Answer Set Programming: Basics

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Slides based on a lecture by Martin Gebser and Torsten Schaub.

Answer Set Programming – Basics: Overview

Outline

Normal logic programs

- A logic program, P, over a set $\mathcal{A}$ of atoms is a finite set of rules
- A (normal) rule, r, is of the form

$$a_0 \leftarrow a_1, \ldots, a_m, \sim a_{m+1}, \ldots, \sim a_n$$

where $0 \leq m \leq n$ and each $a_i \in \mathcal{A}$ is an atom for $0 \leq i \leq n$

- Notation

$$\begin{array}{rcl} head(r) & = & a_0 \\ body(r) & = & \{a_1, \ldots, a_m, \sim a_{m+1}, \ldots, \sim a_n\} \\ body(r)^+ & = & \{a_1, \ldots, a_m\} \\ body(r)^- & = & \{a_{m+1}, \ldots, a_n\} \\ atom(P) & = & \bigcup_{r \in P} \left(\{head(r)\} \cup body(r)^+ \cup body(r)^-\right) \\ body(P) & = & \{body(r) \mid r \in P\} \end{array}$$

- A program P is positive if $body(r)^- = \emptyset$ for all $r \in P$

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Formal Definition

Stable models of positive programs

- A set of atoms X is closed under a positive program P iff for any $r \in P$, $head(r) \in X$ whenever $body(r)^+ \subseteq X$
  - X corresponds to a model of P (seen as a formula)
- The smallest set of atoms which is closed under a positive program P is denoted by $Cn(P)$
  - $Cn(P)$ corresponds to the $\subseteq$-smallest model of P (ditto)
- The set $Cn(P)$ of atoms is the stable model of a *positive* program P

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Stable model of normal programs

- The reduct, P^X, of a program P relative to a set X of atoms is defined by

$$P^X = \{head(r) \leftarrow body(r)^+ \mid r \in P \text{ and } body(r)^- \cap X = \emptyset\}$$

- A set X of atoms is a stable model of a program P, if $Cn(P^X) = X$

- Note $Cn(P^X)$ is the $\subseteq$–smallest (classical) model of P^X
- Note Every atom in X is justified by an *"applying rule from P"*

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A closer look at P^X

- In other words, given a set X of atoms from P,

  P^X is obtained from P by deleting

  1. each rule having $\sim a$ in its body with $a \in X$
     and then
  2. all negative atoms of the form $\sim a$
     in the bodies of the remaining rules

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A first example

$$P = \{p \leftarrow p,\ q \leftarrow \sim p\}$$

X	P^X	$Cn(P^X)$	
$\{\ \ \}$	$p \leftarrow p$ $q \leftarrow$	$\{q\}$	✗
$\{p\ \}$	$p \leftarrow p$	$\emptyset$	✗
$\{\ q\}$	$p \leftarrow p$ $q \leftarrow$	$\{q\}$	
$\{p, q\}$	$p \leftarrow p$	$\emptyset$	✗

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Some properties

- A logic program may have zero, one, or multiple stable models!
- If X is a stable model of a logic program P, then X is a model of P (seen as a formula)
- If X and Y are stable models of a *normal* program P, then $X \not\subset Y$

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Outline

Reasoning Modes

- Satisfiability
- Enumeration†
- Projection†
- Intersection‡
- Union‡
- Optimization
- and combinations of them

† without solution recording

‡ without solution enumeration