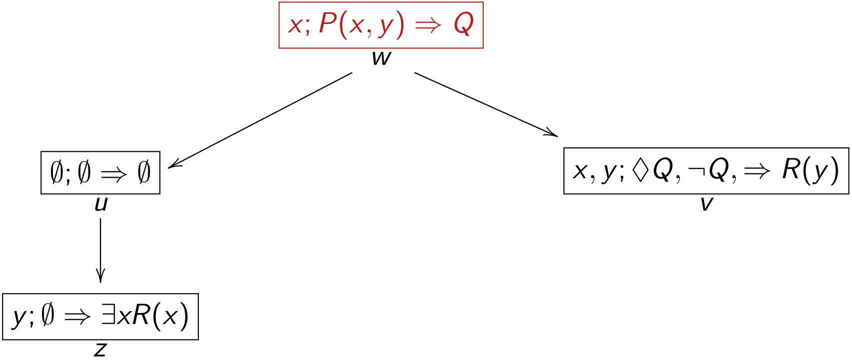

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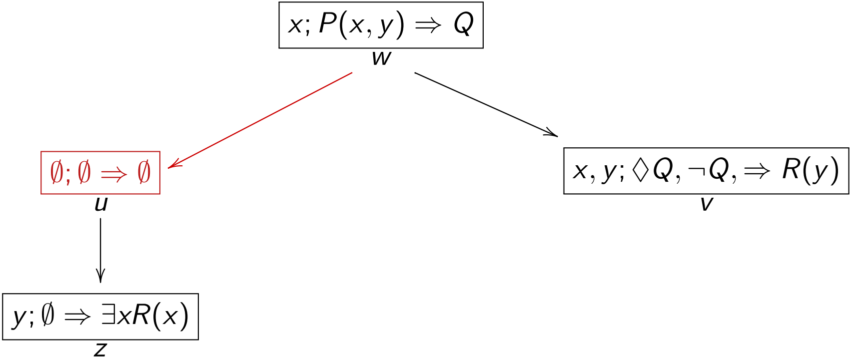

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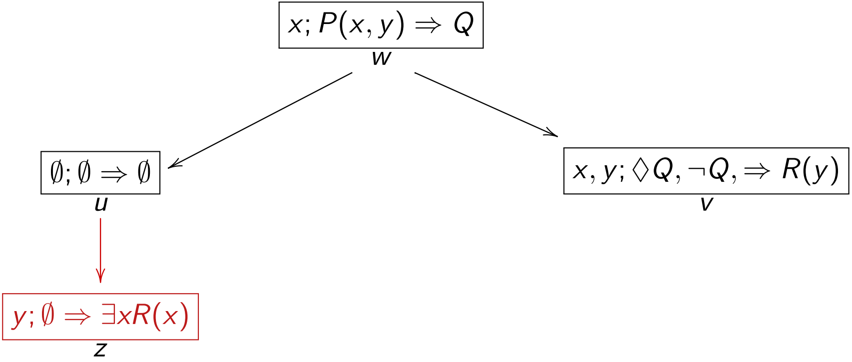

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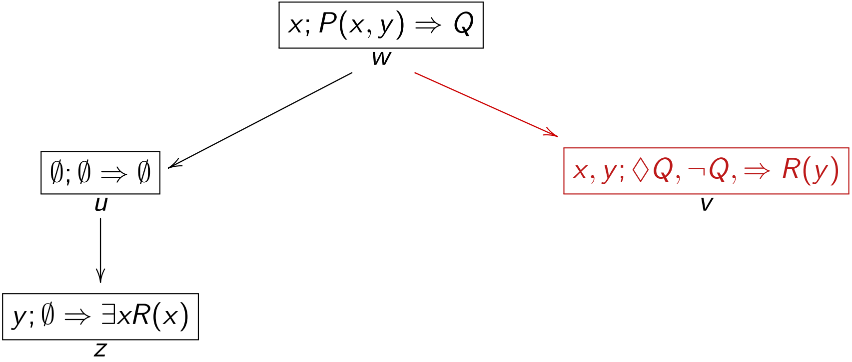

Notation and Interpretation

Notation:

- $X \equiv$ Multiset of Variables $x_1, \ldots, x_\ell$
- $\Gamma, \Delta \equiv$ Multiset of Formulas $A_1, \ldots, A_n$
- $\mathcal{S} \equiv$ Nested Sequent
- $\mathcal{E}x \equiv \exists y(x = y)$

Interpretation:

$\mathtt{fm}(X; \Gamma \Rightarrow \Delta, [\mathcal{S}_1], \ldots, [\mathcal{S}_m]) =$

$$\bigwedge_{x \in X} \mathcal{E}x \wedge \bigwedge_{A \in \Gamma} A \supset \bigvee_{B \in \Delta} B \vee \bigvee_{1 \leq i \leq m} \Box \mathtt{fm}(\mathcal{S}_i)$$

Nested Calculi: Logical Rules

$$\frac{}{\mathcal{S}\{X; \Gamma, R(\vec{x}) \Rightarrow R(\vec{x}), \Delta\}}\ Ax \qquad \frac{}{\mathcal{S}\{X; \Gamma, \bot \Rightarrow \Delta\}}\ L\bot$$

$$\frac{\mathcal{S}\{X; \Gamma \Rightarrow A, \Delta\} \qquad \mathcal{S}\{X; \Gamma, B \Rightarrow \Delta\}}{\mathcal{S}\{X; \Gamma, A \supset B \Rightarrow \Delta\}}\ L\supset \qquad \frac{\mathcal{S}\{X; \Gamma, A \Rightarrow B, \Delta\}}{\mathcal{S}\{X; \Gamma \Rightarrow A \supset B, \Delta\}}\ R\supset$$

$$\frac{\mathcal{S}\{X, y; \Gamma, \forall x A, A(y/x) \Rightarrow \Delta\}}{\mathcal{S}\{X, y; \Gamma, \forall x A \Rightarrow \Delta\}}\ L\forall \quad \frac{\mathcal{S}\{X, y; \Gamma \Rightarrow A(y/x), \Delta\}}{\mathcal{S}\{X; \Gamma \Rightarrow \forall x A, \Delta\}}\ R\forall\ (y \text{ fresh})$$

$$\frac{\mathcal{S}\{X; \Gamma, \Box A \Rightarrow \Delta, [Y; \Pi, A \Rightarrow \Sigma]\}}{\mathcal{S}\{X; \Gamma, \Box A \Rightarrow \Delta, [Y; \Pi \Rightarrow \Sigma]\}}\ L\Box \qquad \frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [\emptyset; \emptyset \Rightarrow A]\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Box A, \Delta\}}\ R\Box$$

Nested Calculi: Identity Rules

$$\frac{\mathcal{S}\{X; \Gamma, x = x \Rightarrow \Delta\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}}\ Ref \qquad \frac{\mathcal{S}\{X, x, y; \Gamma, x = y \Rightarrow \Delta\}}{\mathcal{S}\{X, x; \Gamma, x = y \Rightarrow \Delta\}}\ Repl_X$$

$$\frac{\mathcal{S}\{X; \Gamma, x = y, P(x/z), P(y/z) \Rightarrow \Delta\}}{\mathcal{S}\{X; \Gamma, x = y, P(x/z) \Rightarrow \Delta\}}\ Repl$$

$$\frac{\mathcal{S}\{X; \Gamma, x = y \Rightarrow \Delta\}\{Y; \Pi, x = y \Rightarrow \Sigma\}}{\mathcal{S}\{X; \Gamma, x = y \Rightarrow \Delta\}\{Y; \Pi \Rightarrow \Sigma\}}\ Rig$$

Nested Calculi: Propagation and Structural Rules

$$\frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [\emptyset; \Rightarrow]\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}}\ R_D \qquad \frac{\mathcal{S}\{X; A, \Box A, \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{X; \Box A, \Gamma \Rightarrow \Delta\}}\ R_T$$

$$\frac{\mathcal{S}\{X; A, \Gamma \Rightarrow \Delta, [Y; \Box A, \Pi \Rightarrow \Sigma]\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [Y; \Box A, \Pi \Rightarrow \Sigma]\}}\ R_B$$

$$\frac{\mathcal{S}\{X; \Box A, \Gamma \Rightarrow \Delta, [Y; \Box A, \Pi \Rightarrow \Sigma]\}}{\mathcal{S}\{X; \Box A, \Gamma \Rightarrow \Delta, [Y; \Pi \Rightarrow \Sigma]\}}\ R_4$$

$$\frac{\mathcal{S}\{X; \Box A, \Gamma \Rightarrow \Delta\}\{Y; \Box A, \Pi \Rightarrow \Sigma\}}{\mathcal{S}\{X; \Box A, \Gamma \Rightarrow \Delta\}\{Y; \Pi \Rightarrow \Sigma\}}\ R_5,\ Depth(\mathcal{S}\{\cdot\}\{\emptyset\}) \geq 1$$

Nested Calculi: Domain Rules

$$\frac{\mathcal{S}\{X, x; \Gamma \Rightarrow \Delta, [Y, x; \Pi \Rightarrow \Sigma]\}}{\mathcal{S}\{X, x; \Gamma \Rightarrow \Delta, [Y; \Pi \Rightarrow \Sigma]\}}\ R_{cbf}$$

$$\frac{\mathcal{S}\{X, x; \Gamma \Rightarrow \Delta, [Y, x; \Pi \Rightarrow \Sigma]\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [Y, x; \Pi \Rightarrow \Sigma]\}}\ R_{bf}$$

$$\frac{\mathcal{S}\{X, x; \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}}\ R_{ui}$$

$$\frac{\mathcal{S}\{X, x; \Gamma \Rightarrow \Delta\}\{Y, x; \Pi \Rightarrow \Sigma\}}{\mathcal{S}\{X, x; \Gamma \Rightarrow \Delta\}\{Y; \Pi \Rightarrow \Sigma\}}\ R^{\dagger}_{5dom}$$

$\dagger = Depth(\mathcal{S}\{\emptyset\}\{\cdot\}) \geq 1$ and $Depth(\mathcal{S}\{\cdot\}\{\emptyset\}) \geq 1$

Nice Properties I

1) Height-Preserving Admissibility:

$$\frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{X; \Pi, \Gamma \Rightarrow \Delta, \Sigma\}}\ IW \qquad \frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [Y; \Pi \Rightarrow \Sigma]\}}\ EW$$

$$\frac{\mathcal{S}}{\Rightarrow, [\mathcal{S}]}\ Nec \qquad \frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [Y; \Pi_1 \Rightarrow \Delta_1], [Z; \Pi_2 \Rightarrow \Delta_2]\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [Y, Z; \Pi_1, \Pi_2 \Rightarrow \Delta_1, \Delta_2]\}}\ Merge$$

$$\frac{\mathcal{S}\{X; \Gamma, A, A \Rightarrow \Delta\}}{\mathcal{S}\{X; \Gamma, A \Rightarrow \Delta\}}\ CL \qquad \frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, A, A\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, A\}}\ CR$$

$$\frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{X, x; \Gamma \Rightarrow \Delta\}}\ SW \qquad \frac{\mathcal{S}\{X, x, x; \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{X, x; \Gamma \Rightarrow \Delta\}}\ SC$$

Nice Properties II

2) Syntactic Cut-Elimination:

$$\frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, \Box A\}\{Y_i; \Pi_i \Rightarrow \Sigma_i\}_{i=1}^{n} \qquad \mathcal{S}\{X; \Box A, \Gamma \Rightarrow \Delta\}\{Y_i; \Box A, \Pi_i \Rightarrow \Sigma_i\}_{i=1}^{n}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}\{Y_i; \Pi_i \Rightarrow \Sigma_i\}_{i=1}^{n}} \text{ L-}Cut$$

$$\frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, A\} \quad \mathcal{S}\{X; A, \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}} \; Cut$$

3) Invertible Logical Rules

4) Simplified Syntax & Analytic

5) Modularity/Diverse Coverage

Future Work

1. Generalize to Cover Wider Classes of Logics
   - Bigger Class of Frame Conditions
   - Free logics
   - Additional Modalities, e.g. Converse Modalities
2. Relationships with Other Calculi, e.g. Labeled
3. Can We Further Simplify Sequents, e.g. Linear Nested Sequents?