PROBLEM SOLVING AND SEARCH IN ARTIFICIAL INTELLIGENCE

Lecture 11 Hypertree Decompositions

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Agenda

1. Introduction
2. Uninformed Search versus Informed Search (Best First Search, A* Search, Heuristics)
3. Local Search, Stochastic Hill Climbing, Simulated Annealing
4. Tabu Search
5. Answer-set Programming (ASP)
6. Constraint Satisfaction Problems (CSP)
7. Evolutionary Algorithms/ Genetic Algorithms
8. Structural Decomposition Techniques (Tree/Hypertree Decompositions)
Motivation

• The structure of a large number of problems is more faithfully described by a hypergraph than by a graph

• Several $NP$ complete problems become tractable if restricted to instances with acyclic hypergraphs

• An appropriate notion of hypergraph width should fulfil both of the following conditions
  1. Relevant hypergraph-based problems should be solvable in polynomial time for instances of bounded width
  2. For each constant $k$, one should be able to check in polynomial time whether a hypergraph is of width $k$, and, in the positive case, it should be possible to produce an associated decomposition of width $k$ of the given hypergraph

• The hypertree decomposition is the most general method leading to large tractable classes of important problems such as constraint satisfaction problems or conjunctive queries
A generalized hypertree decomposition (GHD) of $H$ is a tree decomposition of $H$ with the following extension.

- GHD associates additionally to each node of the decomposition tree the set of hyperedges of $H$.
- The set of vertices associated to each node of the tree must be covered by the set of hyperedges associated to that node.
- The width of a generalized hypertree decomposition is the maximum number of hyperedges associated to a same node of the decomposition.
A hypertree for a hypergraph $\mathcal{H} = (V(\mathcal{H}), H(\mathcal{H}))$ is a triple $\langle T, \chi, \lambda \rangle$, where $T = (N, E)$ is a rooted tree, and $\chi$ and $\lambda$ are labeling functions which associate to each vertex $p \in N$ two sets

- $\chi(p) \subseteq V(\mathcal{H})$ and
- $\lambda(p) \subseteq H(\mathcal{H})$.

If $T' = (N', E')$ is a subtree of $T$, we define $\chi(T') = \bigcup_{v \in N'} \chi(v)$. We denote the set of vertices $N$ of $T$ by $\text{vertices}(T)$, and the root of $T$ by $\text{root}(T)$. Moreover, for any $p \in N$, $T_p$ denotes the subtree of $T$ rooted at $p$. 
Hypertree Decomposition

**Definition ([Gottlob et al. (2002)])**

Let $\mathcal{H} = (V(\mathcal{H}), H(\mathcal{H}))$ be a hypergraph. A **hypertree decomposition** of $\mathcal{H}$ is a hypertree $\langle T, \chi, \lambda \rangle$ for $\mathcal{H}$ which satisfies all the following conditions:

1. For each hyperedge $h \in H(\mathcal{H})$, there exists $p \in \text{vertices}(T)$ such that $\text{vertices}(h) \subseteq \chi_p$;
2. For each vertex $y \in V(\mathcal{H})$, the set $\{p \in \text{vertices}(T) \mid y \in \chi_p\}$ induces a (connected) subtree of $T$;
3. For each vertex $p \in \text{vertices}(T)$, $\chi_p \subseteq \text{vertices}(\lambda_p)$;
4. For each vertex $p \in \text{vertices}(T)$, $\text{vertices}(\lambda_p) \cap \chi(T_p) \subseteq \chi_p$.

The **width** of the hypertree decomposition $\langle T, \chi, \lambda \rangle$ is $\max_{p \in \text{vertices}(T)} |\lambda_p|$. The **hypertree width**, $hw(\mathcal{H})$, of $\mathcal{H}$ is the minimum width over all its hypertree decompositions.

**Note:** inclusion in Condition 4 is an equality, as Condition 3 implies the reverse inclusion!
Generalized Hypertree Decomposition

Generalized hypertree decomposition does not include condition 4) of hypertree decomposition.
Generalized Hypertree Decomposition

Special condition violated

Generalized hypertree decomposition of width 2
Hypertree Decomposition

Hypertree decomposition of width 3
Hypertree Decomposition

Hypertree decomposition of width 3
Hypertree Decomposition

Hypertree decomposition of width 3
Hypertree Decomposition

Hypertree decomposition of width 3
Hypertree Decomposition

Hypertree decomposition of width 3
Hypertree Decomposition

Hypertree decomposition of width 3
Hypertree Decomposition

Hypertree decomposition of width 3
Hypertree Width and CSPs

- The smaller the width of the obtained hypertree decomposition, the faster the corresponding CSP instance can be solved.
- A CSP instance can be solved based on its hypertree decomposition as follows:
  - for each node \( t \) of the hypertree, all constraints in \( \lambda(t) \) are “joined” into a new constraint over the variables in \( \chi(t) \)
  - for bounded width, i.e., for bounded cardinality of \( \lambda(t) \), this yields a polynomial time reduction to an equivalent acyclic CSP instance.
Algorithms for Generalized Hypertree Decomposition

- Methods based on tree decomposition
  - Generalized hypertree decomposition can be generated by algorithms for tree decomposition + Set Covering
- Hypertree decomposition based on hypergraph partitioning
- Exact methods
- Literature and benchmark instances for hypertree decomposition:
  http://www.dbai.tuwien.ac.at/proj/hypertree/
  http://wwwinfo.deis.unical.it/~frank/Hypertrees/
Recall, a hypertree decomposition can be divided into two parts:

1. Definition of a tree decomposition $(T, \chi)$
2. Introduction of $\lambda$ such that $\chi(t) \subseteq \bigcup \lambda(t)$ for every node $t$.

$\chi$-labels contain vertices of the hypergraph and $\lambda$-labels contain hyperedges, i.e., sets of vertices, of the hypergraph (covering vertices in $\chi(t)$ by hyperedges in $\lambda(t)$).
Constructing Generalized Hypertree Decomposition from Tree Decomposition ctd.

Apply for each node of tree decomposition set covering

1,11,17,19

1,2,3,4,5,6

3,4,5,6,7,8

5,6,7,8,9

7,9,10

11,12,17,18,19

12,16,17,18,19

12,15,16,18,19

12,13,14,15,18,19
Constructing Generalized Hypertree Decomposition from Tree Decomposition ctd.

Apply for each node of tree decomposition set covering

- h1
  - h2
  - h3
  - h4
  - h5
  - h6
  - h7
  - h8
  - h9
  - h10
  - h11
  - h12
  - h13
  - h14
  - h15
  - h16
  - h17
  - h18

- 1,2,3,4,5,6
  - 3,4,5,6,7,8
  - 5,6,7,8,9
  - 7,9,10

- 1,11,17,19
  - 11,12,17,18,19
  - 12,15,16,18,19
  - 12,13,14,15,18,19
Generalized Hypertree Decomposition

Generalized hypertree decomposition of width 2
Hypertree Decomposition Based on Hypergraph Partitioning

A method for generation of generalized hypertree decompositions based on recursive partitioning of the hypergraph [Dermaku et al.(2008)].

Hypergraph Partitioning

Given a hypergraph $\mathcal{H}(V, H)$ with weighted vertices and hyperedges.

- Find a partition of set $V$ in two (or $k$) disjoint subsets such that the number of vertices in each set $V_i$ is bounded, and the function defined over hyperedges is optimized.
- Most commonly used objective is to minimize the sum of the weights of hyperedges connecting two or more subsets.
Hypergraph partitioning with constraint about the number of vertices in each partition is NP-Complete problem!
Does recursive partitioning of hypergraph lead to "good" hypertree decomposition?

Every cut in hypergraph partitioning can be considered as a node in a hypertree decomposition (called separator)

Add a special hyperedge to each subgraph containing the vertices in the intersection between the subgraphs to enforce joint appearance in the $\chi$-label of a later generated node

Connectedness condition for variables should be ensured!

How to evaluate a cut whose separator contains such hyperedges?
  - associate weights to hyperedges
  - weight 1 for all ordinary hyperedges
  - $(W^+)$ weight of special hyperedge: number of ordinary hyperedges needed to cover the vertex of the special hyperedge
  - other weighting schemes associate different weights to special hyperedges (always weight 1 or weight 2)
  - cut evaluates as the sum of weights of all hyperedges in the separator

Nodes of hypertree are connected at the end of partitioning
From Partitioning to Hypertree

![Diagram of a hypertree with labeled hyperedges and nodes, indicating that all hyperedges have weight 1 and a cut is shown.]
From Partitioning to Hypertree

Node n of hypertree

Cut

h1, h7
From Partitioning to Hypertree

Node of hypertree

h1, h7

P1

P2
From Partitioning to Hypertree

Node n of hypertree

To ensure the connectedness condition nodes 1, 8, 6 should appear together in some node s. To the end this node will be connected to node n above.
From Partitioning to Hypertree

To ensure the connectedness condition nodes 1, 8, 6 should appear together in some node k. To the end this node will be connected to node n above.

Enforce this by introducing new hyperedge which contains all these nodes.
From Partitioning to Hypertree

Node n of hypertree

To ensure the connectedness condition nodes 1, 8, 6 should appear together in some node k. To the end this node will be connected to node n above.

Enforce this by introducing new hyperedge which contains all these nodes.
From Partitioning to Hypertree

Node \( n \) of hypertree

Continue recursively the partitioning.

The weight of hyperedge \( hs1 \) is the sum of weights of hyperedges which cover this hyperedge: in this case \( w(hs1)=2 \).
From Partitioning to Hypertree

Node n of hypertree

h1, h7

h9, h10

Cut
From Partitioning to Hypertree

Node n of hypertree

Cut

h9, h10

h1, h7
From Partitioning to Hypertree

Node n of hypertree

- h1, h7
- h9, h10
From Partitioning to Hypertree

Node n of hypertree

h1, h7

Node s of hypertree

h6, h8

Cut

h9, h10
From Partitioning to Hypertree

Node n of hypertree

Node s of hypertree

Covers hs1 => connect to node n
From Partitioning to Hypertree

![Diagram showing a hypertree structure with nodes labeled h1, h7, h6, h8, h9, h10, and hs1, hs2, h6, h8, h10. The diagram illustrates the process of partitioning and cutting.]
Summary

- Hypertree decomposition is a method leading to a large class of tractable problems such as CSP
- Computation of generalized hypertree decomposition based
  - on tree decompostion + Set Covering
  - hypergraph partitioning
