

COMPLEXITY THEORY

Lecture 2: Turing Machines and Languages

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Knowledge-Based Systems

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More recent versions of this slide deck might be available.
For the most current version of this course, see
https://iccl.inf.tu-dresden.de/web/Complexity_Theory/en

A Model for Computation

Clear

To understand computational problems we need to have a formal understanding of what an **algorithm** is.

Example 2.1 (Hilbert's Tenth Problem):

“Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers.”
(→ Wikipedia)

Question

How can we model the notion of an algorithm?

Answer

With Turing machines.

Turing Machines

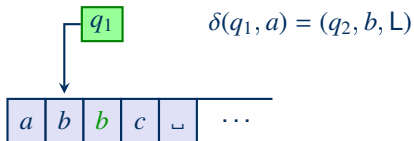
Let us fix a blank symbol \sqcup .

Definition 2.2: A (deterministic) **Turing Machine** $\mathcal{M} = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ consists of

- a finite set Q of **states**,
- an **input alphabet** Σ not containing \sqcup ,
- a **tape alphabet** Γ such that $\Gamma \supseteq \Sigma \cup \{\sqcup\}$.
- a **transition function** $\delta: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$
- an **initial state** $q_0 \in Q$,
- an **accepting state** $q_{\text{accept}} \in Q$, and
- a **rejecting state** $q_{\text{reject}} \in Q$ such that $q_{\text{accept}} \neq q_{\text{reject}}$.

Turing Machines

Example 2.3:



- The tape is bounded on the left, but unbounded on the right; the content of the tape is a finite word over Γ , followed by an infinite sequence of \sqcup .
- The head of the machine is at exactly one position of the tape
- The head can read only one symbol at a time
- The head moves and writes according to the transition function δ ; the current state also changes accordingly
- The head will stay put when attempting to cross the left tape end

Configurations

Observation: to describe the current step of a computation of a TM it is enough to know

- the content of the tape,
- the current state, and
- the position of the head

Definition 2.4: A **configuration** of a TM \mathcal{M} is a word uqv such that

- $q \in Q$,
- $uv \in \Gamma^*$

Some special configurations:

- The **start configuration** for some input word $w \in \Sigma^*$ is the configuration q_0w
- A configuration uqv is **accepting** if $q = q_{\text{accept}}$.
- A configuration uqv is **rejecting** if $q = q_{\text{reject}}$.

Computation

We write

- $C \vdash_{\mathcal{M}} C'$ only if C' can be reached from C by one computation step of \mathcal{M} ;
- $C \vdash_{\mathcal{M}}^* C'$ only if C' can be reached from C in a finite number of computation steps of \mathcal{M} .

We say that \mathcal{M} **halts** on input w if and only if there is a finite sequence of configurations

$$C_0 \vdash_{\mathcal{M}} C_1 \vdash_{\mathcal{M}} \cdots \vdash_{\mathcal{M}} C_\ell$$

such that C_0 is the start configuration of \mathcal{M} on input w and C_ℓ is an accepting or rejecting configuration. Otherwise \mathcal{M} **loops** on input w .

We say that \mathcal{M} **accepts** the input w only if \mathcal{M} halts on input w with an accepting configuration.

Recognisability and Decidability

Definition 2.5: Let \mathcal{M} be a Turing machine with input alphabet Σ . The language accepted by \mathcal{M} is the set

$$\mathbf{L}(\mathcal{M}) := \{ w \in \Sigma^* \mid \mathcal{M} \text{ accepts } w \}.$$

A language $\mathbf{L} \subseteq \Sigma^*$ is called **Turing-recognisable** (recursively enumerable) if and only if there exists a Turing machine \mathcal{M} with input alphabet Σ such that $\mathbf{L} = \mathbf{L}(\mathcal{M})$. In this case we say that \mathcal{M} **recognises** \mathbf{L} .

A language $\mathbf{L} \subseteq \Sigma^*$ is called **Turing-decidable** (decidable, recursive) if and only if there exists a Turing machine \mathcal{M} such that $\mathbf{L} = \mathbf{L}(\mathcal{M})$ and \mathcal{M} halts on every input. In this case we say that \mathcal{M} **decides** \mathbf{L} .

Example

Claim 2.6: The language $\mathbf{L} := \{ a^{2^n} \mid n \geq 0 \}$ is decidable.

Proof: A Turing machine \mathcal{M} that decides \mathbf{L} is

$\mathcal{M} :=$ On input w , where w is a string

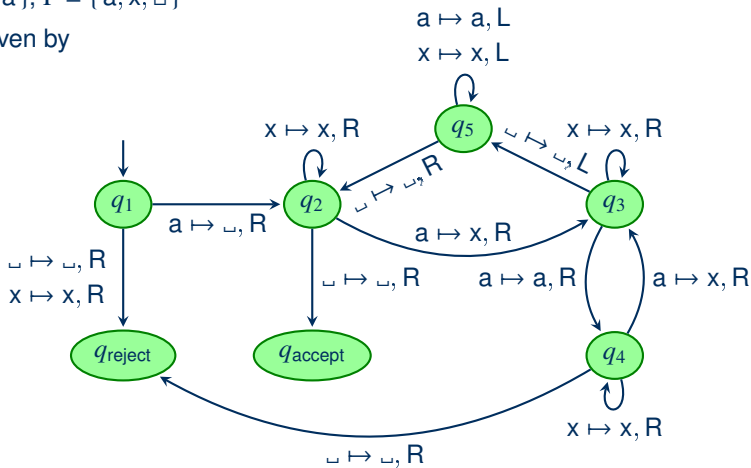
- Go from left to right over the tape and cross off every other a
- If in the first step the tape contained a single a, accept
- If in the first step the number of a's on the tape was odd, reject
- Return the head the beginning of the tape
- Go to the first step

Example (cont'd)

Formally, $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_1, q_{\text{accept}}, q_{\text{reject}})$, where

- $Q = \{q_1, q_2, q_3, q_4, q_5, q_{\text{accept}}, q_{\text{reject}}\}$
- $\Sigma = \{a\}, \Gamma = \{a, x, \sqcup\}$

and δ is given by



Problems as Languages

Observation

- Languages can be used to model computational problems.
- For this, a suitable **encoding** is necessary
- TMs must be able to decode the encoding

Example 2.7 (Graph-Connectedness): The question whether a graph is connected or not can be seen as the **word problem** of the following language

$$\text{GCONN} := \{ \langle G \rangle \mid G \text{ is a connected graph} \},$$

where $\langle G \rangle$ is (for example) the adjacency matrix encoded in binary.

Notation 2.8: The encoding of objects O_1, \dots, O_n we denote by $\langle O_1, \dots, O_n \rangle$.

The Church-Turing Thesis

It turns out that Turing-machines are **equivalent** to a number of formalisations of the intuitive notion of an **algorithm**

- λ -calculus
- while-programs
- μ -recursive functions
- Random-Access Machines
- ...

Because of this it is believed that Turing-machines completely capture the intuitive notion of an algorithm. \leadsto **Church-Turing Thesis:**

“A function on the natural numbers is intuitively computable if and only if it can be computed by a Turing machine.”

(\rightarrow Wikipedia: Church-Turing Thesis)

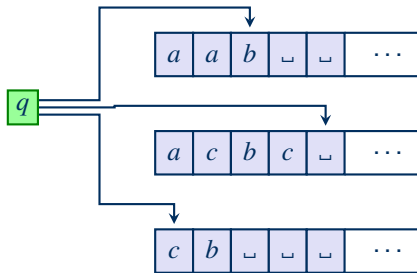
Variations of Turing-Machines

It has also been shown that deterministic, single-tape Turing machines are equivalent to a wide range of other forms of Turing machines:

- Multi-tape Turing machines
- Nondeterministic Turing machines
- Turing machines with doubly-infinite tape
- Multi-head Turing machines
- Two-dimensional Turing machines
- Write-once Turing machines
- Two-stack machines
- Two-counter machines
- ...

Multi-Tape Turing Machines

k -tape Turing machines are a variant of Turing machines that have k tapes.



Multi-Tape Turing Machines

Definition 2.9: Let $k \in \mathbb{N} \setminus \{0\}$. Then a (deterministic) **k -tape Turing machine** is a tuple $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where

- $Q, \Sigma, \Gamma, q_0, q_{\text{accept}}, q_{\text{reject}}$ are as for TMs
- δ is a transition function for k tapes, i.e.,

$$\delta: Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R, N\}^k$$

Running \mathcal{M} on input $w \in \Sigma^*$ means to start \mathcal{M} with the content of the first tape being w and all other tapes blank.

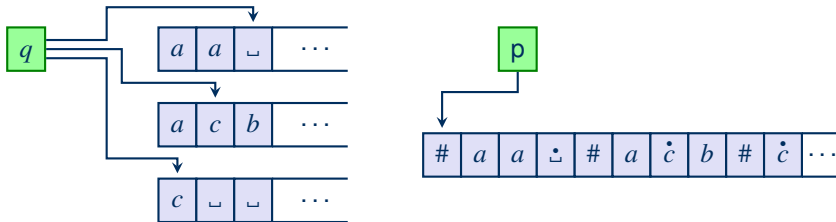
The notions of a **configuration** and of the **language accepted by \mathcal{M}** are defined analogously to the single-tape case.

Multi-Tape Turing Machines

Theorem 2.10: Every multi-tape Turing machine has an equivalent single-tape Turing machine.

Proof: Let \mathcal{M} be a k -tape Turing machine. Simulate \mathcal{M} with a single-tape TM S by

- keeping the content of all k tapes on a single tape, separated by #
- marking the positions of the individual heads using special symbols



Multi-Tape Turing Machines

$S :=$ On input $w = w_1 \dots w_n$

- Format the tape to contain the word

$$\#w_1w_2\dots w_n\#\# \# \# \dots \#$$

- Scan the tape from the first $\#$ to the $(k + 1)$ -th $\#$ to determine the symbols below the markers.
- Update all tapes according to \mathcal{M} 's transition function with a second pass over the tape; if any head of \mathcal{M} moves to some previously unread portion of its tape, insert a blank symbol at the corresponding position and shift the right tape contents by one cell
- Repeat until the accepting or rejecting state is reached.

□

Nondeterministic Turing Machines

Goal

Allow transitions to be **nondeterministic**.

Approach

Change transition function from

$$\delta: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

to

$$\delta: Q \times \Gamma \rightarrow 2^{Q \times \Gamma \times \{L, R\}}.$$

The notions of **accepting** and **rejecting computations** are defined accordingly.

Note: there may be more than one or no computation of a nondeterministic TM on a given input.

A nondeterministic TM \mathcal{M} **accepts** an input w if and only if **there exists** some accepting computation of \mathcal{M} on input w .

Nondeterministic Turing Machines

Theorem 2.11: Every nondeterministic TM has an equivalent deterministic TM.

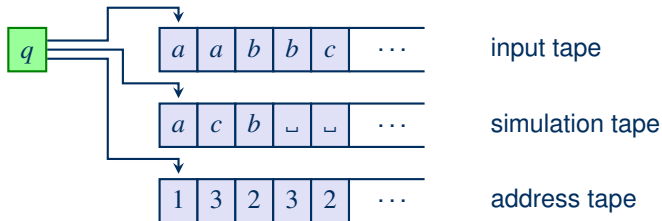
Proof: Let N be a nondeterministic TM. We construct a deterministic TM D that is equivalent to N , i.e., $\mathbf{L}(N) = \mathbf{L}(D)$.

Idea

- D deterministically traverses in breadth-first order the tree of configuration of N , where each branch represents a different possibility for N to continue.
- For this, successively try out all possible choices of transitions allowed by N .

Nondeterministic Turing Machines

Sketch of D :



Let b be the maximal number of choices in δ , i.e.,

$$b := \max\{|\delta(q, x)| \mid q \in Q, x \in \Gamma\}.$$

Nondeterministic Turing Machines

D works as follows:

- (1) Start: input tape contains input w , simulation and address tape empty
- (2) Initialise the address tape with 0.
- (3) Copy w to the simulation tape.
- (4) Simulate one finite computation of N on w on the simulation tape.
 - Interpret the address tape as a list of zero-indexed choices to make during this computation (and abort if the end of the tape is reached).
 - If a choice is invalid, abort simulation.
 - If an accepting configuration is reached at the end of the simulation, accept.
- (5) “Increment” the content of the address tape by 1, intuitively considered as a number in base b **but** $b - 1$ increments to 00, $0b - 1$ to 10 and so on.
Go to step 3.

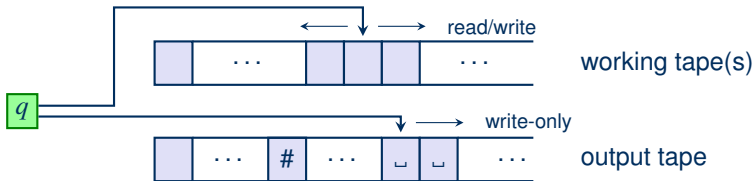


Enumerators

Definition 2.12: A multi-tape Turing machine \mathcal{M} is an **enumerator** if

- \mathcal{M} has a designated write-only **output-tape** on which a symbol, once written, can never be changed and where the head can never move left;
- \mathcal{M} has a **marker symbol** $\#$ separating words on the output tape.

We define the **language generated by \mathcal{M}** to be the set $\mathbf{G}(\mathcal{M})$ of all words that eventually appear between two consecutive $\#$ on the output tape of \mathcal{M} when started on the empty word as input.



Enumerators

Theorem 2.13: A language \mathbf{L} is Turing-recognisable if and only if there exists some enumerator \mathcal{E} such that $\mathbf{G}(\mathcal{E}) = \mathbf{L}$.

Proof: Let \mathcal{E} be an enumerator for \mathbf{L} . Then the following TM accepts \mathbf{L} :

$\mathcal{M} :=$ On input w

- Simulate \mathcal{E} on the empty input. Compare every string output by \mathcal{E} with w
- If w appears in the output of \mathcal{E} , accept

Enumerators

Let $\mathbf{L} = \mathbf{L}(\mathcal{M})$ for some TM \mathcal{M} , and let s_1, s_2, \dots be an enumeration of Σ^* . Then the following enumerator \mathcal{E} enumerates \mathbf{L} :

$\mathcal{E} :=$ Ignore the input.

- Print the first # to initialise the output.
- Repeat for $i = 1, 2, 3, \dots$
 - Run \mathcal{M} for i steps on each input s_1, s_2, \dots, s_i
 - If any computation accepts, print the corresponding s_j followed by #

□

Theorem 2.14: If \mathbf{L} is Turing-recognisable, then there exists an enumerator for \mathbf{L} that prints each word of \mathbf{L} exactly once.

Enumerators

Theorem 2.15: A language L is decidable if and only if there exists an enumerator for L that outputs exactly the words of L in some order of non-decreasing length.

Proof: Suppose L to be decidable, and let M be a TM that decides L .

- Define a TM M' that generates, on some scratch tape, all words over Σ in some order of non-decreasing length. (Exercise!)
- An enumerator \mathcal{E} works as follows:
 - (1) Print the first $\#$ to initialise the output.
 - (2) Run M' (enumerating words), followed by M (to check if the current word is accepted). If M accepts w , then print w followed by $\#$.

Then \mathcal{E} enumerates exactly the words of L in some order of non-decreasing length.

Enumerators

Now suppose \mathbf{L} can be enumerated by some TM \mathcal{E} in some order of non-decreasing length.

- If \mathbf{L} is finite, then \mathbf{L} is accepted by a finite automaton.
- If \mathbf{L} is infinite, then we define a decider \mathcal{M} for it as follows.

$\mathcal{M} :=$ On input w

- Simulate \mathcal{E} until it either outputs w or some word longer than w
- If \mathcal{E} outputs w , then accept, else reject.

Observation: since \mathbf{L} is infinite, for each $w \in \Sigma^*$ the TM \mathcal{E} will eventually generate w or some word longer than w . Therefore, \mathcal{M} always halts and thus decides \mathbf{L} .

□

Summary and Outlook

Turing Machines are a simple model of computation

Recognisable (semi-decidable) = recursively enumerable

Decidable = computable = recursive

Many variants of TMs exist – they normally recognise/decide the same languages

What's next?

- A short look into undecidability
- Recursion and self-referentiality
- Actual complexity classes

Looking for Project or Thesis Topics?

On **Thursday, 24 Oct 2024 at 1pm in APB 3027** we will present possible topics to conduct in the **Knowledge-Based Systems** group as a **study project** (many suitable modules) or **final thesis** (BSc, MSc, Diploma).

Not only **theoretical topics** but also **implementation work**.

We also have student **job opportunities (SHK/WHK)**.

You are especially welcome if you are eager to work with **Rust** or **LEAN** :)

See also: https://iccl.inf.tu-dresden.de/web/Projekte_und_Studienarbeiten_Wissensbasierte_Systeme/en