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Correctness of SLD Resolution

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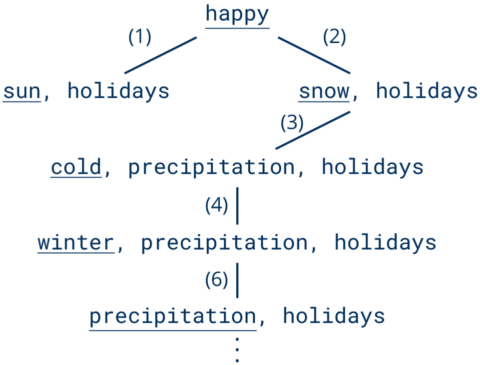
Previously ...

- A proof theory for (definite) logic programs is given by **SLD resolution**.
- A query is resolved with a (variant of a) program clause to another query.
- There are choices to be made (renaming of clause, mgu of query atom and clause, selected atom in query, program clause) with consequences.
- The search space can be visualized by (selection rule-induced) **SLD trees**.

```
(1) happy :- sun, holidays.
(2) happy :- snow, holidays.
(3) snow :- cold, precipitation.
(4) cold :- winter.
(5) precipitation :- holidays.
(6) winter.
(7) holidays.

| ?- happy.
```



Overview

Model Theory: Algebras, Interpretations, and Models

Soundness of SLD Resolution

Completeness of SLD Resolution

Model Theory: Algebras, Interpretations, and Models

Algebras (Semantics of Terms)

Definition

Let V be a set of variables, F be a ranked alphabet of function symbols.
An **algebra** J for F (or F**-algebra** or **pre-interpretation** for F) consists of:

1. A **domain**, a non-empty set D;
2. the assignment of a mapping $f_J \colon D^n \to D$ to every $f \in F^{(n)}$ with $n \geq 0$.

For $f \in F^{(0)}$, the constant function $f_J : D^0 \to D$ maps () to some $d \in D$.

Definition

A **state** σ over D is a mapping $\sigma \colon V \to D$.
The extension of σ to $TU_{F,V}$ by algebra J is the function $\sigma_J \colon TU_{F,V} \to D$ such that for every $f \in F^{(n)}$,

$$\sigma_J(f(t_1, \ldots, t_n)) := f_J(\sigma(t_1), \ldots, \sigma(t_n))$$

For first-order logic, a state is typically called a variable assignment.

Interpretations (Semantics of Programs)

Definition

Let F be a ranked alphabet of function symbols, Π be a ranked alphabet of predicate symbols.

An **interpretation** I for F and Π consists of :

1. An algebra J for F (with domain D);
2. the assignment of a relation

$$p_I \subseteq \underbrace{D \times \cdots \times D}_{n}$$

   to every $p \in \Pi^{(n)}$ with $n \geq 0$.

For $p \in \Pi^{(0)}$, we have $p_I \subseteq \{()\}$, that is, either $p_I = \emptyset$ (false) or $p_I = \{()\}$ (true).

$\rightsquigarrow$ Standard definition of first-order logic interpretations.

Interpretations (Example)

$$add(x, 0, x) \leftarrow$$
$$add(x, s(y), s(z)) \leftarrow add(x, y, z)$$

Consider the addition program, P_{add}:

I_1, I_2, I_3, I_4, I_5, and I_6 are interpretations for $\{s, 0\}$ and $\{add\}$:

I_1: $D_{I_1} = \mathbb{N}$, $0_{I_1} = 0$, $s_{I_1} = \{n \mapsto n+1 \mid n \in \mathbb{N}\}$, $add_{I_1} = \{(m, n, m+n) \mid m, n \in \mathbb{N}\}$

I_2: $D_{I_2} = \mathbb{N}$, $0_{I_2} = 0$, $s_{I_2} = \{n \mapsto n+1 \mid n \in \mathbb{N}\}$, $add_{I_2} = \{(m, n, m * n) \mid m, n \in \mathbb{N}\}$

I_3: $D_{I_3} = HU_{\{s,0\}}$, $0_{I_3} = 0$, $s_{I_3} = \left\{t \mapsto s(t) \,\middle|\, t \in HU_{\{s,0\}}\right\}$,
$add_{I_3} = \{(s^m(0), s^n(0), s^{m+n}(0)) \mid m, n \in \mathbb{N}\}$

I_4: $D_{I_4} = HU_{\{s,0\}}$, $0_{I_4} = 0$, $s_{I_4} = \left\{t \mapsto s(t) \,\middle|\, t \in HU_{\{s,0\}}\right\}$, $add_{I_4} = \emptyset$

I_5: $D_{I_5} = HU_{\{s,0\}}$, $0_{I_5} = 0$, $s_{I_5} = \left\{t \mapsto s(t) \,\middle|\, t \in HU_{\{s,0\}}\right\}$, $add_{I_5} = (HU_{\{s,0\}})^3$

I_6: $D_{I_6} = \{0, 1\}$, $0_{I_6} = 0$, $s_{I_6} = \{0 \mapsto 0, 1 \mapsto 1\}$,
$add_{I_6} = \{(m, n, m) \mid m, n \in \{0, 1\}\}$

Logical Truth (1)

Definition

An **expression** E is an atom, a query, a clause, or a resultant.

Definition

Let E be an expression, I be an interpretation, σ be a state.

We say that E is **true in** I **under** σ and write $I \models_\sigma E$

$$:\Longleftrightarrow$$

by case analysis on E:

$$
\begin{aligned}
I \models_\sigma p(t_1, \ldots, t_n) \quad & :\Longleftrightarrow \quad (\sigma_I(t_1), \ldots, \sigma_I(t_n)) \in p_I \\
I \models_\sigma A_1, \ldots, A_n \quad & :\Longleftrightarrow \quad I \models_\sigma A_i \text{ for every } 1 \leq i \leq n \\
I \models_\sigma A \leftarrow \vec{B} \quad & :\Longleftrightarrow \quad \text{if } I \models_\sigma \vec{B} \text{ then } I \models_\sigma A \\
I \models_\sigma \vec{A} \leftarrow \vec{B} \quad & :\Longleftrightarrow \quad \text{if } I \models_\sigma \vec{B} \text{ then } I \models_\sigma \vec{A}
\end{aligned}
$$

Logical Truth (2)

Definition

Let E be an expression and I be an interpretation.
Furthermore, let $x_1, \ldots, x_k$ be the variables occuring in E.

- $\forall x_1, \ldots, \forall x_k\, E$ is the **universal closure** of E (abbreviated $\forall E$)
- $\exists x_1, \ldots, \exists x_k\, E$ is the **existential closure** of E (abbreviated $\exists E$)
- $I \models \forall E \;:\Longleftrightarrow\; I \models_\sigma E$ for every state σ
- $I \models \exists E \;:\Longleftrightarrow\; I \models_\sigma E$ for some state σ
- E is **true in** I (or: I is a **model of** E), written: $I \models E \;:\Longleftrightarrow\; I \models \forall E$

$\leadsto$ Standard first-order logic definition of logical truth (for expressions).

Logical Truth (III)

Definition

Let S and T be sets of expressions and I be an interpretation.

- I is a **model of** S, written: $I \models S \;:\Longleftrightarrow\; I \models E$ for every $E \in S$
- T is a **logical consequence of** S, written: $S \models T$
  $:\Longleftrightarrow$ every model of S is a model of T

We sometimes refer to logical consequences as **semantic** consequences to stress their model-theoretic definition.

Definition

Let P be a program, Q_0 be a query, and θ be a substitution.

- $\theta|_{Var(Q_0)}$ is a **correct answer substitution** of $Q_0 \;:\Longleftrightarrow\; P \models Q_0\theta$
- $Q_0\theta$ is a **correct instance** of $Q_0 \;:\Longleftrightarrow\; P \models Q_0\theta$

$\rightsquigarrow$ Model-theoretic counterparts to *computed* answer substitutions/instances.

Models (Example)

Consider again P_{add}:

$$add(x, 0, x) \leftarrow$$
$$add(x, s(y), s(z)) \leftarrow add(x, y, z)$$

Furthermore, let I_1, I_2, I_3, I_4, I_5, and I_6 be the interpretations from Slide 7.

- $I_1 \models P_{add}$, since $I_1 \models_\sigma c$ for every clause $c \in P_{add}$ and state $\sigma : V \to \mathbb{N}$:
  1. $(\sigma(x), \sigma(0), \sigma(x)) \in add_{I_1}$ and
  2. if $(\sigma(x), \sigma(y), \sigma(z)) \in add_{I_1}$ then $(\sigma(x), \sigma(y) + 1, \sigma(z) + 1) \in add_{I_1}$.
- $I_2 \not\models P_{add}$:
  (E.g. let $\sigma(x) = 1$, then $I_2 \not\models_\sigma add(x, 0, x)$ since $(\sigma(x), \sigma(0), \sigma(x)) = (1, 0, 1) \notin add_{I_2}$.)
- $I_3 \models P_{add}$ (like for I_1; we call I_3 a (least) Herbrand model)
- $I_4 \not\models P_{add}$ (e.g. let $\sigma(x) = s(0)$, then $I_4 \not\models_\sigma add(x, 0, x)$ since $(\sigma(x), \sigma(0), \sigma(x)) = (s(0), 0, s(0)) \notin add_{I_4}$)
- $I_5 \models P_{add}$ (like for I_1; we call I_5 a Herbrand model)
- $I_6 \models P_{add}$ (like for I_1)

I_1: $D_{I_1} = \mathbb{N}$, $0_{I_1} = 0$, $s_{I_1} = \{n \mapsto n + 1 \mid n \in \mathbb{N}\}$, …

Semantic Consequences (Example)

Consider again the addition program P_{add}.

- $P_{add} \models add(x, 0, x)$
  (For every interpretation I: if $I \models P_{add}$ then $I \models add(x, 0, x)$, since $add(x, 0, x) \in P_{add}$.)
- $P_{add} \models add(x, s(0), s(x))$
  (For every interpretation I: if $I \models P_{add}$ then $I \models add(x, 0, x)$ and $I \models add(x, s(0), s(x)) \leftarrow add(x, 0, x)$ (instance of clause), thus $I \models add(x, s(0), s(x))$.)
- $P_{add} \not\models add(0, x, x)$
  (Consider interpretation I_6 from Slide 7 with $I_6 \models P_{add}$; $I_6 \not\models add(0, x, x)$, since e.g. $I_6 \not\models_\sigma add(0, x, x)$ for $\sigma(x) = 1$, since $(\sigma(0), \sigma(x), \sigma(x)) = (0, 1, 1) \notin add_{I_6}$.)

Quiz: Models and Consequences

Quiz

Consider the following logic program P where only x is a variable: ...

Soundness of SLD Resolution

Towards Soundness of SLD Resolution (1)

Lemma 4.3 (i)

Let $Q \xrightarrow[c]{\theta} Q'$ be an SLD derivation step and $Q\theta \leftarrow Q'$ the resultant associated with it. Then

$$c \models Q\theta \leftarrow Q'$$

Proof.

Let $Q = \vec{A}, B, \vec{C}$ with selected atom B. Let $H \leftarrow \vec{B}$ be the input clause and $Q' = (\vec{A}, \vec{B}, \vec{C})\theta$. Then:

	$c \models H \leftarrow \vec{B}$	(variant of c)
implies	$c \models H\theta \leftarrow \vec{B}\theta$	(instance)
implies	$c \models B\theta \leftarrow \vec{B}\theta$	(θ is a unifier of B and H)
implies	$c \models (\vec{A}, B, \vec{C})\theta \leftarrow (\vec{A}, \vec{B}, \vec{C})\theta$	(“context” unchanged) $\square$

Intuitively: The resultant is a logical consequence of the program clause.

Towards Soundness of SLD Resolution (2)

Lemma 4.3 (ii)

Let ξ be an SLD derivation of $P \cup \{Q_0\}$. For $i \geq 0$, let R_i be the resultant of level i of ξ. Then

$$P \models R_i$$

Proof.

Let $\xi = Q_0 \xrightarrow[c_1]{\theta_1} Q_1 \cdots Q_n \xrightarrow[c_{n+1}]{\theta_{n+1}} Q_{n+1} \cdots$. We use induction on $i \geq 0$:

$i = 0$: $R_0 = Q_0 \leftarrow Q_0$ is equivalent to true, thus $P \models R_0$

$i = 1$: $R_1 = Q_0\theta_1 \leftarrow Q_1$; by Lemma 4.3 (i): $P \models R_1$

$i \leadsto i+1$: By Lemma 4.3 (i), $c_{i+1} \models Q_i\theta_{i+1} \leftarrow Q_{i+1}$, thus $P \models Q_i\theta_{i+1} \leftarrow Q_{i+1}$. By (IH), $P \models R_i$, that is, $P \models Q_0\theta_1 \cdots \theta_i \leftarrow Q_i$ and in particular $P \models Q_0\theta_1 \cdots \theta_i\theta_{i+1} \leftarrow Q_i\theta_{i+1}$. In combination, $P \models Q_0\theta_1 \cdots \theta_i\theta_{i+1} \leftarrow Q_{i+1}$, that is, $P \models R_{i+1}$. $\square$

Soundness of SLD Resolution

Theorem 4.4

If there exists a successful SLD derivation of $P \cup \{Q_0\}$ with cas θ, then

$$P \models Q_0\theta$$

Proof.

Let $\xi = Q_0 \xrightarrow{\theta_1} \cdots \xrightarrow{\theta_n} \square$ be a successful SLD derivation.

Lemma 4.3 (ii) applied to the resultant of level n of ξ implies $P \models Q_0\theta_1 \cdots \theta_n$ and $Q_0\theta_1 \cdots \theta_n = Q_0(\theta_1 \cdots \theta_n \mid_{Var(Q_0)}) = Q_0\theta$. $\square$

Comparison to Intuitive Meaning of Queries

Corollary 4.5

If there exists a successful SLD derivation of $P \cup \{Q_0\}$, then $P \models \exists Q_0$.

Proof.

Theorem 4.4 implies $P \models Q_0\theta$ for some cas θ. Then,

$$
\begin{array}{ll}
 & P \models Q_0\theta \\
\text{implies} & \text{for every interpretation } I\text{: if } I \models P\text{, then } I \models Q_0\theta \\
\text{implies} & \text{for every interpretation } I\text{: if } I \models P\text{, then } I \models \forall(Q_0\theta) \\
\text{implies} & \text{for every interpretation } I\text{: if } I \models P\text{, then } I \models \exists Q_0 \\
\text{implies} & P \models \exists Q_0
\end{array}
$$

$\square$

Completeness of SLD Resolution

Towards Completeness of SLD Resolution

To show completeness of SLD resolution we need to syntactically characterize the set of semantically derivable queries.
The concepts of term models and implication trees serve this purpose.

Definition

Let E be an expression and S be a set of expressions.

- $inst(E) \;:\Longleftrightarrow\;$ set of all instances of E
- $inst(S) \;:\Longleftrightarrow\;$ set of all instances of elements $E \in S$
- $ground(E) \;:\Longleftrightarrow\;$ set of all ground instances of E
- $ground(S) \;:\Longleftrightarrow\;$ set of all ground instances of elements $E \in S$

Term Models

Definition

Let V be a set of variables, F function symbols, Π predicate symbols.

The **term algebra** J for F is defined as follows:

1. domain $D = TU_{F,V}$,
2. mapping $f_J : (TU_{F,V})^n \to TU_{F,V}$ assigned to every $f \in F^{(n)}$ with $f_J(t_1, \ldots, t_n) := f(t_1, \ldots, t_n)$

Definition

A **term interpretation** I for F and Π consists of:

1. term algebra for F,
2. $I \subseteq TB_{\Pi,F,V}$ (set of atoms that are true;
   equivalently: assignment of a relation $p_I \subseteq (TU_{F,V})^n$ to every $p \in \Pi^{(n)}$).

I is a **term model** of a set S of expressions

$:\Longleftrightarrow \quad I$ term interpretation and model of S.

Herbrand Models

Definition

The **Herbrand algebra** J for F is defined as follows:

1. domain $D = HU_F$
2. mapping $f_J : (HU_F)^n \to HU_F$ assigned to every $f \in F^{(n)}$ with
   $f_J(t_1, \ldots, t_n) := f(t_1, \ldots, t_n)$

Definition

A **Herbrand interpretation** I for F and Π consists of:

1. Herbrand algebra for F,
2. $I \subseteq HB_{\Pi,F}$ (set of ground atoms that are true).

I is a **Herbrand model** of a set S of expressions
$\quad :\Longleftrightarrow \quad I$ Herbrand interpretation and model of S

Implication Trees

Definition

$\mathcal{T}$ **implication tree** w.r.t. program P

$$:\Longleftrightarrow$$

- tree $\mathcal{T}$ is finite
- nodes are atoms
- if A is a node with the direct descendants $B_1, \ldots, B_n$ then $A \leftarrow B_1, \ldots, B_n \in inst(P)$
- if A is a leaf, then $A \leftarrow \; \in inst(P)$

$\mathcal{T}$ **ground implication tree** w.r.t. program P

$:\Longleftrightarrow \quad \mathcal{T}$ implication tree w.r.t. P and all nodes are ground atoms

Implication Trees (Example)

Let P_{add} be the addition program, $n \in \mathbb{N}$, V set of variables, $t \in TU_{\{s,0\},V}$.

Consider the tree $\mathcal{T}$ given by

$$
\begin{array}{c}
add(t, s^n(0), s^n(t)) \\
| \\
add(t, s^{n-1}(0), s^{n-1}(t)) \\
\\
\vdots \\
\\
add(t, s(0), s(t)) \\
| \\
add(t, 0, t)
\end{array}
$$

$\mathcal{T}$ is an implication tree w.r.t. P_{add}.
If additionally $t \in HU_{\{s,0\}}$, then $\mathcal{T}$ is a ground implication tree w.r.t. P_{add}.

Implication Trees Constitute Term Models

Lemma 4.7

Consider term interpretation I, atom A, program P.

- $I \models A$ iff $\mathit{inst}(A) \subseteq I$
- $I \models P$ iff for every $A \leftarrow B_1, \ldots, B_n \in \mathit{inst}(P)$,

$$\{B_1, \ldots, B_n\} \subseteq I \text{ implies } A \in I$$

Lemma 4.12

The term interpretation

$$\mathcal{C}(P) := \{\, A \mid A \text{ is the root of some implication tree w.r.t. } P \,\}$$

is a model of P.

Ground Implication Trees Constitute Herbrand Models

Lemma 4.26

Consider Herbrand interpretation I, atom A, program P.

- $I \models A$ iff $\mathit{ground}(A) \subseteq I$
- $I \models P$ iff for every $A \leftarrow B_1, \ldots, B_n \in \mathit{ground}(P)$,

$$\{B_1, \ldots, B_n\} \subseteq I \text{ implies } A \in I$$

Lemma 4.28

The Herbrand interpretation

$$\mathcal{M}(P) := \{\, A \mid A \text{ is the root of some ground implication tree w.r.t. } P \,\}$$

is a model of P.

Constituted Models (Example)

Consider again the addition program P_{add} the a set V of variables.

The term interpretation

$$\begin{aligned}\mathcal{C}(P_{add}) &= \left\{add(t, s^n(0), s^n(t)) \;\middle|\; n \in \mathbf{N}, t \in TU_{\{s,0\},V}\right\} \\ &= \left\{add(s^m(v), s^n(0), s^{n+m}(v)) \;\middle|\; m, n \in \mathbf{N}, v \in V \cup \{0\}\right\}\end{aligned}$$

and the Herbrand interpretation

$$\begin{aligned}\mathcal{M}(P_{add}) &= \left\{add(t, s^n(0), s^n(t)) \;\middle|\; n \in \mathbf{N}, t \in HU_{\{s,0\}}\right\} \\ &= \left\{add(s^m(0), s^n(0), s^{n+m}(0)) \;\middle|\; m, n \in \mathbf{N}\right\}\end{aligned}$$

are models of P_{add}.

Correct vs. Computed Answer Substitutions

Consider P_{add}

$$\begin{aligned} add(x, 0, x) &\leftarrow \\ add(x, s(y), s(z)) &\leftarrow add(x, y, z) \end{aligned}$$

along with the query $Q = add(u, s(0), s(u))$.

- $\theta = \{u/s^2(v)\}$ is a correct answer substitution of Q,
  since $P_{add} \models Q\theta = add(s^2(v), s(0), s^3(v))$ (in analogy to Slide 12 with $x = s^2(v)$)
- SLD derivation of $P_{add} \cup \{Q\}$: $add(u, s(0), s(u)) \xrightarrow[(2)]{\theta_1} add(u, 0, u) \xrightarrow[(1)]{\theta_2} \Box$
  with $\theta_1 = \{x/u, y/0, z/u\}$ and $\theta_2 = \{x/u\}$,
  thus $\eta = (\theta_1\theta_2)|_{\{u\}} = \varepsilon$ is a computed answer substitution of Q.
- We observe that η is strictly more general than θ.
- In fact, no SLD derivation of $P_{add} \cup \{Q\}$ can deliver θ.

Completeness for Implication Trees (1)

Definition

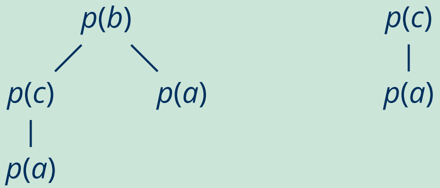
Query Q is n-**deep**

$$:\Longleftrightarrow$$

every atom in Q is the root of an implication tree, and n is the total number of nodes in these trees.

Example

Consider $P = \{p(a) \leftarrow, \quad p(c) \leftarrow p(a), \quad p(b) \leftarrow p(c), p(a)\}$. Then the query $Q = p(b), p(c)$ is 6-deep, as witnessed by these implication trees:



Completeness for Implication Trees (2)

Lemma 4.15

Suppose that query $Q\theta$ is n-deep for some $n \geq 0$, where θ is a correct answer substitution of Q.

Then for every selection rule $\mathcal{R}$, there exists a successful SLD derivation of $P \cup \{Q\}$ with cas η such that $Q\eta$ is more general than $Q\theta$.

Example

Consider $P = \{p(a) \leftarrow, \quad p(c) \leftarrow p(a), \quad p(b) \leftarrow p(c), p(a)\}$ and implication trees

$$\begin{tikzcd}[column sep=small] & p(b) \arrow[dl, dash] \arrow[dr, dash] & \\ p(c) \arrow[d, dash] & & p(a) \\ p(a) & & \end{tikzcd} \qquad \begin{tikzcd} p(c) \arrow[d, dash] \\ p(a) \end{tikzcd}$$

Completeness of SLD Resolution (1)

Theorem 4.13

Suppose that θ is a correct answer substitution of Q.

Then for every selection rule $\mathcal{R}$, there exists a successful SLD derivation of $P \cup \{Q\}$ with cas η such that $Q\eta$ is more general than $Q\theta$.

Proof.

Let $Q = A_1, \ldots, A_m$. Then:

	θ correct answer substitution of $A_1, \ldots, A_m$
implies	$P \models A_1\theta, \ldots, A_m\theta$
implies	for every interpretation I: if $I \models P$, then $I \models A_1\theta, \ldots, A_m\theta$
implies	$\mathcal{C}(P) \models A_1\theta, \ldots, A_m\theta$ (since $\mathcal{C}(P) \models P$ by Lemma 4.12)
implies	$inst(A_i\theta) \subseteq \mathcal{C}(P)$ for every $1 \leq i \leq m$ (by Lemma 4.7)
implies	$A_i\theta \in \mathcal{C}(P)$ for every $1 \leq i \leq m$
implies	$A_1\theta, \ldots, A_m\theta$ is n-deep for some $n \geq 0$ (by def. of $\mathcal{C}(P)$)
implies	claim (by Lemma 4.15) $\square$

Completeness of SLD Resolution (2)

Corollary 4.16

Suppose $P \models \exists Q$.

Then there exists a successful SLD derivation of $P \cup \{Q\}$.

Proof.

$$
\begin{array}{ll}
 & P \models \exists Q \\
\text{implies} & P \models Q\theta \text{ for some substitution } \theta \\
\text{implies} & \theta \text{ correct answer substitution of } Q \\
\text{implies} & \text{claim (by Theorem 4.13)} \qquad \square
\end{array}
$$

Conclusion

Summary

- The semantics of (definite) logic programs is given by a standard first-order model theory.
- SLD resolution is **sound**: For every successful SLD derivation of $P \cup \{Q_0\}$ with *computed* answer substitution θ, we have $P \models Q_0\theta$.
- SLD resolution is **complete**: If θ is a *correct* answer substitution of Q, then
  - for every selection rule
  - there exists a successful SLD derivation of $P \cup \{Q\}$ with cas η
  - such that $Q\eta$ is more general than $Q\theta$.

Suggested action points:

- Compare implication trees to SLD trees
- Clarify the distinction between *computed* and *correct* answer substitutions