# Description Logics

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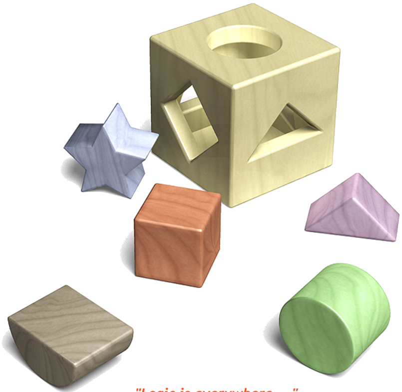


*"Logic is everywhere ..."*

# Alphabet

- We consider an alphabet with
  - constant symbols
  - unary and binary relation symbols
  - the variables $X, Y, \ldots$
  - the connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
  - the quantifiers $\forall, \exists$ and
  - the usual special symbols
- **Notation** $C$ denotes a unary relation symbol
  $R$ denotes a binary relation symbol

# Terms, Role and Concept Formulas

- The set of terms is the set of variables and constant symbols
- The set of role formulas consists of all strings of the form $R(X, Y)$, where $R/2$ ein relation symbol and $X, Y$ are variables
- The set of atomic concept formulas consists of all strings of the form $C(X)$, where $C/1$ is a relation symbol and $X$ a variable
- The set of concept formulas is the smallest set $\mathcal{C}$ satisfying the following properties:
  - All atomic concept formulas are in $\mathcal{C}$
  - If $F(X)$ is in $\mathcal{C}$ then $\neg F(X)$ is in $\mathcal{C}$
  - If $F(X)$ and $G(X)$ are in $\mathcal{C}$ then $(F(X) \wedge G(X))$ and $(F(X) \vee G(X))$ are in $\mathcal{C}$
  - If $R(X, Y)$ is a role fomula and if $F(Y)$ is in $\mathcal{C}$ then $(\exists Y)\,(R(X, Y) \wedge F(Y))$ and $(\forall Y)\,(R(X, Y) \rightarrow F(Y))$ are in $\mathcal{C}$
- Observe  Each concept formula contains precisely one free variable

# Concept Axioms and the T-Box

- Notation $C(X)$ denotes an atomic concept formula
  $F(X)$, $G(X)$ denote concept formulas
- The set of concept axioms consists of all strings of the form
  $(\forall X)\,(C(X) \rightarrow F(X))$ and $(\forall X)\,(C(X) \leftrightarrow F(X))$
- A terminology or T-Box $\mathcal{K}_T$ is a finite set of concept axioms such that
  - each $C$ occurs at most once as left-hand side of an axiom and
  - it does not contain any cycles
- The set of generalized concept axioms consists of all strings of the form
  $(\forall X)\,(F(X) \rightarrow G(X))$ and $(\forall X)\,(F(X) \leftrightarrow G(X))$

# A Simple Terminology

- **Example**

$$
\begin{array}{rcl}
(\forall X)\,(woman(X) & \rightarrow & person(X)) \\
(\forall X)\,(man(X) & \rightarrow & person(X)) \\
(\forall X)\,(mother(X) & \leftrightarrow & (woman(X) \wedge (\exists Y)\,(child(X,Y) \wedge person(Y)))) \\
(\forall X)\,(father(X) & \leftrightarrow & (man(X) \wedge (\exists Y)\,(child(X,Y) \wedge person(Y)))) \\
(\forall X)\,(parent(X) & \leftrightarrow & (mother(X) \vee father(X))) \\
(\forall X)\,(grandparent(X) & \leftrightarrow & (parent(X) \wedge (\exists Y)\,(child(X,Y) \wedge parent(Y)))) \\
(\forall X)\,(father\_without\_son(X) & \leftrightarrow & (father(X) \wedge (\forall Y)\,(child(X,Y) \rightarrow \neg man(Y))))
\end{array}
$$

- **Abbreviations**

$$
\begin{array}{rcl}
woman & \sqsubseteq & person \\
man & \sqsubseteq & person \\
mother & = & woman \sqcap \exists child : person \\
father & = & man \sqcap \exists child : person \\
parent & = & mother \sqcup father \\
grandparent & = & parent \sqcap \exists child : parent \\
father\_without\_son & = & father \sqcap \forall child : \neg man
\end{array}
$$

# Semantics

- Let $I = (\mathcal{D}, \cdot^I)$ be an interpretation
- Concept formulas

$$
\begin{array}{rcl}
C^I & \subseteq & \mathcal{D} \\
(\neg F)^I & = & \mathcal{D} \setminus F^I \\
(F \sqcup G)^I & = & F^I \cup G^I \\
(F \sqcap G)^I & = & F^I \cap G^I \\
R^I(d) & := & \{d' \in \mathcal{D} \mid (d, d') \in R^I\} \\
(\exists R : F)^I & = & \{d \in \mathcal{D} \mid R^I(d) \cap F^I \neq \emptyset\} \\
(\forall R : F)^I & = & \{d \in \mathcal{D} \mid R^I(d) \subseteq F^I\}
\end{array}
$$

- Concept axioms

$$
\begin{array}{lcl}
I \models F \sqsubseteq G & \text{iff} & F^I \subseteq G^I \\
I \models F = G & \text{iff} & F^I = G^I
\end{array}
$$

- **Remark**
  Sometimes the language is extended by $\top$ and $\bot$ with $\top^I = \mathcal{D}$ and $\bot^I = \emptyset$

# Assertions and the A-Box

- The set of **assertions** consists of all ground instances of $C(X)$ and $R(X, Y)$
- An **A-Box** is a finite set $\mathcal{K}_A$ of assertions
- **Semantics**

$$\begin{array}{lll} I \models C(a) & \text{iff} & a^I \in C^I \\ I \models R(a, b) & \text{iff} & b^I \in R^I(a^I) \end{array}$$

- $I \models \mathcal{K}_A$ iff $I \models A$ for all $A \in \mathcal{K}_A$

# A Simple A-Box

- $\mathcal{K}_T$

$$
\begin{array}{rcl}
woman & \sqsubseteq & person \\
man & \sqsubseteq & person \\
mother & = & woman \sqcap \exists child : person \\
father & = & man \sqcap \exists child : person \\
parent & = & mother \sqcup father \\
grandparent & = & parent \sqcap \exists child : parent \\
father\_without\_son & = & father \sqcap \forall child : \neg man
\end{array}
$$

- $\mathcal{K}_A$

$$
\begin{array}{l}
parent(carl) \\
parent(conny) \\
child(conny, joe) \\
child(conny, carl) \\
man(joe) \\
man(carl) \\
woman(conny)
\end{array}
$$

# Subsumption

- **Some Relations**

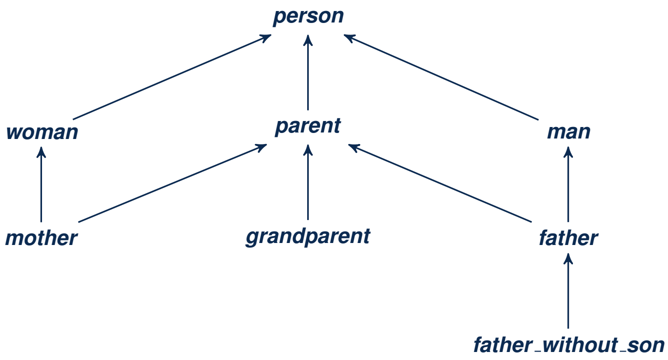
$$
\begin{array}{rcl}
G \text{ subsumes } F \text{ wrt } \mathcal{K}_T & \text{iff} & \mathcal{K}_T \models F \sqsubseteq G \\
G \text{ and } F \text{ are equivalent wrt } \mathcal{K}_T & \text{iff} & \mathcal{K}_T \models F = G \\
G \text{ and } F \text{ are disjoint wrt } \mathcal{K}_T & \text{iff} & \mathcal{K}_T \models F \sqcap G = \bot \\
F \text{ is unsatisfiable wrt } \mathcal{K}_T & \text{iff} & \mathcal{K}_T \models F = \bot
\end{array}
$$

- **Observations**
  - $F \sqsubseteq G \quad \equiv \quad F \sqcap \neg G = \bot$
  - Equivalence, disjointness and unsatisfiability can be reduced to subsumption

# Taxonomies

- We define
  - $F \sqsubseteq_T G$ iff $\mathcal{K}_T \models F \sqsubseteq G$
  - $F \equiv_T G$ iff $\mathcal{K}_T \models F = G$
- Observation Let $\mathcal{C}$ be a set of concept formulas
  - $\equiv_T$ is an equivalence relation on $\mathcal{C}$
  - $\sqsubseteq_T$ is a partial ordering on $\mathcal{C}|_{\equiv_T}$
  - There is a unique, mininal and binary relation $\triangleright_T \subseteq \mathcal{C} \times \mathcal{C}$ with $\overset{*}{\triangleright}_T = \sqsubseteq_T$
- The restriction of $\triangleright_T$ to the set of atomic concept formulas is called taxonomy

# Taxonomy – Example



# Unsatisfiability

- **Logical consequences wrt an A-box like**

$$\mathcal{K}_T \cup \mathcal{K}_A \models C(a)$$

**are equivalent to the question whether**

$$\mathcal{K}_T \cup \mathcal{K}_A \cup \{\neg C(a)\} \text{ is unsatisfiable}$$

- **Many other questions can be reduced to satisfiability testing**

# Some Remarks

- Subsumption and satisfiability are decidable, but intractable in the presented description logic
- Description logics may be extended to include
  - role restrictions
  - complex and/or transitive roles
  - cyclic concept definitions or
  - concrete domains like the reals

  But sometimes they are more restricted
- There are many applications like, for example, within the semantic web, bioinformatics, or medicine