Exercise 14.1. Let \( G = \langle V, E \rangle \) be an undirected graph. A cut \( C = \langle A, B \rangle \) of \( G \) is a partition \( A \cup B = V \) of \( V \) such that there is an edge in \( G \) from a vertex \( a \in A \) to a vertex \( b \in B \). The cut-set of \( C \) is \( S := \{ \{ a, b \} \in E \mid a \in A, b \in B \} \). The normalised cut value of a cut \( C = \langle A, B \rangle \) is

\[
\frac{|\{ \{ a, b \} \in E \mid a \in A, b \in B \}|}{|\{ \{ a, v \} \in E \mid a \in A, v \in V \}|} + \frac{|\{ \{ a, b \} \in E \mid a \in A, b \in B \}|}{|\{ \{ b, v \} \in E \mid b \in B, v \in V \}|},
\]

i.e., the number of endpoints of edges in the cut-set divided by the number of vertices that have an edge with an endpoint in \( A \) or \( B \).

Find the minimal cuts (i) with respect to the cardinality of the cut-set, and (ii) with respect to the normalised cut value of the following graph. Which of these cuts best describes the community structure of the graph?

![Graph](image)

Exercise 14.2. Let \( G = \langle V, E \rangle \) be an undirected graph. A bi-clique in \( G \) consists of two disjoint, nonempty sets \( A, B \subseteq V \) such that the induced subgraph of \( A \cup B \) in \( G \) is a complete bipartite graph, i.e., \( \{ \{ a, b \} \mid a \in A, b \in B \} \subseteq E \), and no two vertices in \( A \) (and \( B \), respectively) are adjacent.

Find all bi-cliques in the following graph. Which bi-cliques are maximal?

![Graph](image)
Exercise 14.3. Let $G = \langle V, E \rangle$ be a directed graph. For two vertices $v, w \in V$, the *distance* $d(v, w)$ is the length of the shortest directed path from $v$ to $w$ (or $\infty$ if there is no such path). For a set $S \subseteq V$ of vertices, the *reachable set* is $R(S) := \{ v \in V \mid \exists w \in S. d(w, v) < \infty \}$. A *point base* of $G$ is a minimal set $B \subseteq V$ such that $R(B) = V$.

Find a point base for the following graph. How does the point base change when adding the edge $\langle B, E \rangle$?

![Graph Diagram]

Exercise 14.4. Write a program that takes as input

- a directed graph in METIS format
- and a dictionary file in the format of Exercise 12.4 mapping vertex IDs to labels,

and uses the Girvan-Newman algorithm to print out all communities on level $k$ of the hierarchical clustering of the input graph.

Modify the program from Exercise 12.4 to extract the subgraph from Wikidata that consists of all edges of the form $t \to s$, where $t$ has a P40 (“child”), a P25 (“mother”), or a P22 (“father”) statement with value $s$, and also occurs as the value of a P35 (“Head of State”) or P6 (“Head of Government”) statement. Use your program to print out the communities on level 5 of the hierarchical clustering of this subgraph.

*Hint:* NetworkX\(^1\) provides an implementation of the GN algorithm.\(^2\)

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\(^1\)https://networkx.github.io/

\(^2\)https://networkx.github.io/documentation/latest/reference/algorithms/generated/networkx.algorithms.community.centrality.girvan_newman.html